Torsional Stress Analysis and Vibration Damping of Three-Layered Rods*

(Characterization of Nonsymmetrical Sandwich Rod and Covered Rod)

Hiromi DEWA**

A torsional stress analysis and the expressions of torsional rigidity for the two types of nonsymmetrical three-layered rectangular rods such as sandwich rods with a viscoelastic core and covered rods having composition opposite to that of the sandwich rods, have been presented. The composite loss factors for these rods in torsional systems are also considered. The expressions of shear stress and torsional rigidity presented are briefly formulated and may be easily applicable to various two- and three-layered rods. Numerical results are discussed for typical sandwich rods and covered ones, and also those having geometrically similar compositions. These results describe how to consider the suitable rod composition in order to reduce the maximum shear stress arising at the interfaces between the layers. Also, they indicate a remarkable difference of the damping characteristics between the symmetrical sandwich rod and the covered one such that the latter holds geometrical similarity, but the former does not presenting instead, an optimal rod composition. Therefore, it was clarified that the damping capacity of the symmetrical sandwich rod may be reduced if the viscoelastic core layer thickness exceeds its optimum.

Key Words: Elasticity, Laminated Construction, Vibration of Continuous System, Torsion, Damping Property, Three-Layered Rod, Stress Analysis, Torsional Rigidity, Composite Loss Factor

1. Introduction

The laminated metal sheets with viscoelastic layers, especially the sandwich structure, have a remarkable ability to dissipate vibratory energy in mechanical systems. These sheet structures have been in rapid development and utilized recently in the aircraft, automobile and other industries.

Since the earliest work for the two-layered beam by Oberst⁽¹⁾ as a basic problem of these sheet structures, many analytical and experimental investigations have been reported, mainly concerning the flexural damping characteristics⁽²⁾⁻⁽⁴⁾. On the other hand, to the author's knowledge, torsional stress analysis or damping characteristics for these sheet structures have not been investigated to any great extent⁽⁵⁾⁻⁽⁷⁾, although these problems are as important as the flexural ones.

Therefore, in this paper, a stress analysis and

vibration damping characteristics in a torsional system are presented for both types of nonsymmetrical three-layered rectangular rods, i. e., a sandwich rod with a viscoelastic core and covered one with viscoelastic layers on both sides of the elastic layer, as shown in Figs. 1(a) and (b). The stress analysis method for a rod with multiplyconnected regions, which had been given by Muskhelishvili⁽⁸⁾ and was also given in detail by Tamate⁽⁹⁾ for two-layered rod, is applied to the present analysis. The analytical

Fig. 1 Nonsymmetrical three-layered rods: (a) Sandwich rod (N-EVE rod), (b) Covered rod (N-VEV rod) (c) Coordinate system

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** Faculty of Engineering, Ibaraki University, 4-12-1 Nakanarusawa-cho, Hitachi, 316, Japan

results for the shear stress and the torsional rigidity of the sandwich rod can be applicable to the covered rod, due to the fact that these rods have the same rod composition except that the shear modulus in each corresponding layer is different.

Numerical results for maximum shear stresses, torsional rigidities, and composite loss factors are discussed for both types of rods with various layer thicknesses and shear modulus ratios of the viscoelastic to elastic layer, respectively.

