Investigation on Path-Integral Expression of the J -Integral Range Using Numerical Simulations of Fatigue Crack Growth*

Shiro KUBO**, Tateki YAFUSO***,
Masatoshi NOHARA****, Toshiya ISHIMARU†
and Kiyotsugu OHJI**

Fatigue crack growth rates under elastic-plastic conditions have been reported to correlate well with the J -integral range, ΔJ , evaluated experimentally from load-point displacement using formula developed for specific specimen configurations. However, the theoretical basis of ΔJ still remains unestablished. In this paper, path-integral expressions of ΔJ and their approximate path-independence in the presence of the effects of strain hardening and crack closure were discussed. Finite element simulations of elastic-plastic crack growth under cyclic loading were conducted. It was found that ΔJ values evaluated by the path-integral were almost path-independent when the minimum load point or the crack opening point was selected as a reference point for evaluation. The examination of the relationship between ΔJ and crack-tip opening displacement or strain distribution near the crack tip led to the conclusion that ΔJ evaluated on the basis of the crack opening point was the most appropriate parameter in characterizing the near-tip stress and strain fields under fatigue conditions.

Key Words: Fracture, Fracture Mechanics, Fatigue, Crack Growth, J -Integral Range, Finite Element Simulation, Crack Closure

1. Introduction

Many experiments have shown that fatigue crack growth rates under elastic-plastic conditions correlate well with the J -integral range, $\Delta J^{(1)-(7)}$. The ΔJ values used in these correlations were evaluated experimentally from load vs displacement curves using formula⁽⁸⁾ which were analogous to simple estimating equations of J -integral under monotonic loading conditions for specific specimen configurations⁽⁹⁾. A theoretical basis of ΔJ has not yet been established. Therefore, no general scheme for evaluating ΔJ for other specimen geometries or practical situations is available, although it has been found that the resistance of fatigue crack growth can be expres-

sed in terms of ΔJ for specific laboratory specimens. In this paper, we discuss path-integral definitions of ΔJ and examine the validity of ΔJ as a fracture mechanics correlating parameter to fatigue crack growth, using finite-element simulations of elastic-plastic fatigue crack growth, in which crack closure is taken into account.

2. Definition of the J -integral range ΔJ

2.1 Path-integral expression of ΔJ

A path-integral expression of ΔJ for a two-dimensional crack under cyclic loading was given by Lamba⁽¹⁰⁾, Wütrich⁽¹¹⁾, and Tanaka⁽¹²⁾ based on certain ideals. It was assumed in these papers that $\Delta\sigma_{ij}$ was a unique function of $\Delta\epsilon_{ij}$, where $\Delta\sigma_{ij}$ was the difference between the stress at minimum load and that at a given load, and $\Delta\epsilon_{ij}$ denoted the corresponding difference in strain during a loading phase. Hence the following quantity can be defined which is equivalent to the strain energy density used in the definition of the J -integral.

$$V = \int_0^{\Delta\epsilon_{ij}} \Delta\sigma_{ij} d\Delta\epsilon_{ij} \quad (1)$$

A path-integral expression of the J -integral range,

* Received 1st September, 1988. Paper No. 87-0526A

** Department of Mechanical Engineering, Osaka University 2-1 Yamadaoka, Suita, Osaka, 565, Japan

*** Department of Mechanical Engineering, University of the Ryukyus, Senbaru, Nishihara, Okinawa, 903-01, Japan

**** Toyota Motor Corp., 1 Toyota-cho, Toyota, Aichi, 471, Japan

† Richo Co. Ltd., 13-1 Himemuro-cho, Ikeda, Osaka, 563, Japan

ΔJ , analogous to the J -integral expression, was presented assuming that no crack closure occurred:

$$\Delta J = \int_{\Gamma} (V \nu_i - \Delta T_i \partial \Delta u_i / \partial x_i) d\Gamma \quad (2)$$

where x_1 and x_2 were the Cartesian coordinates with the x_1 -axis parallel to the crack surface, Γ was a path surrounding the crack tip as shown in Fig. 1, ΔT_i was the range of traction vector on Γ , Δu_i was a difference of the displacement, ν_i was the unit vector perpendicular to Γ , and $d\Gamma$ was an element of arc length along Γ .

