Generalization of Elliott's Solution to Transversely Isotropic Elasticity Problems in Cartesian Coordinates*

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A generalized Elliott solution to transversely isotropic elasticity problems is proposed in Cartesian coordinates. The solution consists of five potential functions and includes some new solutions corresponding to equal roots of a quadratic equation composed of four elastic constants. The solution is reduced to Elliott's solution when three potential functions in the solution are omitted. When elastic constants of transversely isotropic solids are replaced with those of isotropic solids, the solution is exactly coincident with the modified form of the generalized Boussinesq solution to isotropic solids.

Key Words: Elasticity, Solution, Anisotropy, Transverse Isotropy, Rectangular Prism

1. Introduction

With the advance in research on three-dimensional elasticity problems of isotropic solids, several studies on two-dimensional or three-dimensional elasticity problems of anisotropic solids have lately appeared. Though there are various classes of anisotropy, as stated in Love's book⁽¹⁾, transverse isotropy and orthotropy are important from the standpoint of practical considerations in engineering. Since the number of independent elastic constants of transverse isotropy is small, three-dimensional elasticity solutions for this anisotropy seem to be obtained with comparative facility and have been found by some researchers. Lekhnitskii⁽²⁾ has found stress functions for two-dimensional problems and axi-symmetric problems. Michell⁽³⁾ has found a solution as the development of a solution to isotropic solids and has analyzed

a semi-infinite solid. His solution seems, however, to have some difficulty with practical applications to general boundary-value problems. Elliott⁽⁴⁾ has found a simple solution by making use of two potential functions. His solution is ingenious, based on distinct roots of a quadratic equation composed of four elastic constants. However, two potential functions are reduced to one potential function when the roots become equal roots. Then, his solution becomes inapplicable to general boundary-value problems. Lodge⁽⁵⁾ has found a solution by making use of one potential function. His solution corresponds to the second basic solution to isotropic solids proposed by Boussinesq⁽⁶⁾. Hata⁽⁷⁾ has derived Elliott's solution and Lodge's solution by making use of operators in Cartesian coordinates and has fully discussed the propriety in their solutions.

Though Elliott's solution is simple in applications to transversely isotropic elasticity problems and is used by many researchers⁽⁸⁾⁻⁽¹³⁾, it seems to have been hardly applied to three-dimensional problems of finite solids, for instance, finite rectangular prisms, finite thick plates or finite hollow cylinders. Though the reason is not clear, it would appear that the number of

* Received 20th January, 1989. Paper No. 87-0902A

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potential functions in Elliott's solution is seemingly insufficient for the analysis of finite solids. This is evident from the comparison of Elliott's solution accompanied by Lodge's solution with a general solution to isotropic solids, for instance, the generalized Boussinesq solution. The authors do not, however, insist on the deficiency in Elliott's solution from this standpoint. The authors are, however, of the view that isotropy is a special case of anisotropy and strongly believe that all solutions to anisotropic solids have to yield general solutions to isotropic solids when they are specialized for isotropic solids, if they are completely general solutions. Elliott's solution simply yields the first basic solution proposed by Boussinesq when it is specialized for isotropic solids referring to a special case of equal roots. If we examine Elliott's solution closely, we find that it is not a completely general solution and appears to be short of some solutions corresponding to equal roots.

From the above standpoint, this paper proposes new solutions corresponding to equal roots which are not included in Elliott's solution and generalizes Elliott's solution to a more general solution which has wide applicability to transversely isotropic elasticity problems. The generalized Elliott solution presented in this paper consists of five potential functions. If three potential functions are omitted, the solution is reduced to Elliott's solution. Furthermore, the solution is exactly coincident with the modified form of the generalized Boussinesq solution when it is specialized for isotropic solids.