Nomenclature

- a_i =thickness of layer
 b =width of rod
 C, C_i =torsional rigidity of rod and its component
 G_i =shear modulus
 Im =imaginary part of complex quantity
 J_b, J_1, J_2 =angular inertia per unit length of rod and angular inertias of masses at free ends of rod
 $j=\sqrt{-1}$
 l =length of rod
 M, M_i =torsional moment and its component
 Re =real part of complex quantity
 R_i =dimensionless thickness (a_i/b)
 w_i =displacement of z -direction
 α =torsional angle per unit length of rod
 ε_i =shear modulus ratio
 $(\varepsilon_1 = G_2/G_1; \varepsilon_3 = G_2/G_3; \text{for sandwich rod})$
 $(\varepsilon_1 = G_1/G_2; \varepsilon_3 = G_3/G_2; \text{for covered rod})$
 $(\varepsilon = \varepsilon_1 = \varepsilon_3; \text{for symmetrical rod})$
 η =loss factor of viscoelastic layer
 η_c =composite loss factor of rod
 σ_i =normal stress
 $\tau_i, \bar{\tau}_i$ =shear stress and nonnormalized shear stress
 $(\bar{\tau}_i = \tau_i/G_1ab; \text{for sandwich rod})$
 $(\bar{\tau}_i = \tau_i/G_2ab; \text{for covered rod})$
 ω =natural angular frequency of rod
- Suffixes**
 $i(=1, 2, 3)$: quantity of i th layer
 $*$: complex quantity
 k : mode number

2. Stress Analysis and Torsional Rigidity

2.1 Nonsymmetrical sandwich rod

In the following, stress analysis and algebraic treatment are performed for only the nonsymmetrical sandwich rod (N -EVE rod in Fig. 1(a)) in order to obtain the expressions for stress components and torsional rigidity. The similar analytical expressions for the nonsymmetrical covered rod (N -VEW rod in Fig. 1(b)) are easily derived from those of the N -EVE rod, as will be described later.

The origin of Cartesian coordinates is placed at the geometrical center of the core layer of the N -EVE rod, as shown in Fig. 1(c). Taking account of Saint Venant's semi-inverse method in this torsion problem, we then represent the stress components and the fundamental equation of elasticity by the torsion function $\phi_i(x, y)$ as follows.

$$\frac{\partial^2 \phi_i}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial y^2} = 0 \quad (1)$$

$$\left. \begin{aligned} \sigma_{ix} = \sigma_{iy} = \sigma_{iz} = \tau_{ixy} = 0 \\ \tau_{izx} = G_i \alpha \left(\frac{\partial \phi_i}{\partial x} - y \right) \\ \tau_{izy} = G_i \alpha \left(\frac{\partial \phi_i}{\partial y} + x \right) \end{aligned} \right\} \quad (2)$$

The boundary conditions at the outer surfaces and the interfaces of the N -EVE rod are provided as

$$\left. \begin{aligned} \tau_{izy} = 0 & \quad (y = \pm b/2) \\ \tau_{1zx} = 0 & \quad (x = -a_2/2 - a_1) \\ \tau_{3zx} = 0 & \quad (x = a_2/2 + a_3) \\ \tau_{1zx} = \tau_{2zx}, w_1 = w_2 & \quad (x = -a_2/2) \\ \tau_{3zx} = \tau_{2zx}, w_3 = w_2 & \quad (x = a_2/2) \end{aligned} \right\} \quad (3)$$

In order to simplify the boundary value treatment, we introduce a following harmonic function $\phi_i(x, y)$:

$$\phi_i(x, y) = \phi_i(x, y) + xy \quad (4)$$

Assuming that no slippage occurs at the interfaces between each layer, and considering the displacement $w_i = \alpha \phi_i(x, y)$, we can rewrite Eqs. (1) through (3) as follows.

$$\frac{\partial^2 \phi_i}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial y^2} = 0 \quad (5)$$

$$\tau_{izx} = G_i \alpha \left(\frac{\partial \phi_i}{\partial x} - 2y \right), \tau_{izy} = G_i \alpha \frac{\partial \phi_i}{\partial y} \quad (6)$$

$$\frac{\partial \phi_i}{\partial y} = 0 \quad (y = \pm b/2) \quad (7 \cdot a)$$

$$\frac{\partial \phi_1}{\partial x} = 2y \quad (x = -a_2/2 - a_1) \quad (7 \cdot b)$$

$$\frac{\partial \phi_3}{\partial x} = 2y \quad (x = a_2/2 + a_3) \quad (7 \cdot c)$$