2.2 Path-independence of ΔJ

The path-independence of ΔJ can be proven when V is given by a unique function of $\Delta \varepsilon_{ij}$ and crack closure does not occur^{(10),(11)}. The path-independence of ΔJ does not hold in general because of the involvement of crack closure and the effect of the hardening parameter which violates the one-to-one correspondence between V and $\Delta \varepsilon_{ij}$. In this section, we will investigate the path-independence of ΔJ defined by Eq. (2), when the minimum load point or the crack-opening point in a cyclic stress-strain hysteresis loop is used as a reference point for the evaluation of $\Delta \sigma_{ij}$, $\Delta \varepsilon_{ij}$, ΔT_i , and Δu_i . It is assumed that loading is almost proportional at any point of the body in the same manner as is assumed in the discussion of the J -integral⁽⁸⁾.

In Fig. 1, Γ_t is an integration path taken in the vicinity of the crack tip, and S_1 and S_2 are paths on the crack surfaces. A domain A is enclosed by the paths Γ , Γ_t , S_1 and S_2 . Equation (2) evaluated along a closed path ($\Gamma + S_1 - \Gamma_t + S_2$) can be transformed to an area integral by the divergence theorem, as in the following.

$$\begin{aligned} & \int_{\Gamma + S_1 - \Gamma_t + S_2} (V \nu_i - \Delta T_i \partial \Delta u_i / \partial x_i) d\Gamma \\ &= \int_A \left(\frac{\partial V}{\partial x_i} - \frac{\partial \Delta \sigma_{ij}}{\partial x_j} \frac{\partial \Delta u_i}{\partial x_i} - \Delta \sigma_{ij} \frac{\partial^2 \Delta u_i}{\partial x_j \partial x_i} \right) dA \quad (3) \end{aligned}$$

when the cyclic stress-strain relationship is stable and the range of strain $\Delta \varepsilon_{ij}$ corresponds uniquely to the stress range $\Delta \sigma_{ij}$, the first term of the integrand in Eq. (3) is given by

Fig. 1 Integration path of ΔJ

$$\partial V / \partial x_i = \Delta \sigma_{ij} \partial \Delta \varepsilon_{ij} / \partial x_i \quad (4)$$

If such a one-to-one correspondence between $\Delta \varepsilon_{ij}$ and $\Delta \sigma_{ij}$ does not exist, the expression for $\partial V / \partial x_i$ is much more complicated, as follows.

Figure 2 shows the stress-strain relationship at a material point in the body. Point A in the diagram is a reference point (to be called a base point) of evaluation of $\Delta \varepsilon_{ij}$ and $\Delta \sigma_{ij}$. We may choose the minimum load point or the crack opening point as the base point. Yielding takes place initially at point B during loading. The difference between the strains at A and B is denoted by β_{ij} .

We suppose that the relationship between strain-increment and stress-increment beyond the yielding point B is independent of the loading history. The $\partial V / \partial x_i$ is given by a summation of the contributions due to the change in $\Delta \varepsilon_{ij}$ and β_{ij} . If β_{ij} changes by $d\beta_{ij}$, keeping $\Delta \varepsilon_{ij}$ constant, the value of V is shown as the area bounded by the broken lines in Fig. 2. Thus, the change of V can be expressed in the form:

$$(-\Delta \sigma_{ij} d\beta_{ij} + \partial \Delta \sigma_{ij} / \partial \Delta \varepsilon_{mn} |_A d\beta_{mn} \Delta \varepsilon_{ij})$$

where $\partial \Delta \sigma_{ij} / \partial \Delta \varepsilon_{mn} |_A$ is the partial differential evaluated at point A. Therefore, $\partial V / \partial x_i$ can be written as follows using an exchange of dummy subscript:

