Generalized Thermoelasticity for an Infinite Body with a Circular Cylindrical Hole*

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A one-dimensional generalized thermoelasticity is presented based on Lord and Shulman's theory, which involves one relaxation time of the thermoelastic process. This theory has been developed in an attempt to eliminate the paradox of an infinite velocity of thermoelastic propagation inherent in the classical theory. The analytical object of this paper is an infinite body with a circular cylindrical hole. The boundary condition is that a constant heat flux is flowing into the infinite body from a circular cylindrical hole but the displacement at the hole is constrained. An approximate analysis for short periods of time is carried out because it is very difficult to obtain an exact solution. The numerical results of the effect of the relaxation time on displacement, temperature and stress distributions are shown.

Key Words: Elasticity, Thermal Stress, Generalized Thermoelasticity, Lord and Shulman's Theory, Relaxation Time, Infinite Body, Circular Cylindrical Hole, Short-Time Solution

1. Introduction

Chester⁽¹⁾ used macroscopic equations for energy conservation and heat current to discuss the possibility of second sound (thermal waves) in solids. Subsequently, the phenomenon of second sound in solids was observed in experiments on temperature pulse propagation in solid helium by Ackerman et al.⁽²⁾ and Ackerman and Overton, Jr.⁽³⁾

When an isotropic and homogeneous elastic solid is subjected to a thermal disturbance, the effect is caused at a location far distant from the source instantaneously in the classical linear thermoelastic theory. This means that the thermal wave propagates at infinite velocity, and it is a physically unreasonable

result. Two generalized thermoelastic theories are proposed to eliminate the paradox and correct the classical theory on the assumption that a thermal wave propagates at finite velocity. These theories are by Lord and Shulman⁽⁴⁾ and Green and Lindsay⁽⁵⁾. The former is a thermoelastic theory including the finite velocity of a thermal wave by correcting Fourier's thermal conduction law and introducing one relaxation time of the thermoelastic process. The latter is a theory that considers the finite velocity of a thermal wave by correcting the energy equation and Duhamel-Neumann's relation and introducing two relaxation times of the thermoelastic process. Many papers have been concerned with the fundamental considerations, e. g., a reciprocal theorem, a uniqueness theorem of a solution, an energy law, etc., for both theories. However, there have been few papers which analyze theoretically by use of these theories, perform numerical calculations and investigate the effects of the thermal wave on temperature and thermal stresses concretely.

Francis⁽⁶⁾ gives the values of the one relaxation time in Lord and Shulman's theory for some materials. However, the existence of the two relaxation times in Green and Lindsay's theory cannot be

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confirmed at present⁽⁷⁾. Therefore, this paper deals with a one-dimensional generalized thermoelasticity based on Lord and Shulman's theory and investigates the effects of the thermal wave on such properties as temperature and thermal stresses. The analytical object is an infinite body with a circular cylindrical hole, and the boundary condition is that a constant heat flux is flowing into the infinite body, but the displacement at the hole is constrained. An approximate analysis for short periods of time is carried out because it is very difficult to obtain an exact solution in such a problem. The numerical results of the effect of the relaxation time on displacement, temperature and thermal stress distributions are shown.

2. Analysis

We consider the one-dimensional generalized thermoelasticity of an infinite body with a circular cylindrical hole whose radius is a . The boundary condition is that a constant heat flux q_0 is flowing into the infinite body from the hole, but the displacement at the hole is constrained.

The radial strain ε_{rr} and the hoop strain $\varepsilon_{\theta\theta}$ are given by

$$\varepsilon_{rr} = u_{,r}, \quad \varepsilon_{\theta\theta} = \frac{u}{r}, \quad (1)$$

where u is a radial displacement, and a comma denotes the differentiation with respect to the following variable.

The stress-strain relations are

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \end{bmatrix} = 2\mu \begin{bmatrix} u_{,r} \\ u/r \end{bmatrix} + \lambda \left(u_{,r} + \frac{u}{r} \right) - (3\lambda + 2\mu)\alpha T, \quad (2)$$

where σ_{rr} and $\sigma_{\theta\theta}$ are the radial stress and the hoop stress, respectively, λ and μ are Lamé's constants, and α is a coefficient of linear thermal expansion.

The equation of motion presented by the displacement component is

$$u_{,rr} + \frac{1}{r}u_{,r} - \frac{u}{r^2} - \zeta\alpha T_{,r} = \frac{1}{c_1^2}u_{,tt}, \quad (3)$$

where c_1 is the propagation velocity of a longitudinal wave, and $\zeta = (1+\nu)/(1-\nu)$ with Poisson's ratio ν .