2. Displacement Equations of Equilibrium

If Cartesian coordinates (x, y, z) such that the axis of z is taken parallel to the axis of elastic symmetry are used, the generalized Hooke's law of transversely isotropic solids is as follows:

$$\left. \begin{aligned} \sigma_{xx} &= C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + C_{13}\varepsilon_{zz} \\ \sigma_{yy} &= C_{12}\varepsilon_{xx} + C_{11}\varepsilon_{yy} + C_{13}\varepsilon_{zz} \\ \sigma_{zz} &= C_{13}\varepsilon_{xx} + C_{13}\varepsilon_{yy} + C_{33}\varepsilon_{zz} \\ \sigma_{yz} &= 2C_{44}\varepsilon_{yz}, \quad \sigma_{zx} = 2C_{44}\varepsilon_{zx} \\ \sigma_{xy} &= 2C_{66}\varepsilon_{xy} \end{aligned} \right\}, \quad (1)$$

where

$$C_{66} = \frac{C_{11} - C_{12}}{2} \quad (2)$$

and C_{ij} , σ_{ij} and ε_{ij} are the elastic constants, stress tensor and strain tensor, respectively. Strain tensor ε_{ij} is expressed in components of displacement u , v and w as

$$\left. \begin{aligned} \varepsilon_{xx} &= u_{,x}, \quad \varepsilon_{yy} = v_{,y}, \quad \varepsilon_{zz} = w_{,z} \\ \varepsilon_{yz} &= \frac{1}{2}(v_{,z} + w_{,y}), \quad \varepsilon_{zx} = \frac{1}{2}(w_{,x} + u_{,z}) \\ \varepsilon_{xy} &= \frac{1}{2}(u_{,y} + v_{,x}) \end{aligned} \right\}, \quad (3)$$

in which $u_{,x}$, $w_{,z}$ and so on denote $\partial u/\partial x$, $\partial w/\partial z$ and so on, respectively. If we substitute components of stress obtained by substituting Eq. (3) into Eq. (1) into the equations of equilibrium in the absence of body forces, we obtain displacement equations of equilibrium in the form

$$\left. \begin{aligned} C_{11}u_{,xx} + C_{66}u_{,yy} + C_{44}u_{,zz} \\ + (C_{12} + C_{66})v_{,xy} + (C_{13} + C_{44})w_{,xz} &= 0 \\ C_{66}v_{,xx} + C_{11}v_{,yy} + C_{44}v_{,zz} \\ + (C_{12} + C_{66})u_{,xy} + (C_{13} + C_{44})w_{,yz} &= 0 \\ C_{44}\nabla_1^2 w + C_{33}w_{,zz} + (C_{13} + C_{44}) \\ \times (u_{,x} + v_{,y})_{,z} &= 0 \end{aligned} \right\}, \quad (4)$$

where

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

3. Generalization of Elliott's Solution

We define the following quadratic equation composed of four elastic constants as seen in Elliott's solution:

$$C_{11}C_{44}\nu^2 + [C_{13}(C_{13} + 2C_{44}) - C_{11}C_{33}]\nu + C_{33}C_{44} = 0 \quad (5)$$

and denote the roots in ν by ν_1 and ν_2 . We also define the following parameter expressed in the roots and three elastic constants:

$$k_\alpha = \frac{C_{11}\nu_\alpha - C_{44}}{C_{13} + C_{44}} \quad (\alpha = 1, 2). \quad (6)$$

Then, we obtain the following relation from Eqs. (5) and (6):

$$\left. \begin{aligned} \nu_1\nu_2 &= \frac{C_{33}}{C_{11}}, \quad k_1k_2 = 1 \\ \frac{C_{33}k_\alpha}{C_{44}k_\alpha + (C_{13} + C_{44})} &= \nu_\alpha \quad (\alpha = 1, 2) \end{aligned} \right\}. \quad (7)$$

Hereafter, we shall derive the solutions to Eq. (4), dividing them into two forms.