$$\begin{aligned} \frac{\partial V}{\partial x_i} &= \frac{\partial V}{\partial \Delta \varepsilon_{ij}} \frac{\partial \Delta \varepsilon_{ij}}{\partial x_i} + \frac{\partial V}{\partial \beta_{ij}} \frac{\partial \beta_{ij}}{\partial x_i} \\ &= \Delta \sigma_{ij} \frac{\partial \Delta \varepsilon_{ij}}{\partial x_i} - \left(\Delta \sigma_{ij} - \frac{\partial \Delta \sigma_{mn}}{\partial \Delta \varepsilon_{ij}} |_A \Delta \varepsilon_{mn} \right) \frac{\partial \beta_{ij}}{\partial x_i} \quad (5) \end{aligned}$$

Substituting Eq. (5) into Eq. (3) and using the symmetry of stress tensor $\sigma_{ij} = \sigma_{ji}$, the equilibrium equation $\partial \sigma_{ij} / \partial x_j = 0$ and infinitesimal strain-displacement relation $\varepsilon_{ij} = (\partial u_i / \partial x_j + \partial u_j / \partial x_i) / 2$, we obtain the following equation.

$$\begin{aligned} & \int_{\Gamma_t} (V \nu_i - \Delta T_i \partial \Delta u_i / \partial x_i) d\Gamma \\ &= \int_{\Gamma} (V \nu_i - \Delta T_i \partial \Delta u_i / \partial x_i) d\Gamma \end{aligned}$$

Fig. 2 Evaluation of V

$$\begin{aligned}
& - \int_A \left(\frac{\partial \Delta \sigma_{mn}}{\partial \Delta \varepsilon_{ij}} \bigg|_A \Delta \varepsilon_{mn} - \Delta \sigma_{ij} \right) \frac{\partial \beta_{ij}}{\partial x_1} dA \\
& - \int_{S_1 + S_2} \Delta T_i \partial \Delta u_i / \partial x_1 d\Gamma
\end{aligned} \quad (6)$$

Equation (6) gives the relation between ΔJ evaluated along the path Γ_i in the vicinity of crack tip and ΔJ evaluated along an arbitrary path Γ :

First we will discuss the path-independence of the minimum load point based ΔJ , which is calculated by taking the minimum load point as the base point for evaluating the $\Delta \sigma_{ij}$, $\Delta \varepsilon_{ij}$, ΔT_i , and Δu_i . Under the kinematic hardening law, the shape and size of yield surface remain the same even after cyclic deformation. When the cyclic stress-strain relation is stable, yielding takes place at the minimum load as well as at the maximum load in the region where cyclic plastic deformation occurs. As a consequence, the $\Delta \sigma_{ij} - \Delta \varepsilon_{ij}$ relationship during the load-increasing phase coincides with the $\sigma_{ij} - \varepsilon_{ij}$ relationship under monotonic loading with doubled yield stress. This indicates that V is a unique function of $\Delta \varepsilon_{ij}$: therefore, the second term of the right-hand side of Eq. (5) does not contribute to $\partial V / \partial x_1$ and that of Eq. (6) does not contribute to ΔJ .

Furthermore, in the region where the deformation is elastic ($\partial \Delta \sigma_{mn} / \partial \Delta \varepsilon_{ij} \big|_A \Delta \varepsilon_{ij} - \Delta \sigma_{ij}$) Eq. (6) becomes zero due to the linear relationship between the stress increment and the strain increment. The second term of the right-hand side, therefore, becomes zero. As a result, the second term of the right-hand side of Eq. (6) does not contribute to ΔJ under the kinematic hardening law, irrespective of whether the material is elastic-plastic or elastic.

We will now consider the third term of the right-hand side of Eq. (6). When the crack is closed, $u_2 = 0$ and $T_1 = 0$; when it is open $T_1 = T_2 = 0$. Thus, the contribution of the third term to ΔJ is expected to be small.