$$\left. \begin{aligned} G_1 \frac{\partial \phi_1}{\partial x} - G_2 \frac{\partial \phi_2}{\partial x} = 2y(G_1 - G_2) \\ \phi_1 = \phi_2 \end{aligned} \right\} \quad (x = -a_2/2) \quad (7 \cdot d)$$

$$\left. \begin{aligned} G_3 \frac{\partial \phi_3}{\partial x} - G_2 \frac{\partial \phi_2}{\partial x} = 2y(G_3 - G_2) \\ \phi_3 = \phi_2 \end{aligned} \right\} \quad (x = a_2/2) \quad (7 \cdot e)$$

Taking account of the separation of variables in the solution of Eq. (5), we easily give the odd function $\phi_i(x, y)$ (clarified from referring to Eq. (7·a)) as

$$\begin{aligned} \phi_i(x, y) = \sum_{n=0}^{\infty} \{ p_i \sinh(k_n \frac{x}{b}) \\ + q_i \cosh(k_n \frac{x}{b}) \} \sin k_n \frac{y}{b}, \end{aligned} \quad (8)$$

where $k_n = (2n+1)\pi$.

The unknown coefficients of p_i and q_i in Eq. (8) can be determined by substituting Eq. (8) into Eqs.

(7•b) through (7•e). In this process, the value of $2y$ is expanded into Fourier series between the interval $(b/2, -b/2)$ of the y coordinate as follows.

$$2y = \sum_{n=0}^{\infty} \frac{8(-1)^n}{k_n^2} \sin k_n \frac{y}{b} \quad (9)$$

Then the following simultaneous algebraic equations for p_i and q_i are given as

$$\left. \begin{aligned} p_1(1+t_2 T_1) - q_1(T_1+t_2) &= \theta_n/Y_1 y_2 \\ p_3(1+t_2 T_3) + q_3(T_3+t_2) &= \theta_n/Y_3 y_2 \\ p_1 t_1 - q_1 &= p_2 t_2 - q_2 \\ p_3 t_2 + q_3 &= p_2 t_2 + q_2 \\ p_1 - q_1 t_2 &= \varepsilon_1(p_2 - q_2 t_2) + (1 - \varepsilon_1)\theta_n/y_2 \\ p_3 + q_3 t_2 &= \varepsilon_3(p_2 + q_2 t_2) + (1 - \varepsilon_3)\theta_n/y_2 \end{aligned} \right\}, \quad (10)$$

where $T_i = \tanh(k_n R_i)$, $t_i = \tanh(k_n R_i/2)$, $Y_i = \cosh(k_n R_i)$, $y_i = \cosh(k_n R_i/2)$, $\theta_n = 8b^2(-1)^n/k_n^3$.

By solving Eqs.(10), the coefficients of p_i and q_i of the function ψ_i are given as

$$\left. \begin{aligned} p_2 &= \frac{\theta_n}{y_2} \left\{ \frac{(\varepsilon_1 - t_1 T_1)(T_3 + \varepsilon_3 t_2) + (\varepsilon_3 - t_3 T_3)(T_1 + \varepsilon_1 t_2)}{(\varepsilon_1 + t_2 T_1)(T_3 + \varepsilon_3 t_2) + (\varepsilon_3 + t_2 T_3)(T_1 + \varepsilon_1 t_2)} \right\} \\ &= \frac{\theta_n}{y_2} p' \\ q_2 &= \frac{\theta_n}{y_2} \left\{ \frac{(\varepsilon_1 + t_2 T_1)\varepsilon_3 - t_3 T_3 - (\varepsilon_3 + t_2 T_3)(\varepsilon_1 - t_1 T_1)}{(\varepsilon_1 + t_2 T_1)(T_3 + \varepsilon_3 t_2) + (\varepsilon_3 + t_2 T_3)(T_1 + \varepsilon_1 t_2)} \right\} \\ &= \frac{\theta_n}{y_2} q' \\ p_1 &= y_2 \theta_n \{ (T_1 + t_2)(q' - p' t_2) + 1/Y_1 \} \\ q_1 &= y_2 \theta_n \{ (t_2 T_1 + 1)(q' - p' t_2) + t_2/Y_1 \} \\ p_3 &= -y_2 \theta_n \{ (T_3 + t_2)(q' + p' t_2) - 1/Y_3 \} \\ q_3 &= y_2 \theta_n \{ (t_2 T_3 + 1)(q' + p' t_2) - t_2/Y_3 \} \end{aligned} \right\} \quad (11)$$