3.1 A solution of the first form

We assume a solution corresponding to Boussinesq's solution to isotropic solids in the form

$$\left. \begin{aligned} u &= (x\phi_{1,x})_{,x} + \xi_1 \\ v &= (x\phi_{1,x})_{,y} + \eta_1 \\ w &= k_2(x\phi_{1,x})_{,z} + \zeta_1 \end{aligned} \right\}. \quad (8)$$

If we substitute Eq. (8) into Eq. (4) and take ϕ_1 as a potential function satisfying the following equation:

$$\nabla_1^2 \phi_1 + \nu_2 \phi_{1,zz} = 0, \quad (9)$$

we obtain a system of partial differential equations with three unknowns in the form

$$\left. \begin{aligned} C_{11}\xi_{1,xx} + C_{66}\xi_{1,yy} + C_{44}\xi_{1,zz} + (C_{12} + C_{66}) \\ \times \eta_{1,xy} + (C_{13} + C_{44})\zeta_{1,xz} &= -2C_{11}\phi_{1,xxx} \\ C_{66}\eta_{1,xx} + C_{11}\eta_{1,yy} + C_{44}\eta_{1,zz} + (C_{12} + C_{66}) \\ \times \xi_{1,xy} + (C_{13} + C_{44})\zeta_{1,yz} &= -2C_{11}\phi_{1,xyx} \\ C_{44}\nabla_1^2 \zeta_1 + C_{33}\zeta_{1,zz} + (C_{13} + C_{44})(\xi_{1,x} \\ + \eta_{1,y})_{,z} &= -2[C_{44}k_2 + (C_{13} + C_{44})]\phi_{1,xxx} \end{aligned} \right\}. \quad (10)$$

Setting

$$\left. \begin{aligned} \xi_1 &= (\phi'_{01} + a_0 \phi_1)_{,x} \\ \eta_1 &= (\phi'_{01} + a_0 \phi_1)_{,y} \\ \zeta_1 &= (k_1 \phi'_{01} + \zeta)_{,z} \end{aligned} \right\} \quad (11)$$

in which a_0 is an unknown constant to be determined later, and substituting Eq.(11) into Eq.(10) under Eq. (9), we obtain

$$\left. \begin{aligned} &C_{11}(\nabla_1^2 \phi'_{01} + \nu_1 \phi'_{01,zz}) + (C_{13} + C_{44}) \zeta_{,zz} \\ &= [a_0(C_{11}\nu_2 - C_{44}) + 2C_{11}\nu_2] \phi_{1,zz} + 2C_{11} \phi_{1,yy} \\ &[C_{44}k_1 + (C_{13} + C_{44})](\nabla_1^2 \phi'_{01} + \nu_1 \phi'_{01,zz}) \\ &+ C_{44}\nabla_1^2 \zeta + C_{33}\zeta_{,zz} \\ &= -\{a_0(C_{13} + C_{44}) + 2[C_{44}k_2 + (C_{13} + C_{44})]\} \phi_{1,xx} \\ &- a_0(C_{13} + C_{44}) \phi_{1,yy} \end{aligned} \right\} \quad (12)$$

If we determine a_0 such that the coefficient of $\phi_{1,zz}$ in the right-hand side of the first equation of Eq.(12) becomes zero, we have

$$a_0 = -\frac{2C_{11}\nu_2}{C_{11}\nu_2 - C_{44}} \quad (13)$$

By substituting the above value into Eq.(12), we obtain the following two equations:

$$\nabla_1^2 \phi'_{01} + \nu_1 \phi'_{01,zz} = -\frac{C_{13} + C_{44}}{C_{11}} \zeta_{,zz} + 2\phi_{1,yy} \quad (14)$$

$$\begin{aligned} &[C_{44}k_1 + (C_{13} + C_{44})](\nabla_1^2 \phi'_{01} + \nu_1 \phi'_{01,zz}) \\ &+ C_{44}\nabla_1^2 \zeta + C_{33}\zeta_{,zz} \\ &= 2C_{44}(k_1 - k_2) \phi_{1,xx} + \frac{2C_{11}\nu_2}{k_2} \phi_{1,yy} \end{aligned} \quad (15)$$

By substituting Eq.(14) into Eq.(15) and eliminating ϕ'_{01} , we obtain

$$\nabla_1^2 \zeta + \nu_2 \zeta_{,zz} = 2(k_1 - k_2) \phi_{1,xx} \quad (16)$$

The general solution to Eq.(16) is expressed in the sum of complementary function ζ_c and particular integral ζ_p . Complementary function ζ_c shall, however, be excluded here, because it is dependent upon function ϕ_{03} in a solution of the second form which shall be stated later. Particular integral ζ_p is

$$\zeta = \zeta_p = (k_1 - k_2)x\phi_{1,x} \quad (17)$$

By substituting Eq.(17) into the right-hand side of Eq. (14), we obtain

$$\begin{aligned} &\nabla_1^2 \phi'_{01} + \nu_1 \phi'_{01,zz} \\ &= -(\nu_1 - \nu_2)x\phi_{1,zz} + 2\phi_{1,yy} \end{aligned} \quad (18)$$

