Transverse Bending of Rectangular Plates with Single- or Double-Edge Cracks*

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The stress intensity factors of rectangular plates with single- or double-edge cracks under transverse bending are evaluated by the body-force doublet method. The stress field due to a pair of point moments (doublets) in a semi-infinite plate is used to solve these problems. According to the results, the effect of plate length on K_I is less under transverse bending than under uniform tension in rectangular plates of the same form.

Key Words: Elasticity, Stress Intensity Factor, Rectangular Plate, Edge Crack, Transverse Bending

1. Introduction

To date, as examples of the analysis of transverse bending of strips with single- or double-edge cracks based on classical theory⁽¹⁾, Hasebe et al.⁽²⁾, Hasebe⁽³⁾ and Hasebe and Takemura⁽⁴⁾ have cited stress intensity factors of strips with a step and an edge crack, and strips with double-edge cracks. On the other hand, Boduroglu and Erdogan⁽⁵⁾ performed analyses of strips with double-edge cracks based on Reissner's theory⁽⁶⁾. However, the analysis of rectangular plates with single- or double-edge cracks under transverse bending is rarely conducted.

In this paper, the authors obtain the stress intensity factors of rectangular plates with single- or double-edge cracks under transverse bending (Fig. 5). The analysis is based on classical theory and uses the body force doublet method⁽⁷⁾. The effect of the plate

length on the stress intensity factor is investigated. An analysis of semi-infinite plates with an edge crack (Fig. 4) is also performed with the object of checking the accuracy of the present method. Though the present method is similar to the method of analysis of finite plates under transverse bending based on the body force method⁽⁸⁾ by Murakami and Araki⁽⁹⁾, it differs from theirs in that the solution of a pair of point moments applied on a semi-infinite plate is used as a fundamental solution in order to improve the accuracy and reduce the number of elements.

2. Method of Analysis

2.1 Boundary condition

In the analysis of the plane stress problem by the body force doublet method, boundary conditions can be satisfied by distributing pairs of point forces on the imaginary boundaries in an infinite plate. Similarly, in the analysis of transverse bending, the boundary conditions can be satisfied by distributing pairs of point moments in an infinite plate. However, the torsional moment boundary condition and the shearing stress boundary condition cannot be satisfied independently⁽⁴⁾. This point must be taken into consid-

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eration in the analysis.

2.2 Fundamental solution

In this paper, the solution of a pair of point moments applied to a semi-infinite plate as shown in Fig. 1(b) is used as a fundamental solution. First, we derive the solution of a point moment applied to a semi-infinite plate as shown in Fig. 1(a) and then obtain the solution of a pair of point moments by differentiation.

In the classical theory, displacement w perpendicular to the x - y plane can be expressed as Eq. (1) by complex functions $\phi(z)$ and $\bar{\phi}(z)$. Bending moments M_x , M_y and M_{xy} , and shearing forces Q_x and Q_y can also be expressed as Eqs. (2)~(6).

$$Dw = \text{Re}[\bar{z}\phi(z) + \psi(z)] \quad (1)$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \quad (2)$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \quad (3)$$

$$M_{xy} = -D(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \quad (4)$$

$$Q_x = -D \frac{\partial}{\partial x} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \quad (5)$$

$$Q_y = -D \frac{\partial}{\partial y} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \quad (6)$$

where D denotes bending stiffness and $z = x + iy$.

$$D = \frac{Et^3}{12(1-\nu^2)} \quad (7)$$

E : Young's modulus, t : thickness of plate, ν : Poisson's ratio

2.2.1 Definition of resultant moment M_x , M_y and M_{xy} in Eqs. (2)~(4) denote bending or torsional moments per unit length. Q_x and Q_y in Eqs. (5) and (6) denote shearing forces per unit length. Thus

(a) Point moment

(b) Pair of point moments (doublet)