The heat conduction equation based on Fourier's law yields an infinite velocity of a thermal wave and is at variance with the reality. Therefore, Lord and Shulman corrected Fourier's law and developed a new theory which eliminates this inconvenience. The one-dimensional heat conduction equation with respect to the generalized thermoelasticity based on their theory is given by

$$\begin{aligned} x \left(T_{,rr} + \frac{1}{r}T_{,r} \right) &= T_{,t} + t_0 T_{,tt} \\ &+ \frac{\delta}{\zeta\alpha} \left[u_{,r} + \frac{u}{r} + t_0 \left(u_{,r} + \frac{u}{r} \right)_{,t} \right]_{,t}, \end{aligned} \quad (4)$$

where x is a thermal diffusivity, t_0 is a relaxation time,

and δ is a thermomechanical coupling parameter.

We introduce the following nondimensional quantities:

$$\left. \begin{aligned} \rho &= \frac{r}{a}, \quad \tau = \frac{xt}{a^2}, \quad \theta = \frac{T}{T_0} \\ \left[\frac{\rho\rho}{\theta\theta} \right] &= \frac{1}{(\lambda+2\mu)\alpha T_0} \left[\frac{\sigma_{rr}}{\sigma_{\theta\theta}} \right], \quad \beta = \frac{x}{c_1 a} \\ U &= \frac{u}{aaT_0}, \quad \tau_0 = \frac{xt_0}{a^2}, \end{aligned} \right\} \quad (5)$$

where T_0 is a reference temperature. Equations (2) ~ (4) are nondimensionalized as follows:

$$\left[\frac{\rho\rho}{\theta\theta} \right] = \left[\frac{1}{\eta} \right] \left(U_{,\rho} + \frac{U}{\rho} \right) \mp (1-\eta) \frac{U}{\rho} - \zeta\theta \quad (6)$$

$$U_{,\rho\rho} + \frac{1}{\rho}U_{,\rho} - \frac{U}{\rho^2} - \zeta\theta_{,\rho} = \beta^2 U_{,\tau\tau} \quad (7)$$

$$\begin{aligned} \theta_{,\rho\rho} + \frac{1}{\rho}\theta_{,\rho} - (\theta + \tau_0\theta_{,\tau})_{,\tau} \\ = \frac{\delta}{\zeta} \left[U_{,\rho} + \frac{U}{\rho} + \tau_0 \left(U_{,\rho} + \frac{U}{\rho} \right)_{,\tau} \right]_{,\tau}, \end{aligned} \quad (8)$$

where $\eta = \nu/(1-\nu)$.

We perform the Laplace transform on Eqs. (6) ~ (8) under the initial condition

$$\tau = 0; \quad U = U_{,\tau} = 0, \quad \theta = \theta_{,\tau} = 0; \quad (9)$$

we obtain

$$\left[\frac{\rho\rho^*}{\theta\theta^*} \right] = \left[\frac{1}{\eta} \right] \left(U_{,\rho}^* + \frac{U^*}{\rho} \right) \mp (1-\eta) \frac{U^*}{\rho} - \zeta\theta^* \quad (10)$$

$$U_{,\rho\rho}^* + \frac{1}{\rho}U_{,\rho}^* - \left(\beta^2 p^2 + \frac{1}{\rho^2} \right) U^* - \zeta\theta_{,\rho}^* = 0 \quad (11)$$

$$\begin{aligned} \theta_{,\rho\rho}^* + \frac{1}{\rho}\theta_{,\rho}^* - (\tau_0 p + 1)p\theta^* \\ = \frac{\delta}{\zeta} p(\tau_0 p + 1) \left(U_{,\rho}^* + \frac{U^*}{\rho} \right), \end{aligned} \quad (12)$$

where the sign $*$ denotes the Laplace transform and p is a transformation parameter.

We solve Eqs. (11) and (12) and obtain

$$U^* = \zeta [A_1 \xi_1 K_1(\xi_1 \rho) + A_2 \xi_2 K_1(\xi_2 \rho)] \quad (13)$$

$$\begin{aligned} \theta^* = -(\xi_1^2 - \beta^2 p^2) A_1 K_0(\xi_1 \rho) \\ - (\xi_2^2 - \beta^2 p^2) A_2 K_0(\xi_2 \rho), \end{aligned} \quad (14)$$

where A_1, A_2 are integral constants, $K_i(\cdot)$ is a modified Bessel function of the second kind of order i , and ξ_1, ξ_2 are the positive roots of the following transcendental equation.

$$\begin{aligned} \xi^4 - [1 + (\beta^2 + \tau_0)p + \delta(\tau_0 p + 1)]p\xi^2 \\ + (\tau_0 p + 1)\beta^2 p^3 = 0 \end{aligned} \quad (15)$$