where p' and q' represent the parts in braces of the first and second equations of Eqs.(11).

Consequently, the shear stresses τ_{ixx} and τ_{ixy} in each layer can be obtained completely from Eqs.(6) in conjunction with the determined function ψ_i in Eq.(8) as follows.

$$\begin{aligned} \tau_{ixx}/G_i \alpha &= \sum_{n=0}^{\infty} \frac{k_n}{b} \left\{ p_i \cosh k_n \frac{x}{b} + q_i \sinh k_n \frac{x}{b} \right\} \sin k_n \frac{y}{b} - 2y \\ &\quad (12) \end{aligned}$$

$$\begin{aligned} \tau_{ixy}/G_i \alpha &= \sum_{n=0}^{\infty} \frac{k_n}{b} \left\{ p_i \sinh k_n \frac{x}{b} + q_i \cosh k_n \frac{x}{b} \right\} \cos k_n \frac{y}{b} \\ &\quad (13) \end{aligned}$$

Subsequently, we consider the torsional rigidity of the N -EVE rod. The resultant torsional moment M is determined by the shear stresses as the surface integral over each layer cross section (D_i);

$$M = \sum_{i=1}^3 M_i = \sum_{i=1}^3 \int_{D_i} (x\tau_{ixy} - y\tau_{ixx}) dD_i. \quad (14)$$

Integrating Eq.(14) after substituting Eqs.(12) and (13) into Eq.(14) in conjunction with Eqs.(11), and performing complicated integrating operations involving hyperbolic terms, we obtain the torsional rigidity

of each layer $C_i(M_i/\alpha)$ and the resultant torsional rigidity $C(M/\alpha)$ of the N -EVE rod as the following arranged expressions:

$$\begin{aligned} \frac{C_1}{G_1 b^4} &= \left(\frac{R_1}{3} - \sum_{n=0}^{\infty} \frac{64 t_1}{k_n^5} \right) + \varepsilon_1 R_2 \sum_{n=0}^{\infty} \frac{8 T_1 N_1}{k_n^4 D} \\ &\quad + \varepsilon_1 \sum_{n=0}^{\infty} \frac{32 t_1 T_1 N_1}{k_n^5 D} \end{aligned} \quad (15)$$

$$\frac{C_2}{G_2 b^4} = \left(\frac{R_2}{3} - \sum_{n=0}^{\infty} \frac{64 t_2}{k_n^5} \right) - R_2 \sum_{n=0}^{\infty} \frac{16 N_2}{k_n^4 D} - \sum_{n=0}^{\infty} \frac{64 t_2 N_2}{k_n^5 D} \quad (16)$$

$$\begin{aligned} \frac{C_3}{G_3 b^4} &= \left(\frac{R_3}{3} - \sum_{n=0}^{\infty} \frac{64 t_3}{k_n^5} \right) + \varepsilon_3 R_2 \sum_{n=0}^{\infty} \frac{8 T_3 N_3}{k_n^4 D} \\ &\quad + \varepsilon_3 \sum_{n=0}^{\infty} \frac{32 t_3 T_3 N_3}{k_n^5 D} \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{C}{G_1 b^4} &= \left(\frac{R_1}{3} - \sum_{n=0}^{\infty} \frac{64 t_1}{k_n^5} \right) + \varepsilon_1 \left(\frac{R_2}{3} - \sum_{n=0}^{\infty} \frac{64 t_2}{k_n^5} \right) \\ &\quad + \frac{\varepsilon_1}{\varepsilon_3} \left(\frac{R_3}{3} - \sum_{n=0}^{\infty} \frac{64 t_3}{k_n^5} \right) + \varepsilon_1 \sum_{n=0}^{\infty} \frac{32 N}{k_n^5 D}, \end{aligned} \quad (18)$$