Fig. 1 Fundamental solution

when we imagine a route which connects two points A and B in an infinite plate, resultant moments due to transverse bending R_x and R_y exerted through the route from the right-hand side to the left-hand side can be given by the equations:

$$R_x = \int_A^B \left[M_x dy - \left\{ \int (Q_y dx - Q_x dy) + M_{xy} \right\} dx \right] \quad (8)$$

$$R_y = \int_A^B \left[\left\{ \int (Q_x dy - Q_y dx) + M_{xy} \right\} dy - M_y dx \right] \quad (9)$$

where the directions of M_x , M_y and others are taken to be equal to those of σ_x , σ_y and others in the two-dimensional problem, and R_x and R_y denote the components of resultant moment in the x and y directions, respectively. The complex functions $\phi(z)$ and $\bar{\phi}(z)$, from which R_x , R_y , M_x , M_y , M_{xy} and M_{xy} are derived, are given by

$$- [R_y + iR_x]_A^B = [-(3+\nu)\bar{\phi}(z) + (1-\nu)\{\bar{z}\phi(z) + \psi''(z)\}]_A^B \quad (10)$$

$$M_x = -2(1+\nu)\text{Re}[\phi'(z)] - (1-\nu)\text{Re}[\bar{z}\phi''(z) + \psi''(z)] \quad (11)$$

$$M_y = -2(1+\nu)\text{Re}[\bar{\phi}'(z)] + (1-\nu)\text{Re}[\bar{z}\phi''(z) + \psi''(z)] \quad (12)$$

$$M_{xy} = (1-\nu)\text{Im}[\bar{z}\phi''(z) + \psi''(z)] \quad (13)$$

In the above equations, dashes indicate differentiation, and Re and Im denote the real and imaginary parts of the expression, respectively.

2.2.2 The solution of a point moment applied to a semi-infinite plate by the complex functions Noda et al.⁽¹⁰⁾ have already obtained the solution of a point moment applied to a semi-infinite plate expressed by a real function. However, in order to obtain the resultant bending moment, it is convenient to express the solution as complex functions. Based on the principle of reflection⁽¹¹⁾, the solution can be derived as follows.

By using complex functions $\phi_1(z)$ and $\bar{\phi}_1(z)$, the resultant bending moment can be written as Eq. (10). Now, so as to satisfy the boundary condition that the straight boundary L , the x axis, be free, we introduce the following auxiliary function $\chi_1(z)$:

$$\chi_1(z) = (1-\nu)\{\bar{z}\phi_1'(z) + \bar{\phi}_1'(z)\} \quad (14)$$

and define the analytical continuation over the boundary L as follows.

$$\phi_1(z) = \frac{1}{3+\nu} \bar{\chi}_1(z) \quad (15)$$

$$\chi_1(z) = (3+\nu)\bar{\phi}_1(z) \quad (16)$$

When a point moment $Me^{i\alpha}$ is applied at $z=z_0$ to an infinite plate, $\phi_1(z)$ and $\bar{\phi}_1(z)$ have a singular point at $z=z_0$. Their principal parts are

$$\phi_1(z) = N_1 \log(z-z_0) \quad (17)$$

$$\bar{\phi}_1(z) = N_2(z-z_0)\log(z-z_0) - \bar{z}_0 N_1 \log(z-z_0) \quad (18)$$

where

$$N_1 = \frac{Me^{ia}}{8\pi} \quad (19)$$

$$N_2 = \frac{Me^{-ia}}{8\pi} \quad (20)$$

The principal part of the function $\chi_1(z)$ at $z=z_0$ defined by Eq.(14) is

$$\chi_1(z) = (1-\nu) \left\{ N_1 \frac{z_0 - \bar{z}_0}{z - \bar{z}_0} + N_2 \log(z - z_0) \right\} \quad (21)$$

When $\phi_1(z)$ and $\psi_1(z)$ are continued analytically over the x axis, the straight boundary, a singular point appears at $z=z_0$. From Eqs.(15) and (16), the principal parts of it can be written as

$$\phi_1(z) = \frac{1-\nu}{3+\nu} \left\{ \bar{N}_1 \frac{\bar{z}_0 - z_0}{z - \bar{z}_0} + \bar{N}_2 \log(z - \bar{z}_0) \right\} \quad (22)$$

$$\chi_1(z) = (3+\nu) \bar{N}_1 \log(z - \bar{z}_0) \quad (23)$$

$\phi_1(z)$ and $\psi_1(z)$ have no singular points any more except the above points. Thus adding them, we find

$$\phi_1(z) = N_1 \log(z - z_0) + \frac{1-\nu}{3+\nu} \left\{ \bar{N}_1 \frac{\bar{z}_0 - z_0}{z - \bar{z}_0} + \bar{N}_2 \log(z - \bar{z}_0) \right\} \quad (24)$$