We obtain from Eq. (15)

$$\begin{aligned} \xi_{1,2}^2 = (p/2) \{ B_0 p + B_1 \pm [(B_0^2 - 4\tau_0 \beta^2)p^2 \\ + 2(B_0 B_1 - 2\beta^2)p + B_1^2]^{1/2} \}, \end{aligned} \quad (16)$$

where

$$B_0 = \beta^2 + (1 + \delta)\tau_0, \quad B_1 = 1 + \delta. \quad (17)$$

We substitute Eqs. (13) and (14) into Eq. (10) and obtain

$$\begin{aligned} \left[\frac{\rho\rho^*}{\theta\theta^*} \right] = -\zeta \left[\frac{1}{\eta} \right] [A_1 \xi_1^2 K_0(\xi_1 \rho) + A_2 \xi_2^2 K_0(\xi_2 \rho)] \\ \mp (1-\eta) U^*/\rho - \zeta\theta^*. \end{aligned} \quad (18)$$

The boundary conditions are given by

$$r=a; u=0, -kT, r=q_0H(t), \quad (19)$$

where k is a thermal conductivity, and $H(\cdot)$ is Heaviside's unit step function. Nondimensionalizing Eq.(19) and performing the Laplace transform, we get the following equation:

$$\rho=1; U^*=0, \theta_s^*=-Q_0/p, \quad (20)$$

where $Q_0=aq_0/kT_0$.

Substituting Eqs.(13) and (14) into Eq.(20), we obtain

$$\left. \begin{aligned} A_1\xi_1K_1(\xi_1)+A_2\xi_2K_1(\xi_2) &=0 \\ A_1(\xi_1^2-\beta^2p^2)\xi_1K_1(\xi_1) \\ &+A_2(\xi_2^2-\beta^2p^2)\xi_2K_1(\xi_2)=-Q_0/p \end{aligned} \right\} \quad (21)$$

From this equation, the unknown constants A_1 and A_2 are determined as follows.

$$\left. \begin{aligned} A_1 &=-\frac{Q_0}{p(\xi_1^2-\xi_2^2)\xi_1K_1(\xi_1)} \\ A_2 &=\frac{Q_0}{p(\xi_1^2-\xi_2^2)\xi_2K_1(\xi_2)} \end{aligned} \right\} \quad (22)$$

Introducing Eq.(22) into Eqs.(13),(14), and (18), we obtain

$$U^*=-\frac{\zeta Q_0}{p(\xi_1^2-\xi_2^2)}\left[\frac{K_1(\xi_1\rho)}{K_1(\xi_1)}-\frac{K_1(\xi_2\rho)}{K_1(\xi_2)}\right] \quad (23)$$

$$\theta^*=\frac{Q_0}{p(\xi_1^2-\xi_2^2)}\left[(\xi_1^2-\beta^2p^2)\frac{K_0(\xi_1\rho)}{\xi_1K_1(\xi_1)}-(\xi_2^2-\beta^2p^2)\frac{K_0(\xi_2\rho)}{\xi_2K_1(\xi_2)}\right] \quad (24)$$

$$\left[\frac{\rho\rho^*}{\theta\theta^*}\right]=\frac{\zeta Q_0}{p(\xi_1^2-\xi_2^2)}\left[\frac{\xi_1K_0(\xi_1\rho)}{K_1(\xi_1)}-\frac{\xi_2K_0(\xi_2\rho)}{K_1(\xi_2)}\right]\left[\frac{1}{\eta}\mp(1-\eta)U^*/\rho-\zeta\theta^*\right] \quad (25)$$

We will seek the short-time solution because it is very difficult to obtain an exact solution. As the parameter p is large enough in this case, Eq.(16) is modified as follows:

$$\xi_{1,2}^2=\frac{p^2}{2}\left[B_0\pm C_0+\frac{B_1\pm C_1}{p}\pm\frac{C_2\pm C_3}{p^2}\pm\frac{C_3}{p^3}\right], \quad (26)$$

where

$$\left. \begin{aligned} C_0 &=\sqrt{B_0^2-4\tau_0\beta^2}, C_1=(B_0B_1-2\beta^2)/C_0 \\ C_2 &=2\beta^2(B_0B_1-\tau_0B_1^2-\beta^2)/C_0^3 \\ C_3 &=-C_1C_2/C_0 \end{aligned} \right\} \quad (27)$$

and the number of the series-expanding terms is limited to four. Then we obtain

$$\left. \begin{aligned} \frac{K_0(\xi_{1,2}\rho)}{K_1(\xi_{1,2})} &=\frac{1}{\sqrt{\rho}}e^{-\xi_{1,2}(\rho-1)}\left(1-\frac{\rho_{01}}{\xi_{1,2}}+\frac{\rho_{02}}{\xi_{1,2}^2}\right) \\ \frac{K_1(\xi_{1,2}\rho)}{K_1(\xi_{1,2})} &=\frac{1}{\sqrt{\rho}}e^{-\xi_{1,2}(\rho-1)}\left(1-\frac{\rho_{11}}{\xi_{1,2}}+\frac{\rho_{12}}{\xi_{1,2}^2}\right) \end{aligned} \right\} \quad (28)$$