The general solution to Eq.(18) is expressed in the sum of complementary function ϕ'_{01c} and particular integral ϕ'_{01p} as

$$\phi'_{01} = \phi'_{01c} + \phi'_{01p} \quad (19)$$

If we notice the right-hand side of Eq.(18), we find that particular integral ϕ'_{01p} is divided into two expressions corresponding to equal roots ($\nu_1 = \nu_2$) and distinct roots ($\nu_1 \neq \nu_2$). If we use the method of undetermined coefficients under Eq.(9), we obtain the following particular integral:

$$\phi'_{01p} = \begin{cases} y\phi_{1,y} & \text{for } \nu_1 = \nu_2 \\ -x\phi_{1,x} - \frac{2\nu_2}{\nu_1 - \nu_2} \phi_1 & \text{for } \nu_1 \neq \nu_2 \end{cases} \quad (20)$$

If we substitute Eqs.(13), (17) and (19) into Eq.(11),

we obtain the following functions:

$$\left. \begin{aligned} \xi_1 &= (\phi'_{01c} + \phi'_{01p} - \frac{2C_{11}\nu_2}{C_{11}\nu_2 - C_{44}} \phi_1)_{,x} \\ \eta_1 &= (\phi'_{01c} + \phi'_{01p} - \frac{2C_{11}\nu_2}{C_{11}\nu_2 - C_{44}} \phi_1)_{,y} \\ \zeta_1 &= [k_1(\phi'_{01c} + \phi'_{01p}) + (k_1 - k_2)x\phi_{1,x}]_{,z} \end{aligned} \right\} \quad (21)$$

By substituting the above functions into Eq.(8) and making use of Eq.(20), we obtain

$$\left. \begin{aligned} u &= [\phi'_{01c} + \gamma_1(x\phi_{1,x} + y\phi_{1,y}) - \gamma_2\phi_1]_{,x} \\ v &= [\phi'_{01c} + \gamma_1(x\phi_{1,x} + y\phi_{1,y}) - \gamma_2\phi_1]_{,y} \\ w &= k_1[\phi'_{01c} + \gamma_1(x\phi_{1,x} + y\phi_{1,y}) - \gamma_3\phi_1]_{,z} \end{aligned} \right\} \quad (22)$$

where

$$\left. \begin{aligned} \nabla_1^2 \phi'_{01c} + \nu_1 \phi'_{01c,zz} &= 0 \\ \nabla_1^2 \phi_1 + \nu_2 \phi_{1,zz} &= 0 \end{aligned} \right\} \quad (23)$$

$$\left. \begin{aligned} \gamma_1 &= \begin{cases} 1 & \text{for } \nu_1 = \nu_2 \\ 0 & \text{for } \nu_1 \neq \nu_2 \end{cases} \\ \gamma_2 &= \begin{cases} \frac{2C_{11}\nu_2}{C_{11}\nu_2 - C_{44}} & \text{for } \nu_1 = \nu_2 \\ \frac{2\nu_2}{\nu_1 - \nu_2} \cdot \frac{C_{11}\nu_1 - C_{44}}{C_{11}\nu_2 - C_{44}} & \text{for } \nu_1 \neq \nu_2 \end{cases} \\ \gamma_3 &= \begin{cases} 0 & \text{for } \nu_1 = \nu_2 \\ \frac{2\nu_2}{\nu_1 - \nu_2} & \text{for } \nu_1 \neq \nu_2 \end{cases} \end{aligned} \right\} \quad (24)$$