Although several assumptions or approximations are involved in the above discussion, it is expected that the second and third terms of the right-hand side of Eq. (6) are sufficiently small when the cyclic deformations under the kinematic hardening law are stable. Hence, the approximate path-independence of the minimum load point based ΔJ holds. When the size of the yield surface depends on cyclic history, the area integral given by the second term of the right-hand side of Eq. (6) contributes to ΔJ .

The crack opening and closure of a fatigue crack have a significant effect on the fatigue crack growth. Keeping this in mind, we will discuss the path-independence of the opening point based ΔJ , which is calculated by employing the opening point as the base point for evaluating $\Delta \sigma_{ij}$, $\Delta \varepsilon_{ij}$, ΔT_i and Δu_i . In this

case, ΔT_i is equal to zero since the crack remains open at any point above the base point. Consequently, the third term of the right-hand side of Eq. (6) becomes zero. The path-independence of ΔJ may possibly be lost due to the contribution of the second term. Because the singularity of stresses and strains at the crack tip is lost when the crack is not open, the strain difference between the minimum load point and the opening point is expected to be sufficiently small in the vicinity of the crack tip. Thus, the second term of the right-hand side of Eq. (6) may be negligibly small under the kinematic hardening law. The path-independence of the opening point based ΔJ , therefore, may hold as for the minimum load point based ΔJ .

As shown in the above discussion, the approximate path-independence of ΔJ is expected to hold under certain conditions, although path-independence of ΔJ is generally lost when the influence of hardening or crack closure is involved. The ΔJ value in the vicinity of the crack tip can be evaluated for any path enclosing the crack tip if the surface integral and the path-integral along the crack surface as well as the path-integral along Γ are calculated in accordance with Eq. (6), even when the path-independence does not hold.

2.3 Simple estimation of ΔJ

According to Dowling⁽²⁾, the ΔJ of deeply center-cracked specimens can be evaluated using the hysteresis loop of the load P vs load point displacement δ as in the following:

$$\Delta J_{DO-\delta} = \Delta K_{eff}^2 / E + F / (Bb) \quad (7)$$

As illustrated in Fig. 3, the crack opening point DO is assumed to be on the smooth extension of the unloading curve, and F is half of the area hatched by vertical lines. ΔK_{eff} is the range of the effective stress intensity factor; E is Young's modulus; B is the specimen thickness, and b is a ligament length. In investigating the changes of $\Delta J_{DO-\delta}$ in a loading cycle, the $\Delta J_{DO-\delta}$ values are calculated using Eq. (7), where the area F' is substituted for F (see Fig. 3) and ΔK_{eff} is evaluated from the load increase above DO point.

Fig. 3 Simple estimation of ΔJ

3. Simulations of fatigue crack growth

3.1 Method of calculation

Finite element simulations of elastic-plastic fatigue crack growth were conducted, and the behavior of ΔJ in the presence of crack opening and closure was investigated. The method of finite element analysis used in the simulations is the same as used by Ohji and Ogura^{(13)~(14)} for the analysis of crack opening and closure. The crack extension was simulated by releasing the connection of a current crack-tip node at the minimum load. Geometrically similar center-cracked specimens with half-widths of 7.5 mm, 15 mm and 30 mm were used for the finite element analyses (Fig. 4). Uniform tensile load was applied at both ends of the specimens. Only one-quarter of the specimen was analyzed because of the symmetry. A triangular constant strain element was used (Fig. 5). Smaller elements were arranged in the vicinity of the crack tip. The minimum element size was set at 0.2 mm for the three specimens in order to make a meaningful comparison possible among the near-tip quantities for these specimens. The paths for the evaluation of ΔJ and J are shown by heavy lines Fig. 5. The initial value of the half-crack length a_0 was about 0.4 W .