where

$$\left. \begin{aligned} \rho_{01} &=\frac{1}{8}\left(3+\frac{1}{\rho}\right), \rho_{02}=\frac{3}{128}\left(11+\frac{2}{\rho}+\frac{3}{\rho^2}\right) \\ \rho_{11} &=\frac{3}{8}\left(1-\frac{1}{\rho}\right), \rho_{12}=\frac{3}{128}\left(11-\frac{6}{\rho}-\frac{5}{\rho^2}\right) \end{aligned} \right\} \quad (29)$$

and the number of the series-expanding terms is limited to three.

After this, we examine two cases, that is, one in which the value of the relaxation time τ_0 is not zero and one in which it is zero.

$$(1) \quad \tau_0>0$$

Equation (26) is modified into the following forms:

$$\xi_{1,2}^2=\left[\frac{a_0^2}{b_0^2}\right]p^2\left(1+\left[\frac{a_1}{b_1}\right]\frac{1}{p}+\left[\frac{a_2}{b_2}\right]\frac{1}{p^2}\right), \quad (30)$$

where

$$\left. \begin{aligned} \left[\frac{a_0}{b_0}\right] &=\sqrt{\frac{B_0\pm C_0}{2}}, \left[\frac{a_1}{b_1}\right]=\frac{B_1\pm C_1}{B_0\pm C_0} \\ \left[\frac{a_2}{b_2}\right] &=\frac{\pm C_2}{B_0\pm C_0} \end{aligned} \right\} \quad (31)$$

From Eq.(30), we obtain

$$\xi_{1,2}=\left[\frac{a_0}{b_0}\right]p\left(1+\left[\frac{a_3}{b_3}\right]\frac{1}{p}+\left[\frac{a_4}{b_4}\right]\frac{1}{p^2}\right) \quad (32)$$

$$\xi_{1,2}^2-\beta^2p^2=\frac{p^2}{2}\left(\left[\frac{a_5}{b_5}\right]+\left[\frac{a_6}{b_6}\right]\frac{1}{p}+\left[\frac{a_7}{b_7}\right]\frac{1}{p^2}\right), \quad (33)$$

where

$$\left. \begin{aligned} \left[\frac{a_3}{b_3}\right] &=\frac{1}{2}\left[\frac{a_1}{b_1}\right], \left[\frac{a_4}{b_4}\right]=-\frac{1}{8}\left[\frac{a_1^2}{b_1^2}\right]+\frac{1}{2}\left[\frac{a_2}{b_2}\right] \\ \left[\frac{a_5}{b_5}\right] &=B_0\pm C_0-2\beta^2, \left[\frac{a_6}{b_6}\right]=B_1\pm C_1 \\ \left[\frac{a_7}{b_7}\right] &=\pm C_2 \end{aligned} \right\} \quad (34)$$

Substituting Eq.(32) into Eq.(28), we obtain

$$\left. \begin{aligned} \frac{K_0(\xi_{1,2}\rho)}{K_1(\xi_{1,2})} &=X_{1,2}Y_{1,2}\left(1+\left[\frac{a_{\rho 0}}{b_{\rho 0}}\right]\frac{1}{p}+\left[\frac{a_{\rho 1}}{b_{\rho 1}}\right]\frac{1}{p^2}\right) \\ \frac{K_1(\xi_{1,2}\rho)}{K_1(\xi_{1,2})} &=X_{1,2}Y_{1,2}\left(1+\left[\frac{a_{\rho 2}}{b_{\rho 2}}\right]\frac{1}{p}+\left[\frac{a_{\rho 3}}{b_{\rho 3}}\right]\frac{1}{p^2}\right) \end{aligned} \right\} \quad (35)$$