Thus, Eqs.(22), (23) and (24) constitute a required solution. However, when the concrete expression of potential function ϕ_1 is substituted into Eq.(22), needless terms to boundary-value problems in the theory of elasticity arise and are to be excluded by adding to ϕ'_{01c} . To avoid this complexity, we newly put ϕ'_{01c} in the form

$$\phi'_{01c} = \phi_{01} - \gamma_1(x\phi_{1,x} + y\phi_{1,y}) \quad (25)$$

By substituting Eq.(25) into Eq.(22) and the first equation of Eq.(23), we lastly obtain

$$\left. \begin{aligned} u &= (\phi_{01} - \gamma_2\phi_1)_{,x} \\ v &= (\phi_{01} - \gamma_2\phi_1)_{,y} \\ w &= k_1(\phi_{01} - \gamma_3\phi_1)_{,z} \end{aligned} \right\} \quad (26)$$

where

$$\left. \begin{aligned} \nabla_1^2 \phi_{01} + \nu_1 \phi_{01,zz} &= -2\gamma_1\nu_2\phi_{1,zz} \\ \nabla_1^2 \phi_1 + \nu_2 \phi_{1,zz} &= 0 \end{aligned} \right\} \quad (27)$$

and coefficients γ_1 , γ_2 and γ_3 in Eq.(26) and the first equation of Eq.(27) are shown in Eq.(24). The first terms in the right-hand side of Eq.(26), as the complementary function of ϕ_{01} , are Elliott's solution. The second terms are new solutions obtained here.

3.2 A solution of the second form

We again assume a solution corresponding to Boussinesq's solution in the form

$$\left. \begin{aligned} u &= (z\phi'_3)_{,x} + \xi_3 \\ v &= (z\phi'_3)_{,y} + \eta_3 \\ w &= k_1(z\phi'_3)_{,z} + \zeta_3 \end{aligned} \right\} \quad (28)$$

If we substitute Eq.(28) into Eq.(4) and take ϕ'_3 as a potential function satisfying the following equation:

$$\nabla_1^2 \phi'_3 + \nu_1 \phi'_{3,zz} = 0 \quad (29)$$

we obtain a system of partial differential equations

with three unknowns in the form

$$\left. \begin{aligned} &C_{11}\xi_{3,xx} + C_{66}\xi_{3,yy} + C_{44}\xi_{3,zz} + (C_{12} + C_{66}) \\ &\quad \times \eta_{3,xy} + (C_{13} + C_{44})\xi_{3,xz} = -2C_{11}\nu_1\phi'_{3,xz} \\ &C_{66}\eta_{3,xx} + C_{11}\eta_{3,yy} + C_{44}\eta_{3,zz} + (C_{12} + C_{66}) \\ &\quad \times \xi_{3,xy} + (C_{13} + C_{44})\xi_{3,yz} = -2C_{11}\nu_1\phi'_{3,yz} \\ &C_{44}\nabla_1^2\xi_3 + C_{33}\xi_{3,zz} + (C_{13} + C_{44})(\xi_{3,x} + \eta_{3,y})_{,z} \\ &\quad = -2C_{33}k_1\phi'_{3,zz} \end{aligned} \right\} \quad (30)$$

Setting

$$\left. \begin{aligned} \xi_3 &= (\phi'_{03} + \xi)_{,x} \\ \eta_3 &= (\phi'_{03} + \xi)_{,y} \\ \xi_3 &= k_2\phi'_{03,z} + b_0\phi'_{3,z} \end{aligned} \right\}, \quad (31)$$

in which b_0 is an unknown constant to be determined later, and substituting Eq. (31) into Eq. (30) under Eq. (29), we obtain

$$\left. \begin{aligned} &C_{11}(\nabla_1^2\phi'_{03} + \nu_2\phi'_{03,zz}) + C_{11}\nabla_1^2\xi \\ &\quad + C_{44}\xi_{,zz} = -[b_0(C_{13} + C_{44}) + 2C_{11}\nu_1]\phi'_{3,z} \\ &[C_{44}k_2 + (C_{13} + C_{44})](\nabla_1^2\phi'_{03} + \nu_2\phi'_{03,zz}) \\ &\quad + (C_{13} + C_{44})\nabla_1^2\xi \\ &\quad = -[b_0(C_{33} - C_{44}\nu_1) + 2C_{33}k_1]\phi'_{3,z} \end{aligned} \right\} \quad (32)$$