where

$$\begin{aligned} N_1 &= 2(t_1 + t_2)(T_3 + \varepsilon_3 t_2) - T_3(t_1 - t_3)(1 - t_2^2) \\ N_2 &= T_1(t_1 + t_2)(T_3 + \varepsilon_3 t_2) + T_3(t_2 + t_3)(T_1 + \varepsilon_1 t_2) \\ N_3 &= 2(t_2 + t_3)(T_1 + \varepsilon_1 t_2) + T_1(t_1 - t_3)(1 - t_2^2) \\ N &= 2T_1(t_1 + t_2)^2(T_3 + \varepsilon_3 t_2) + 2T_3(t_2 + t_3)^2(T_1 + \varepsilon_1 t_2) \\ &\quad - T_1 T_3(1 - t_2^2)(t_1 - t_3)^2 \\ D &= (\varepsilon_1 + t_2 T_1)(T_3 + \varepsilon_3 t_2) + (\varepsilon_3 + t_2 T_3)(T_1 + \varepsilon_1 t_2) \end{aligned}$$

In Eq.(18), if we take the shear modulus ratios ε_1 and ε_3 and the dimensionless thicknesses R_1 and R_3 (in the following, the dimensionless thickness R_i of each layer is abbreviated simply to thickness R_i) as respectively equivalent, then Eq.(18) coincide with the torsional rigidity of the symmetrical sandwich rod, which was reported previously⁽⁷⁾, solved by means of introducing a different function in order to treat its boundary value problem.

Each first term in Eqs.(15) through (17) represents the torsional rigidity of each layer itself before bonding, and the sum of the second and the third terms represents the increment of torsional rigidity produced by bonding each layer. In the same way, the first through the third terms in Eq.(18) represent the torsional rigidity of each layer itself, and the fourth term represents the total increment of the rod by bonding.

2.2 Nonsymmetrical covered rod

The analytical results for the N -VEV rod can be easily derived from the results of the N -EVE rod. It is clear from the inspection of the boundary conditions in Eqs.(3) that the simultaneous algebraic equations of Eqs.(10) are retained completely for the N -VEV rod if the shear modulus ratios are taken to be the same ones as for the N -EVE rod (then $\varepsilon_i > 1$). But if the shear modulus ratios for the N -VEV rod are taken to be the viscoelastic to elastic ratio in the usual manner (then $\varepsilon_i < 1$), the expressions of shear stresses and torsional rigidity for the N -VEV rod can be

obtained by replacing ε_1 and ε_3 with their reciprocal values in Eq.(11) and Eqs.(15) through (18). Therefore, these expressions are easily utilized for the N -VEV rod.

3. Composite Loss Factor

The dynamic behavior of viscoelastic materials can be explained by the dynamic behavior of elastic materials, replacing the viscoelastic problems with the corresponding elastic ones. This procedure, known as the elastic-viscoelastic correspondence principle, may be justified by linear viscoelastic theory⁽¹⁰⁾. This principle is applied to determination of the composite loss factors for the present rods.

For a low-frequency torsional vibration, the frequency equation⁽¹¹⁾ of a homogeneous and uniform cross section rod which has masses of angular inertias (J_1 and J_2) at both free ends, is represented by

$$(mn\beta^2 - 1)\tan\beta - (m+n)\beta = 0, \quad (19)$$

where $m = J_1/J_0$, $n = J_2/J_0$, $J_0 = J_b$, $\beta = \omega l \sqrt{J_b/C}$.