where

$$\left. \begin{aligned} X_1 &=\frac{1}{\sqrt{\rho}}e^{-a_0a_3(\rho-1)}, X_2=\frac{1}{\sqrt{\rho}}e^{-b_0b_3(\rho-1)} \\ Y_1 &=e^{-a_0(\rho-1)p}, Y_2=e^{-b_0(\rho-1)p} \\ \left[\frac{a_{\rho 0}}{b_{\rho 0}}\right] &=-\left[\frac{a_0a_4}{b_0b_4}\right](\rho-1)-\left[\frac{1/a_0}{1/b_0}\right]\rho_{01} \\ \left[\frac{a_{\rho 1}}{b_{\rho 1}}\right] &=\frac{1}{2}\left[\frac{a_0^2a_4^2}{b_0^2b_4^2}\right](\rho-1)^2 \\ &+\left[\left[\frac{a_4}{b_4}\right](\rho-1)+\left[\frac{a_3/a_0}{b_3/b_0}\right]\rho_{01}+\left[\frac{1/a_0^2}{1/b_0^2}\right]\rho_{02}\right] \\ \left[\frac{a_{\rho 2}}{b_{\rho 2}}\right] &=-\left[\frac{a_0a_4}{b_0b_4}\right](\rho-1)-\left[\frac{1/a_0}{1/b_0}\right]\rho_{11} \\ \left[\frac{a_{\rho 3}}{b_{\rho 3}}\right] &=\frac{1}{2}\left[\frac{a_0^2a_4^2}{b_0^2b_4^2}\right](\rho-1)^2 \\ &+\left[\left[\frac{a_4}{b_4}\right](\rho-1)+\left[\frac{a_3/a_0}{b_3/b_0}\right]\rho_{11}+\left[\frac{1/a_0^2}{1/b_0^2}\right]\rho_{12}\right] \end{aligned} \right\} \quad (37)$$

We then introduce the following variables.

$$\left. \begin{aligned} \left[\frac{a_{\rho 4}}{b_{\rho 4}}\right] &=\left[\frac{a_{\rho 0}}{b_{\rho 0}}\right]-\frac{C_1}{C_0}, \left[\frac{a_{\rho 5}}{b_{\rho 5}}\right]=\left[\frac{a_{\rho 1}}{b_{\rho 1}}\right]-\frac{C_2}{C_0} \\ \left[\frac{a_{\rho 6}}{b_{\rho 6}}\right] &=\left[\frac{a_{\rho 2}}{b_{\rho 2}}\right]-\frac{C_1}{C_0}, \left[\frac{a_{\rho 7}}{b_{\rho 7}}\right]=\left[\frac{a_{\rho 3}}{b_{\rho 3}}\right]-\frac{C_2}{C_0} \end{aligned} \right\} \quad (38)$$

Substituting the above equations into Eq.(23) ~ (25), and performing the inverse Laplace transform, we obtain the following equations for displacement, temperature, and thermal stress distributions:

$$U=-\frac{\zeta Q_0}{2C_0}\left\{X_1\tau_a^2\left[1+\frac{1}{3}\tau_a a_{\rho 6}\right]\right.$$

$$\begin{aligned}
& + \frac{1}{12} \tau_a^2 \left(a_{\rho 7} - \frac{C_1}{C_0} a_{\rho 6} \right) \Big] H(\tau_a) \\
& - X_2 \tau_b^2 \left[1 + \frac{1}{3} \tau_b b_{\rho 6} + \frac{1}{12} \tau_b^2 \left(b_{\rho 7} - \frac{C_1}{C_0} b_{\rho 6} \right) \right] H(\tau_b) \Big\} \\
& \quad (39) \\
\theta = & \frac{Q_0}{2C_0} \left[\frac{X_1}{a_0} \tau_a \left[a_5 + \frac{1}{2} \tau_a [a_5(a_{\rho 4} - a_3) + a_6] \right. \right. \\
& + \frac{1}{6} \tau_a^2 \left\{ a_5 a_{\rho 5} - \left[a_5 \left(a_3 + \frac{C_1}{C_0} \right) - a_6 \right] a_{\rho 4} \right. \\
& + a_5(a_3^2 - a_4) - a_3 a_6 + a_7 \Big\} \Big] H(\tau_a) \\
& - \frac{X_2}{b_0} \tau_b \left[b_5 + \frac{1}{2} \tau_b [b_5(b_{\rho 4} - b_3) + b_6] \right. \\
& + \frac{1}{6} \tau_b^2 \left\{ b_5 b_{\rho 5} - \left[b_5 \left(b_3 + \frac{C_1}{C_0} \right) - b_6 \right] b_{\rho 4} \right. \\
& + b_5(b_3^2 - b_4) - b_3 b_6 + b_7 \Big\} \Big] H(\tau_b) \Big] \\
\left[\frac{\rho \rho}{\theta \theta} \right] = & \left[\frac{1}{\eta} \right] \frac{\zeta Q_0}{C_0} \left[a_0 X_1 \tau_a \left\{ 1 + \frac{1}{2} \tau_a (a_{\rho 4} + a_3) \right. \right. \\
& + \frac{1}{6} \tau_a^2 \left[a_{\rho 5} + \left(a_3 - \frac{C_1}{C_0} \right) a_{\rho 4} + a_4 \right] \Big] H(\tau_a) \\
& - b_0 X_2 \tau_b \left\{ 1 + \frac{1}{2} \tau_b (b_{\rho 4} + b_3) + \frac{1}{6} \tau_b^2 \left[b_{\rho 5} \right. \right. \\
& + \left. \left. \left(b_3 - \frac{C_1}{C_0} \right) b_{\rho 4} + b_4 \right] \right\} H(\tau_b) \Big] \mp (1 - \eta) \frac{U}{\rho} - \zeta \theta, \\
& \quad (41)
\end{aligned}$$

where

$$\begin{aligned}
\tau_a = \tau - a_0(\rho - 1), \quad \tau_b = \tau - b_0(\rho - 1). \quad (42) \\
(2) \quad \tau_0 = 0
\end{aligned}$$