If we determine b_0 such that the right-hand side of the second equation of Eq. (32) becomes zero, we have

$$b_0 = -\frac{2C_{11}\nu_2k_1}{C_{11}\nu_2 - C_{44}} \quad (33)$$

By substituting the above value into Eq. (32), we obtain the following two equations:

$$\begin{aligned} &C_{11}(\nabla_1^2\phi'_{03} + \nu_2\phi'_{03,zz}) + C_{11}\nabla_1^2\xi + C_{44}\xi_{,zz} \\ &\quad = \frac{2C_{11}C_{44}(\nu_1 - \nu_2)}{C_{11}\nu_2 - C_{44}}\phi'_{3,z} \end{aligned} \quad (34)$$

$$\nabla_1^2\phi'_{03} + \nu_2\phi'_{03,zz} = -\frac{\nu_2(C_{13} + C_{44})}{C_{33}k_2}\nabla_1^2\xi \quad (35)$$

By substituting Eq. (35) into Eq. (34) and eliminating ϕ'_{03} , we obtain

$$\nabla_1^2\xi + \nu_1\xi_{,zz} = \frac{2\nu_1(k_1 - k_2)}{k_2}\phi'_{3,z} \quad (36)$$

The general solution to Eq. (36) is expressed in the sum of complementary function ξ_c and particular integral ξ_p . Complementary function ξ_c shall, however, be excluded, because it is dependent upon the complementary function of ϕ_{01} in the solution of the first form stated before. Particular integral ξ_p is

$$\xi = \xi_p = \frac{k_1 - k_2}{k_2}z\phi'_3 \quad (37)$$

If we put ϕ'_3 in the form

$$\phi'_3 = k_2^2\phi_{3,z} \quad (38)$$

and substitute Eqs. (37) and (38) into the right-hand side of Eq. (35), we have

$$\nabla_1^2\phi'_{03} + \nu_2\phi'_{03,zz} = (\nu_1 - \nu_2)z\phi_{3,zzz} \quad (39)$$

The general solution to Eq. (39) is expressed in the sum of complementary function $\phi'_{03c} = \phi_{03}$ and particular integral ϕ'_{03p} as

$$\phi'_{03} = \phi_{03} + \phi'_{03p} \quad (40)$$

Particular integral ϕ'_{03p} is divided into two expressions corresponding to equal roots and distinct roots. If we use the method of undetermined coefficients under Eq.

(29), we obtain the following particular integral:

$$\phi'_{03p} = \begin{cases} 0 & \text{for } \nu_1 = \nu_2 \\ -z\phi_{3,z} - \frac{2\nu_2}{\nu_1 - \nu_2}\phi_3 & \text{for } \nu_1 \neq \nu_2 \end{cases} \quad (41)$$

If we substitute Eqs. (33), (37), (38) and (40) into Eq. (31), we obtain the following functions:

$$\left. \begin{aligned} \xi_3 &= [\phi_{03} + \phi'_{03p} + k_2(k_1 - k_2)z\phi_{3,z}]_{,x} \\ \eta_3 &= [\phi_{03} + \phi'_{03p} + k_2(k_1 - k_2)z\phi_{3,z}]_{,y} \\ \xi_3 &= k_2(\phi_{03} + \phi'_{03p} - \frac{2C_{11}\nu_2}{C_{11}\nu_2 - C_{44}}\phi_3)_{,z} \end{aligned} \right\} \quad (42)$$

By substituting the above functions and Eq. (38) into Eq. (28) and making use of Eq. (41), we obtain

$$\left. \begin{aligned} u &= (\phi_{03} + \gamma_1 z\phi_{3,z} - \gamma_3\phi_3)_{,x} \\ v &= (\phi_{03} + \gamma_1 z\phi_{3,z} - \gamma_3\phi_3)_{,y} \\ w &= k_2(\phi_{03} + \gamma_1 z\phi_{3,z} - \gamma_2\phi_3)_{,z} \end{aligned} \right\} \quad (43)$$

where

$$\left. \begin{aligned} \nabla_1^2\phi_{03} + \nu_2\phi_{03,zz} &= 0 \\ \nabla_1^2\phi_3 + \nu_1\phi_{3,zz} &= 0 \end{aligned} \right\} \quad (44)$$

and coefficients γ_1 , γ_2 and γ_3 are shown in Eq. (24). The first terms in the right-hand side of Eq. (43) are Elliott's solution. The second and third terms are new solutions obtained here.