Then the frequency equations with various boundary conditions are obtained by taking account of the values for the inertia moment ratios m and n in Eq. (19) as follows.

- (1) Free-free rod: ($m=n=0$),
- (2) Cantilever rod: ($m=\infty, n=0$),
- (3) Fixed-fixed rod: ($m=n=\infty$),
- (4) Cantilever rod with an inertia mass at the free end: ($m=\infty, n=J_2/J_0$).

In the case of (4), for example, the frequency equation is given as

$$\beta \tan \beta = J, \quad (20)$$

where $J = 1/n = J_0/J_2$.

After making an approximation⁽¹¹⁾, $\tan\beta = \beta + \beta^3$, we expand the solution of Eq.(20) into Taylor series; the first- and k th-order approximate solutions (up to the second terms) are given as follows⁽⁹⁾.

$$\beta_1 = J \left(1 - \frac{J}{3}\right), \quad \beta_k = \left\{ (k-1)\pi + \frac{J(1-J/3)}{(k-1)\pi} \right\}_{k=2,3,\dots} \quad (21)$$

When the above solution and also the correspondence principle are applied to the present three-layered rods, the complex angular frequency ω^* and the complex torsional rigidity C^* can be obtained by replacing the static shear modulus G of the viscoelastic layer with the complex shear modulus $G^* = G + jG'$ defined as $\eta = G'/G$. Therefore, the first- and k th-order complex angular frequencies ω_1 and ω_k , and the composite loss factor η_c of the rods are given as follows.

$$\omega_1^2 = \frac{C^* J}{J_b l^2} \left(1 - \frac{J}{3}\right) \quad (22)$$

$$\omega_k^2 = \frac{C^* J}{J_b l^2} \left\{ (k-1)\pi + \frac{J(1-J/3)}{(k-1)\pi} \right\}^2_{k=2,3,\dots} \quad (23)$$

$$\eta_c = \left\{ \frac{\text{Im}(\omega_k^{*2})}{\text{Re}(\omega_k^{*2})} \right\}_{k=1,2,\dots} = \frac{\text{Im}(C^*)}{\text{Re}(C^*)} \quad (24)$$

For the other cases of (1) through (3), each complex angular frequency is determined by Eq.(19) in a similar manner, and the composite loss factor may be determined approximately from the ratio of the imaginary to real part of the complex torsional rigidity regardless of the boundary conditions, if the rod length is sufficiently long.

4. Numerical Results and Discussion

4.1 Shear stress distributions of symmetrical rods

In the following discussion, the shear stresses are normalized as $\bar{\tau}_i = \tau_i/G_1 ab$ for the sandwich rod and $\bar{\tau}_i = \tau_i/G_2 ab$ for the covered rod in Eqs.(12) and (13), and series number n is considered up to 51 terms.

Typical shear stress distributions in the interfaces of the viscoelastic layers for the symmetrical sand-

Fig. 2 Distribution of $\bar{\tau}_2$ in the interface of the viscoelastic layer of the EVE rod; $\varepsilon_1 = G_2/G_1$, $\varepsilon_3 = G_2/G_3$

Fig. 3 Distribution of $\bar{\tau}_3$ in the outer surface of the viscoelastic layer of the VEV rod; $\varepsilon_1 = G_1/G_2$, $\varepsilon_3 = G_3/G_2$

wich rod (EVE rod: $R_1=R_3$, $\epsilon_1=\epsilon_3$) and in the interface and outer surface of the symmetrical covered rod (VEV rod: $R_1=R_3$, $\epsilon_1=\epsilon_3$) are shown in Figs. 2 and 3, respectively, as a function of middle layer thickness. The maximum stress $\bar{\tau}_{2zx(\max)}$ of the EVE rod always arises at the top and bottom edges in both the interfaces which are evaluated to have low fracture toughness, and its value increases with decrease of the viscoelastic layer thicknesses. On the other hand, the maximum values of $\bar{\tau}_{2zy(\max)}$ arise at the centers of the interfaces, but $\bar{\tau}_{2zy(\max)}$ does not show in Fig. 2 as its value is sufficiently low.