Equation (26) is modified into the following forms.

$$\left. \begin{aligned} \xi_1^2 &= \beta^2 p^2 \left\{ 1 + \frac{\delta}{\beta^2 p} \left[1 + \frac{1}{\beta^2 p} + (1 - \delta) \frac{1}{\beta^4 p^2} \right] \right\} \\ \xi_2^2 &= p \left\{ 1 - \frac{\delta}{\beta^2 p} \left[1 + (1 - \delta) \frac{1}{\beta^2 p} \right] \right\} \end{aligned} \right\} \quad (43)$$

From this equation, we obtain

$$\begin{aligned} \xi_1 &= \beta p \left\{ 1 + \frac{\delta}{2\beta^2 p} \left[1 + \left(1 - \frac{1}{4} \delta \right) \frac{1}{\beta^2 p} \right] \right\} \\ \xi_2 &= \sqrt{p} \left\{ 1 - \frac{\delta}{2\beta^2 p} \left[1 + \left(1 - \frac{3}{4} \delta \right) \frac{1}{\beta^2 p} \right] \right\} \end{aligned} \quad (44)$$

Substituting Eq. (44) into Eq. (28), we obtain the following equation:

$$\left. \begin{aligned} \frac{K_0(\xi_1 \rho)}{K_1(\xi_1)} &= X_1 Y_1 \left(1 + \frac{a_{\rho 0}}{p} + \frac{a_{\rho 1}}{p^2} \right) \\ \frac{K_1(\xi_1 \rho)}{K_1(\xi_1)} &= X_1 Y_1 \left(1 + \frac{a_{\rho 2}}{p} + \frac{a_{\rho 3}}{p^2} \right) \\ \frac{K_0(\xi_2 \rho)}{K_1(\xi_2)} &= X_2 Y_2 \left(1 + \frac{b_{\rho 0}}{\sqrt{p}} + \frac{b_{\rho 1}}{p} + \frac{b_{\rho 2}}{p\sqrt{p}} + \frac{b_{\rho 3}}{p^2} \right) \\ \frac{K_1(\xi_2 \rho)}{K_1(\xi_2)} &= X_2 Y_2 \left(1 + \frac{b_{\rho 4}}{\sqrt{p}} + \frac{b_{\rho 5}}{p} + \frac{b_{\rho 6}}{p\sqrt{p}} + \frac{b_{\rho 7}}{p^2} \right) \end{aligned} \right\} \quad (45)$$

where

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{\rho}} \exp \left[-\frac{\delta}{2\beta} (\rho - 1) \right], \quad X_2 = \frac{1}{\sqrt{\rho}} \\ Y_1 &= \exp \left[-\beta p (\rho - 1) \right], \quad Y_2 = \exp \left[-\sqrt{p} (\rho - 1) \right] \\ \left[\frac{a_{\rho 0}}{a_{\rho 2}} \right] &= -\frac{1}{\beta} \left\{ \frac{\delta}{2\beta^2} \left(1 - \frac{1}{4} \delta \right) (\rho - 1) + \left[\frac{\rho_{01}}{\rho_{11}} \right] \right\} \end{aligned}$$