3.3 A generalized Elliott solution

Now, Lodge's solution⁽⁵⁾ to axially asymmetric problems is as follows:

$$u = \phi_{,y}, \quad v = -\phi_{,x}, \quad w = 0, \quad (45)$$

where

$$\left. \begin{aligned} \nabla_1^2\phi + \nu_3\phi_{,zz} &= 0 \\ \nu_3 &= \frac{C_{44}}{C_{66}} = \frac{2C_{44}}{C_{11} - C_{12}} \end{aligned} \right\} \quad (46)$$

If we make a linear combination of this solution and two solutions of Eqs. (26) and (43) of the first and second forms, we obtain a generalized Elliott solution in the form

$$\left. \begin{aligned} u &= (\phi_{01} + \phi_{03} + \gamma_1 z\phi_{3,z} - \gamma_2\phi_1 - \gamma_3\phi_3)_{,x} \\ &\quad + \phi_{,y} \\ v &= (\phi_{01} + \phi_{03} + \gamma_1 z\phi_{3,z} - \gamma_2\phi_1 - \gamma_3\phi_3)_{,y} \\ &\quad - \phi_{,x} \\ w &= [k_1(\phi_{01} - \gamma_3\phi_1) + k_2(\phi_{03} + \gamma_1 z\phi_{3,z} \\ &\quad - \gamma_2\phi_3)]_{,z} \end{aligned} \right\} \quad (47)$$

in which ϕ_{01} , ϕ_1 , ϕ_{03} , ϕ_3 and ϕ are the potential functions governed by Eqs. (27), (44) and (46), respectively, and k_1 , k_2 , γ_1 , γ_2 and γ_3 are the parameters in Eq. (6) and the coefficients in Eq. (24), respectively.

By excluding ϕ_1 , ϕ_3 and ϕ from the solution of Eq. (47), we obtain Elliott's solution in the form

$$\left. \begin{aligned} u &= (\phi_{01} + \phi_{03})_{,x} \\ v &= (\phi_{01} + \phi_{03})_{,y} \\ w &= (k_1\phi_{01} + k_2\phi_{03})_{,z} \end{aligned} \right\} \quad (48)$$

where

$$\left. \begin{aligned} \nabla_1^2\phi_{01} + \nu_1\phi_{01,zz} &= 0 \\ \nabla_1^2\phi_{03} + \nu_2\phi_{03,zz} &= 0 \end{aligned} \right\} \quad (49)$$

We shall now consider the specialization of the solution of Eq. (47) for the case of such equal roots as

occur in isotropic solids. Since the elastic constants of transversely isotropic solids are replaced with those of isotropic solids as $c_{33}=c_{11}$, $c_{13}=c_{12}$ and $c_{44}=c_{66}=(c_{11}-c_{12})/2$, we have $\nu_1=\nu_2=\nu_3=1$ and $k_1=k_2=1$. Therefore, the coefficients in Eq.(24) become

$$\left. \begin{aligned} \gamma_1 &= 1, \quad \gamma_3 = 0 \\ \gamma_2 &= \frac{2c_{11}}{c_{11}-c_{44}} = \frac{4c_{11}}{c_{11}+c_{12}} = 4(1-\nu) \end{aligned} \right\}, \quad (50)$$

in which ν denotes Poisson's ratio of isotropic solids. When we set

$$\left. \begin{aligned} \phi_{01} + \phi_{03} &= \frac{\phi'_0}{2G}, \quad \phi_1 = \frac{\phi'_1}{2G} \\ \phi_3 &= \frac{\phi'_3}{2G}, \quad \psi = \frac{\partial_3}{G} \end{aligned} \right\}, \quad (51)$$