For the VEV rod, the stress distribution of $\bar{\tau}_{3zy}$ ($\bar{\tau}_{3zy}=\bar{\tau}_{1zy}$) in the outer surface shows flat distribution, and that of $\bar{\tau}_{3zx}$ ($\bar{\tau}_{3zx}=\bar{\tau}_{1zx}$) in the interface shows the maximum values at the top and bottom edges. These maximum values are almost equivalent, though $\bar{\tau}_{3zy(\max)}$ is slightly higher than $\bar{\tau}_{3zx(\max)}$. In comparison with the stress distributions of both rods, the maximum stress in the viscoelastic layer of the EVE rod is much greater than that of the VEV rod.

4.2 Maximum shear stresses of nonsymmetrical rods

Subsequently, we discuss the maximum stresses of both the N -EVE and the N -VEV rods. In these cases, typical nonsymmetrical rods, having a constant sum of the outer layer thicknesses and a shifting location of the middle layer between the outer layers are considered. The maximum stress variations in these rods are shown in Figs. 4 and 5, as a function of the thickness ratio R_3/R_1 . In these figures, the solid line and the broken one indicate that the maximum stress arises in the first layer and the third one or at

their sides, respectively.

In the N -EVE rod, the maximum stress $\bar{\tau}_{1zy(\max)}$ or $\bar{\tau}_{3zy(\max)}$ in the elastic layers arises at the center of the thicker layer surface, and the maximum stress $\bar{\tau}_{2zx(\max)}$ in the viscoelastic layer arises at the top and the bottom edges in the interface of the same thicker elastic layer side.

Furthermore, it is clarified that the maximum stress in the elastic layers is maximized in the two-layered rod ($R_1=0$ or $R_3=0$) and minimized at the EVE rod ($R_3/R_1=1$), and shows the opposite feature in the core layer. These maximum values increase with decrease of the viscoelastic layer thickness.

In the same way, the variation of $\bar{\tau}_{2zy(\max)}$ for the N -VEV rod in Fig. 5 is similar to $\bar{\tau}_{1,3zy(\max)}$ of the N -EVE rod, though $\bar{\tau}_{2zy(\max)}$ arises at the thinner viscoelastic layer side and is sufficiently low. It is interesting that maximum stresses in both the viscoelastic layer and the elastic one of the N -VEV rod are maximized in the two-layered rod and minimized in the VEV rod as shown in Fig. 5.

4.3 Composite loss factor and torsional rigidity

Figure 6 shows the loss factor ratio of η_c/η and the torsional rigidity $C/G_1 b^4$ of the N -EVE rod, which has a constant sum of both elastic layer thicknesses. The solid line and the broken one indicate rods composed of the same elastic layer materials and of different ones, respectively.

The composite loss factor of rods composed of the same elastic layers takes its peak value at the EVE rod and its minimum value at the two-layered rod, but its torsional rigidity exhibits opposite features. If the N -EVE rods are composed of different

Fig. 4 Variation of $\bar{\tau}_{i(\max)}$ in the interface and outer surface of the N -EVE rod; $\epsilon_1=G_2/G_1$, $\epsilon_3=G_2/G_3$

Fig. 5 Variation of $\bar{\tau}_{i(\max)}$ in the interface and outer surface of the N -VEV rod; $\epsilon_1=G_1/G_2$, $\epsilon_3=G_3/G_2$

elastic layer materials, as shown by the broken line, then the peak value becomes lower than that of the N -VEV rod and also its optimal thickness ratio shifts to a different value; and then its torsional rigidity gradually becomes lower according to the increase of R_3/R_1 . These facts for the composite loss factors indicate that the layer with low shear modulus should be made thicker than the other elastic one in order to obtain higher damping capacity.