$$\left. \begin{aligned} \left[\frac{a_{\rho 1}}{a_{\rho 3}} \right] &= \frac{1}{\beta^2} \left\{ \frac{\delta^2}{8\beta^4} \left(1 - \frac{1}{4} \delta \right)^2 (\rho - 1)^2 \right. \\ &+ \frac{\delta}{2\beta} \left[1 + \frac{1}{\beta} \left(1 - \frac{1}{4} \delta \right) (\rho - 1) \right] \left[\frac{\rho_{01}}{\rho_{11}} \right] + \left[\frac{\rho_{02}}{\rho_{12}} \right] \Big\} \\ \left[\frac{b_{\rho 0}}{b_{\rho 4}} \right] &= \frac{\delta}{2\beta^2} (\rho - 1) - \left[\frac{\rho_{01}}{\rho_{11}} \right], \\ \left[\frac{b_{\rho 1}}{b_{\rho 5}} \right] &= \frac{\delta^2}{8\beta^4} (\rho - 1)^2 - \frac{\delta}{2\beta^2} (\rho - 1) \left[\frac{\rho_{01}}{\rho_{11}} \right] + \left[\frac{\rho_{02}}{\rho_{12}} \right] \\ \left[\frac{b_{\rho 2}}{b_{\rho 6}} \right] &= \frac{\delta}{2\beta^2} \left\{ \frac{\rho - 1}{\beta^2} \left[1 - \frac{3}{4} \delta + \frac{\delta^2}{24\beta^2} (\rho - 1)^2 \right] \right. \\ &- \left[1 + \frac{\delta}{4\beta^2} (\rho - 1)^2 \right] \left[\frac{\rho_{01}}{\rho_{11}} \right] + (\rho - 1) \left[\frac{\rho_{02}}{\rho_{12}} \right] \Big\} \\ \left[\frac{b_{\rho 3}}{b_{\rho 7}} \right] &= \frac{\delta}{2\beta^2} \left\{ \frac{\delta (\rho - 1)^2}{2\beta^4} \left[1 - \frac{3}{4} \delta + \frac{\delta^2}{96\beta^2} (\rho - 1)^2 \right] \right. \\ &- \frac{\rho - 1}{\beta^2} \left[1 - \frac{1}{4} \delta + \frac{\delta^2}{24\beta^2} (\rho - 1)^2 \right] \left[\frac{\rho_{01}}{\rho_{11}} \right] \\ &+ \left[2 + \frac{\delta}{4\beta^2} (\rho - 1)^2 \right] \left[\frac{\rho_{02}}{\rho_{12}} \right] \Big\} \end{aligned} \right\} \quad (46)$$

After this, we can obtain displacement, temperature, and thermal stress distributions by the same procedure as for the previous case, but we will omit this on account of space considerations.

3. Numerical Examples and Discussion

Numerical examples are carried out under the following conditions:

$$\nu = 0.3, \quad \delta = 0.1, \quad \tau = 10^{-5}, \quad Q_0 = 1$$

$$\beta = 0.001 \sim 0.1, \quad \tau_0 = 0 \sim 2 \times 10^{-4}$$

Figures 1~4 show the displacement, temperature, radial stress, and hoop stress distributions in and around the circular cylindrical hole for various values of the relaxation time τ_0 in the case $\beta = 0.01$, respectively. Table 1 shows the positions of stress wave and thermal wave and the comparison of the velocity between stress wave and thermal wave for various values of the relaxation time in the case $\beta = 0.01$.

The displacement distributions in Fig. 1 show that the displacement magnitudes decrease gradually and the peak point moves on toward the hole when τ_0 becomes large. The reason for this is that the rate of heat conduction is slow because the propagation velocity of a thermal wave is slow when τ_0 becomes large.

The temperature distributions in Fig. 2 show the following. The rise in temperature occurs in the region through which the thermal wave passed in the range of $\tau_0 = 0 \sim 5 \times 10^{-5}$. The propagation velocity of the stress wave is faster than that of the thermal wave in the case of $\tau_0 = 2 \times 10^{-4}$. Because the rise in temperature due to the stress wave occurs, the broken point occurs at the position of the thermal wavefront which becomes clear.

Table 1 Positions of stress wavefront and thermal wavefront and the comparison of the velocity between stress wave and thermal wave for various values of the relaxation time in the case of $\beta=0.01$.

Relaxation time τ_0	Position of wavefront ρ		Velocity v	
	(a)	(b)	(a)	(b)
0	1.001000	∞	$v_e < v_t$	
5×10^{-6}	1.000997	1.004484		
1×10^{-5}	1.000994	1.003180		
2×10^{-5}	1.000988	1.002264		
5×10^{-5}	1.000957	1.001478		
2×10^{-4}	1.001084	1.000653	$v_e > v_t$	

(a) stress wave (b) thermal wave

Fig. 1 Displacement distributions in and around the circular cylindrical hole in the case of $\beta=0.01$.

Fig. 2 Temperature distributions in and around the circular cylindrical hole in the case of $\beta=0.01$.

Figures 3 and 4 show the radial stress $\hat{\rho\rho}$ and the hoop stress $\hat{\theta\theta}$, respectively. The thermal stresses occur in the region through which the thermal wave passed in the range of $\tau_0=0\sim5\times10^{-5}$. The thermal stresses occur in the region through which the stress wave passed because the stress wave is faster than the thermal wave in the case of $\tau_0=2\times10^{-4}$. The broken points occur at the stress wavefront in the range of $\tau_0=0\sim5\times10^{-5}$ and at the thermal wavefront in the case where $\tau_0=2\times10^{-4}$.