It was assumed that the cyclic deformation behavior of the materials was described by the kinematic hardening law, which was expressed by a bilinear

Fig. 4 Specimen geometry

Fig. 5 Finite-element mesh

stress-strain relation. The mechanical properties of the materials were assumed as follows: Young's modulus $E=210$ GPa, strain hardening coefficient $H'=12$ GPa, yield stress $\sigma_y=300$ MPa, Poisson's ratio $\nu=0.3$. All analyses were carried out under the plane stress condition. The completely reversed loads were applied with the maximum nominal stress of $\sigma_{\max}=198$ MPa, 180 MPa, 162 MPa and 144 MPa, calculated for an uncracked cross-sectional area.

In order to calculate ΔJ by the path-integral, the values of stress σ_{ij} , strain ε_{ij} and u_{ij} at the base point were stored, and the differences, $\Delta\sigma_{ij}$, $\Delta\varepsilon_{ij}$, and Δu_{ij} , were evaluated referring to these values and the corresponding current values. The minimum load point based ΔJ evaluated by the path-integral was denoted by $\Delta J_{\text{MIN-path}}$. Since there was an ambiguity in the definition of the opening point in the finite element simulations of fatigue crack growth, we defined the opening point in three ways: (1) when the nearest node to the crack tip on the crack surface becomes free, [TO(tip-opening)]; (2) when all nodes on crack surfaces become free, [FO(full-opening)]; and (3) when the nodal force acting on the crack tip node becomes zero, [NF(node-free)]. The ΔJ values calculated based on the TO-point, FO-point and NF-point were denoted by $\Delta J_{\text{TO-path}}$, $\Delta J_{\text{FO-path}}$ and $\Delta J_{\text{NF-path}}$, respectively.

In order to evaluate $\Delta J_{\text{DO-s}}$ by the simple estimating equation (Eq. (7)), the intersection Do of the smooth extension of the unloading curve the loading line was determined automatically in the simulation program. For comparison with the ΔJ of the various definitions mentioned in the foregoing, conventional values of the J -integral were also calculated, by using the conventional ordinary path-integral expression⁽⁸⁾ and the current values of stress, strain and displacement.

3.2 Results and discussion

Figure 6 shows examples of the shape of plastic zone obtained under the conditions employed in the simulations. At lower applied stress levels, the plastic zone was well contained by the elastic zone as shown in Fig. 6(a), while the plastic zone developed through the ligament area at higher applied stress levels, as shown in Fig. 6(b).

Figure 7 shows an example of nominal stress vs load point displacement hysteresis loop in the case where the plastic zone is large. Fairly large plastic displacement was included in the total displacement. In the case shown in Fig. 7, the crack tip opening point designated by TO coincided with the full opening point FO. Furthermore, the TO-point is not far from the NF-point. The three opening points based ΔJ , ($\Delta J_{\text{TO-path}}$, $\Delta J_{\text{FO-path}}$ and $\Delta J_{\text{NF-path}}$) were therefore,

almost equal. Point DO was considerably lower than TO and was almost on the same nominal stress level as the crack closure point CL.

The values of $\Delta J_{\text{MIN-path}}$, $\Delta J_{\text{TO-path}}$, $\Delta J_{\text{NF-path}}$ and J evaluated by the path-integral at several load levels during a loading phase are shown in Fig. 8(a), (b), (c) and (d), respectively. Since FO agreed with TO for the case shown, $\Delta J_{\text{FO-path}}$ vs path-number relationship was equivalent to $\Delta J_{\text{TO-path}}$ vs path-number relationship. The values of $\Delta J_{\text{MIN-path}}$ evaluated for the four paths were almost constant, as was suggested in section 2.2, while $\Delta J_{\text{TO-path}}$ and $\Delta J_{\text{NF-path}}$ showed the tendency of gradual decrease for small path-numbers. However, the differences in ΔJ for the paths adopted were within 10% of the maximum value of ΔJ . This was true for other conditions employed in the calculations. It can be concluded that the path-independence of ΔJ holds approximately. On the other hand, the differences in J were more than 20% of the maximum value of J in some cases.