in which G denotes the shear modulus of isotropic solids, Eq.(47) becomes

$$\left. \begin{aligned} 2Gu &= [\phi'_0 + z\phi'_{3,z} - 4(1-\nu)\phi'_1]_{,x} + 2\partial_{3,y} \\ 2Gv &= [\phi'_0 + z\phi'_{3,z} - 4(1-\nu)\phi'_1]_{,y} - 2\partial_{3,x} \\ 2Gw &= [\phi'_0 + z\phi'_{3,z} - 4(1-\nu)\phi'_1]_{,z} \end{aligned} \right\}, \quad (52)$$

where

$$\left. \begin{aligned} \nabla^2 \phi'_0 &= -2\phi'_{1,zz}, \quad \nabla^2 \phi'_1 = 0 \\ \nabla^2 \phi'_3 &= 0, \quad \nabla^2 \partial_3 = 0 \\ \nabla^2 &\equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \end{aligned} \right\}. \quad (53)$$

Now, the generalized Boussinesq solution⁽⁶⁾ to isotropic solids is

$$2Gu = \text{grad}(\phi_0 + \mathbf{r} \cdot \boldsymbol{\lambda}) - 4(1-\nu)\boldsymbol{\lambda} + 2\text{rot} \boldsymbol{\vartheta}, \quad (54)$$

where

$$\left. \begin{aligned} \mathbf{u} &= [u, v, w], \quad \mathbf{r} = [x, y, z] \\ \boldsymbol{\lambda} &= [\lambda_1, \lambda_2, \lambda_3], \quad \boldsymbol{\vartheta} = [\vartheta_1, \vartheta_2, \vartheta_3] \\ \nabla^2 \phi_0 &= 0, \quad \nabla^2 \lambda_1 = \nabla^2 \lambda_2 = \nabla^2 \lambda_3 = 0 \\ \nabla^2 \vartheta_1 &= \nabla^2 \vartheta_2 = \nabla^2 \vartheta_3 = 0 \end{aligned} \right\}. \quad (55)$$

When we put vector potentials $\boldsymbol{\lambda}$, $\boldsymbol{\vartheta}$ and scalar potential ϕ_0 in the form

$$\left. \begin{aligned} \boldsymbol{\lambda} &= \text{grad} \phi'_1 + \mathbf{k}(\phi'_3 - \phi'_1)_{,z} \\ &= [\phi'_{1,x}, \phi'_{1,y}, \phi'_{3,z}] \\ \boldsymbol{\vartheta} &= [0, 0, \partial_3] \\ \phi_0 &= \phi'_0 - x\phi'_{1,x} - y\phi'_{1,y} \end{aligned} \right\}, \quad (56)$$

in which $\mathbf{k}$ is the unit vector, the solution of Eq.(54) is exactly coincident with the solution of Eq.(52).

4. Concluding Remarks

When quadratic equation (5) yields equal roots, Elliott's solution to transversely isotropic elasticity problems, heretofore in use, becomes inapplicable to general boundary-value problems because of the contraction of the solution, as stated in section 1. Noting this, this paper proposed the generalized Elliott solution of Eq.(47), including new solutions corresponding to equal roots. The solution has high validity for transversely isotropic elasticity problems and has extensive applicability to three-dimensional elasticity problems in Cartesian coordinates without distinction

of finite solids or infinite solids. It was shown that Elliott's solution of Eq.(48) falls under a part of the solution. It was also shown in Eq.(50) to Eq.(56) that the solution is coincident with the modified form of the generalized Boussinesq solution, when the solution is specialized for isotropic solids. This confirms that the authors' notion stated in section 1 is satisfied by the solution. Though in the case of applications of Elliott's solution to isotropic solids, a means of replacing isotropy with approximate isotropy has been used so far, the solution does not need this device and can exactly analyze isotropic solids because there is no contraction of the solution. The solution was derived with division into two forms for brevity's sake. The derivation of the solution of the first form may yield simultaneously the solution of the second form, if the complementary function ζ_p of Eq.(16) is also taken. As an application of the solution, a three-dimensional stress analysis for a short rectangular prism subjected to partial pressure on two end faces has been achieved thus far. It shall be presented on another occasion.

In consideration of the premises, the authors may conclude that the generalized Elliott solution presented in this paper is highly useful for transversely isotropic elasticity problems.

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