The displacement, the radial stress, and the hoop stress distributions in and around the hole for various values of the reciprocal of the nondimensionalized propagation velocity of longitudinal wave β in the case of $\tau_0=10^{-5}$ are shown in Figs.5~7, respectively, and the positions of wavefronts for those cases are shown in Table 2.

Figure 5 shows that the displacement magnitudes decrease and the peak point moves toward the hole when β becomes large. Figures 6 and 7, representing the stress distributions, show that the peak point moves toward the hole because the propagation velocity of the stress wave is slow when β becomes large.

Fig. 3 Radial stress distributions in and around the circular cylindrical hole in the case of $\beta=0.01$.

Fig. 4 Hoop stress distributions in and around the circular cylindrical hole in the case of $\beta=0.01$.

The propagation velocity of the thermal wave is always faster than that of the stress wave in all ranges of β for these figures. Table 2 shows that the positions of the thermal wave slightly change for these ranges of β . Therefore, the displacement and stresses disappear at a slightly altered position. The figure for temperature has been omitted because there are no distinct differences between these β .

To examine the effect of the magnitudes between the propagation velocity of the thermal wave and

Table 2 Positions of stress wavefront and thermal wavefront and the comparison of the velocity between stress wave and thermal wave for various values of the reciprocal of propagation velocity of longitudinal wave in the case of $\tau_0 = 10^{-5}$.

Reciprocal of propagation velocity β	Position of wavefront ρ		Velocity v	
	(a)	(b)	(a)	(b)
0.10	1.0001000	1.003162	$v_e < v_t$	
0.03	1.0003331	1.003164		
0.02	1.0004994	1.003166		
0.01	1.0009945	1.003180

Fig. 5 Displacement distributions in and around the circular cylindrical hole in the case of $\tau_0 = 10^{-5}$.

Fig. 6 Radial stress distributions in and around the circular cylindrical hole in the case of $\tau_0 = 10^{-5}$.

stress wave on displacement, temperature and stress distributions, the results for the case of $\tau_0 = 5 \times 10^{-5}$ are shown in Figs. 8~11, and the positions of wavefronts for this case are shown in Table 3. It is clear from Table 3 that the positions at which the displacement, temperature, and thermal stresses vanish vary considerably because the propagation velocity of the stress wave is faster than that of the thermal wave when β is under 0.005. The temperature in the region through which the thermal wave passed varies slightly. The stress causes large variations to occur in this region when the stress wave and thermal wave reach the region.

As $\tau = 10^{-5}$ is a very short period of time, the magnitude of the thermomechanical coupling parameter affects the displacement, temperature, and thermal stresses only slightly.

4. Conclusions

A short-time solution of a one-dimensional generalized thermoelasticity for an infinite body with a circular cylindrical hole which is supplied with a constant heat flux is presented. The results of numerical calculations of displacement, temperature, and

Fig. 7 Hoop stress distributions in and around the circular cylindrical hole in the case of $\tau_0 = 10^{-5}$.

Fig. 8 Displacement distributions in and around the circular cylindrical hole in the case of $\tau_0 = 5 \times 10^{-5}$.

Table 3 Positions of stress wavefront and thermal wavefront and the comparison of the velocity between stress wave and thermal wave for various values of the reciprocal of propagation velocity of longitudinal wave in the case of $\tau_0 = 5 \times 10^{-5}$.

Reciprocal of propagation velocity β	Position of wavefront ρ		Velocity v	
	(a)	(b)	(a)	(b)
0.05	1.0001998	1.001416	$v_e < v_t$	
0.02	1.0004965	1.001424		
0.01	1.0009569	1.001478		
0.005	1.002167	1.001305	$v_e > v_t$	
0.002	1.005263	1.001344		
0.001	1.010497	1.001347		

(a) stress wave (b) thermal wave

Fig. 9 Temperature distributions in and around the circular cylindrical hole in the case of $\tau_0 = 5 \times 10^{-5}$.

thermal stress distributions in and around the hole in the case of time $\tau = 10^{-5}$ show the following.

- (1) In the case in which β is fixed
 - (i) The magnitudes of displacement, temperature and thermal stresses decrease gradually when τ_0 becomes long.
 - (ii) The positions of stress and thermal wavefronts appear clearly in the radial stress and hoop stress distributions.
- (2) In the case in which τ_0 is fixed
 - (i) The displacement magnitude decreases when β becomes large.
 - (ii) The magnitudes of radial stress and hoop stress increase when β becomes large.
 - (iii) Clear variances occur in radial and hoop stress distributions when the stress wave and the thermal wave are reached.

Fig. 10 Radial stress distributions in and around the circular cylindrical hole in the case of $\tau_0 = 5 \times 10^{-5}$.

Fig. 11 Hoop stress distributions in and around the circular cylindrical hole in the case of $\tau_0 = 5 \times 10^{-5}$.

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