Figure 9 illustrates the changes of the averaged values of $\Delta J_{\text{MIN-path}}$, $\Delta J_{\text{TO-path}}$, $\Delta J_{\text{NF-path}}$ and J with respect to all the paths selected, and also $\Delta J_{\text{DO-}\delta}$ in

Fig. 6 Plastic zone shape

relation to load point displacement, during the load-increasing phase. The values of $\Delta J_{\text{MIN-path}}$ and $\Delta J_{\text{DO-}\delta}$ are larger than the others. The opening point based $\Delta J_{\text{TO-path}}$ and $\Delta J_{\text{NF-path}}$, exhibit a rapid increase after the TO point.

One of the representative values of the cyclic deformations in the vicinity of the crack tip may be the varying range of the crack tip opening displacement. The vertical displacement at the nearest node to the crack tip on the crack faces was taken, for the sake of convenience, as the crack tip opening displacement CTOD in the simulations. The relationships between the range of CTOD, Δ CTOD and ΔJ or J are shown in Fig. 10. The numerals attached to the plotted data represent conditions of analysis: the first two digits of the numerals represent the width of the specimen in mm and the following three digits designate σ_{max} in MPa. The solid lines are regression lines by means of the least squares method.

In the relationships between CTOD and opening point based ΔJ (i. e. $\Delta J_{\text{TO-path}}$, $\Delta J_{\text{FO-path}}$ and $\Delta J_{\text{NF-path}}$), data plots for various combinations of W and σ_{max} fall within the narrow band around the regression lines. A fairly good correlation is observed between $\Delta J_{\text{DO-}\delta}$ and Δ CTOD. Because the load at point was almost equal to that of the crack closure as mentioned above, a similar result could be obtained by using ΔJ based on the closure load, instead of $\Delta J_{\text{DO-}\delta}$. As for the relationships of $\Delta J_{\text{MIN-path}}$ vs Δ CTOD and J vs Δ CTOD, however, the slopes of the data shown by $\blacktriangle$, $\triangle$, $\bullet$, $\circ$ were smaller than that of the regression line for the entire plot. Hence, the opening point based ΔJ and $\Delta J_{\text{DO-}\delta}$ are considered to be better than $\Delta J_{\text{MIN-path}}$ or J as a parameter for controlling the amplitude of stress and strain. Moreover, the amount of scatter is more or less smaller when the opening point was used than when the closure point was used.

Figure 11 shows the relationships of CTOD with $\Delta J_{\text{MIN-path}}$, $\Delta J_{\text{TO-path}}$, and J during the loading phases. The three solid lines in Fig. 11 are regression lines showing the relationship between Δ CTOD and ΔJ or

Fig. 7 Nominal stress σ -load point displacement δ hysteresis loop

J at maximum loads. The relationship between CTOD and $\Delta J_{\text{TO-path}}$ during the loading phase agrees closely with that for the maximum value obtained under various conditions. Similarly, close agreement was also recognized for the $\Delta J_{\text{FO-path}}$ -CTOD or $\Delta J_{\text{NF-path}}$ -CTOD relationship. On the other hand, the relationship between $\Delta J_{\text{MIN-path}}$ and CTOD for small CTODs did not agree with the $\Delta J_{\text{MIN-path}}$ -CTOD relationship.

The range of strain $\Delta \varepsilon_y$ in the $x_2(y)$ -direction in front of the crack tip was plotted against the distance r_x from the crack tip as in Fig. 12. In this figure, it is found that, at a given $\Delta J_{\text{TO-path}}$ level, the $\Delta \varepsilon_y$ - r_x relationships near the crack tip are almost identical for three specimen types. Also, one may conclude that singularity in $\Delta \varepsilon_y$ exists, considering that the solution is disturbed in the very vicinity of the crack tip due to discretization errors.