The numerical results for the N -VEV rod are shown in Fig. 7. In this case, both composite loss factor and torsional rigidity of the two-layered rod show higher values than those of the VEV rod. These facts are interesting from the viewpoint of a covered rod structure design, even though the maximum stress in the viscoelastic layer of the two-layered rod is slightly higher than that of the VEV rod, as shown in Fig. 5.

Finally, we discuss the composite loss factors of geometrically similar EVE and VEV rods, which have constant thickness ratios (R_2/R_1 for the EVE rod and R_1/R_2 for the VEV rod). These results are shown in Figs. 8 and 9 as a function of the elastic layer thickness. The composite loss factor of the EVE rod has the peak values for each shear modulus ratio as shown in Fig. 8. The optimal value of R_1 corresponding to each peak value increases with an increase of the

shear modulus ratio as well as with a decrease of the viscoelastic layer thickness. But these peak values retain almost unchanged for various rod compositions, except for the high shear modulus ratio ($\epsilon=10^{-1}$). Therefore these results indicate that the optimal damping is dependent on each rod composition.

Fig. 7 Variations of η_c/η and C of the N -VEV rod ($R_1 + R_3 = \text{constant}$)

Fig. 6 Variations of η_c/η and C of the N -EVE rod ($R_1 + R_3 = \text{constant}$)

Fig. 8 Variation of η_c/η of geometrically similar the EVE rod ($R_2/R_1 = \text{constant}$, $R_1 = R_3$)

Fig. 9 Variation of η_c/η of the geometrically similar VEV rod ($R_1/R_2=\text{constant}$, $R_1=R_3$)

However, the same results for the VEV rod do not have the peak values, and clearly show the tendency of maintaining a constant value, as shown in Fig. 9, except for the larger value of R_2 . This fact may be clarified by considering that the damping characteristics of the VEV rods hold a relation of geometrical similarity.

In order to further consider the EVE rod, we discuss the relationship between the composite loss factor and the viscoelastic layer thickness. Then, in the case of Fig. 8, $R_2/R_1=10^{-1}$ and 10^{-2} at $\varepsilon=10^{-3}$ are rewritten in Fig. 10.

As shown in Fig. 10, if the elastic layer thicknesses of the EVE rod are below the critical value R_1 which corresponds to the intersection of the two solid curves, then, a thicker viscoelastic layer results in a higher composite loss factor (illustrated by the left-hand arrow). However, above the critical value, thickening results in a lower composite loss factor (illustrated by the right-hand arrow). A control factor which determines the optimal R_1 may be suggested by evaluating the ratio $\Delta E/E$ shown as the broken lines, where E and ΔE represent the sum of torsional rigidity of the elastic layers before bonding and that of the increment produced by bonding, respectively. Each value of R_1 rendering the ratio $\Delta E/E$ unity clearly agrees well with the optimal R_1 which makes each composite loss factor a peak value.

Fig. 10 Relationship of $\eta_{c(\max)}$ vs $\Delta E/E$ in the geometrically similar EVE rod

5. Conclusions

Shear stress, torsional rigidity, and composite loss factor of typical nonsymmetrical sandwich and covered rods were investigated numerically by means of the analytical expressions developed. These expressions have been represented briefly, and are easily applicable to torsion problems of the various two- or three-layered rods. It is indicated that the two-layered rod is superior to the covered rod with the same total viscoelastic layer thickness in both damping and torsional rigidity. Furthermore, it was revealed that the composite loss factor of the symmetrical covered rod exhibits geometrical similarity. However, the symmetrical sandwich rod does not; it presents the optimal rod composition for the damping characteristics. Therefore it may be said for the symmetrical sandwich rod that thickening the viscoelastic core layer does not always make the composite loss factor higher. A controlling factor evaluating such an optimal rod composition was also presented in this paper.

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