From the above discussion, it can be concluded that the strain range near the crack tip is determined

Fig. 8 Examination of path independence of ΔJ and J

Fig. 9 Change of ΔJ and J during a loading phase

Fig. 10 $\Delta CTOD$ - ΔJ and $\Delta CTOD$ - J relationships

Fig. 11 CTOD- ΔJ relationship during loading phases

Fig. 12 The range of strain in front of a crack tip

by the opening point based ΔJ .

4. Conclusions

Path-integral expressions of the J -integral range, ΔJ , were discussed, and the evaluation of ΔJ was made by finite element simulation of fatigue crack growth. The path-independence of ΔJ was demonstrated when the minimum load point or the crack-opening point was employed as a reference point for the evaluation. The relationship between ΔJ and crack tip opening displacement CTOD and the strain fields near the crack tip showed that ΔJ , based on the opening point or closure load, was an appropriate parameter in characterizing the range of the stress, strain and displacement in the vicinity of the crack tip.

Acknowledgement

The authors are grateful to Dr. Yoshikazu Nakai

of Osaka University for valuable discussions. This research was partly supported by the Ministry of Education under Grant in-Aid for Scientific Research.

References

- (1) Dowling, N.E. and Begley, J., Fatigue Crack Growth During Gross Plasticity and the J -Integral, ASTM STP, 590 (1976), p. 82.
- (2) Dowling, N.E., Geometry Effects and the J -Integral Approach to Elastic-Plastic Fatigue Crack Growth, ASTM STP, 601, (1976), p. 19.
- (3) Taira, S., Tanaka, K. and Ogawa, S., Fatigue Crack Propagation Law under Elastic-Plastic and General Yield Conditions, J. Soc. Mater. Sci., Jpn. (in Japanese), Vol. 26, No. 280 (1977), p. 93.
- (4) Ando, K. and Ogura, N., Evaluation of Fatigue Crack Propagation Rate by Means of J -Integral, J. Soc. Mater. Sci., Jpn., (in Japanese), Vol. 29, No. 325 (1978), p. 767.
- (5) Kobayashi, H., Shibuya, E. and Nakazawa, H., Elastic-Plastic Fracture Mechanics Approach to Fatigue Crack Growth, J. Soc. Mater. Sci., Jpn., (in Japanese), Vol. 29, No. 325 (1980), p. 1049.
- (6) Hoshide, T. and Tanaka, K., Analysis of Fatigue Crack Propagation from Surface Flaws by Elastic Plastic Fracture Mechanics, Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 48, No. 433, A (1982), p. 1102.
- (7) Hatanaka, K., Fujimitsu, T. and Watanabe, H., Growth Behaviours of Small Surface Cracks in Low Carbon Steel Fatigued under Rotating Bending, Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 50, No. 452, A (1984), p. 737.
- (8) Rice, J.R., A Path Independent Integral and the Approximate Analysis of Strain Concentration by Notches and Cracks, Trans. ASME, J. Appl. Mech., Vol. 35, No. 2 (1968), p. 379.
- (9) Rice, J.R., Paris, P.C. and Merkle, J.G., Some Further Results of J -Integral Analysis and Estimates, ASTM STP, 536, (1976), p. 231.
- (10) Lamba, H.S., The J -Integral Applied to Cyclic Loading, Eng. Fract. Mech., Vol. 7 (1975), p. 693.
- (11) Wüthrich, C., The Expression of the J -Integral Concept to Fatigue Cracks, Int. J. Fract., Vol. 20 (1982), R35.
- (12) Tanaka, K., The Cyclic J -Integral as a Criterion for Fatigue Crack Growth, Int., J. Fract., Vol. 22 (1983), p. 91.
- (13) Ohji, K., Ogura, K. and Okubo, Y., Cyclic Analysis of a Propagating Crack and Its Correlation with Fatigue Crack Growth, Eng. Fract. Mech., Vol. 7 (1975), p. 457.
- (14) Ogura, K., Ohji, K., Okubo, Y. and Shibano Y., Analysis of the Fatigue Crack Closure Variable Load and Its Relation to the Crack Growth Rate, Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 42, No. 358 (1976), p. 1615.