$$\chi_1(z) = (1-\nu) \left\{ N_1 \frac{z_0 - \bar{z}_0}{z - \bar{z}_0} + N_2 \log(z - z_0) \right\} + (3+\nu) \bar{N}_1 \log(z - \bar{z}_0) \quad (25)$$

$$\psi_1' = \frac{1}{1-\nu} \chi_1(z) - z \phi_1'(z) \quad (26)$$

Substituting Eqs.(24)~(26) into Eqs.(10)~(13), we find the solution of a point moment applied on an infinite plate.

2.2.3 The solution of a pair of point moments applied to a semi-infinite plate expressed by complex functions From Eqs.(24)~(26), the solution of a pair of point moments $\pm Me^{ia}$ acting with an infinitesimal distance $\Delta z_0 (= \delta e^{i\gamma})$ can be obtained as shown in Fig. 1. The complex functions $\phi_2(z)$ and $\psi_2(z)$ expressing the displacements due to the pair of point moments become

$$\phi_2(z) = \lim_{\delta \rightarrow 0} \left[\frac{\partial \phi_1}{\partial z_0} \Delta z_0 + \frac{\partial \phi_1}{\partial \bar{z}_0} \Delta \bar{z}_0 \right] \quad (27)$$

$$\chi_2(z) = \lim_{\delta \rightarrow 0} \left[\frac{\partial \chi_1}{\partial z_0} \Delta z_0 + \frac{\partial \chi_1}{\partial \bar{z}_0} \Delta \bar{z}_0 \right] \quad (28)$$

The pairs of point moments (doublets) which should really be applied are the x -directional bending moment doublet X [Fig. 2(a)], the y -directional bending moment doublet Y [Fig. 2(b)] and the torsional doublet T [Fig. 2(c)]. Bending moment doubles X and Y are combined with doublets νM perpendicular to them. This is to cancel out the bending deformation in the y direction due to the x -directional doublet or the bending deformation in the x direction due to the y -directional doublet.

Substituting $\alpha=0, \gamma=0$ or $\alpha=\pi/2, \gamma=\pi/2$ into Eqs.(27) and (28), we find the complex function expressing the elastic field due to X or Y . Substituting $\alpha=0, \gamma=\pi/2$ or $\alpha=\pi/2, \gamma=0$ into Eqs.(27) and (28),

we find the complex function expressing the elastic field due to T . From these, the complex functions expressing the elastic field due to three kinds of unit moment doublets become

$$\begin{aligned} -[R_\nu + iR_x]_A^B \Big|_{x=1} &= \frac{1-\nu}{8\pi} \left[-2(1+\nu) \frac{1}{z - z_0} + (3+\nu) \frac{1}{\bar{z} - \bar{z}_0} + (1-\nu) \frac{\bar{z} - \bar{z}_0}{(z - \bar{z}_0)^2} + 2(1+\nu) \frac{1}{\bar{z} - z_0} \right. \\ &\quad - (3+\nu) \frac{1}{z - \bar{z}_0} - (1-\nu) \frac{\bar{z} - \bar{z}_0}{(\bar{z} - z_0)^2} \\ &\quad - \frac{(1-\nu)(1+3\nu)}{3+\nu} \frac{z - \bar{z}}{(z - \bar{z}_0)^2} - 2 \frac{(1-\nu)^2}{3+\nu} \\ &\quad \times \left. \frac{(z - \bar{z})(z_0 - \bar{z}_0)}{(z - \bar{z}_0)^3} \right]_A^B \\ -[R_\nu + iR_x]_A^B \Big|_{y=1} &= \frac{1-\nu}{8\pi} \left[-2(1+\nu) \frac{1}{z - z_0} \right. \\ &\quad - (3+\nu) \frac{1}{\bar{z} - \bar{z}_0} - (1-\nu) \frac{\bar{z} - \bar{z}_0}{(z - \bar{z}_0)^2} \\ &\quad + 2(1+\nu) \frac{1}{\bar{z} - z_0} \\ &\quad \left. + (3+\nu) \frac{1}{z - \bar{z}_0} + (1-\nu) \frac{\bar{z} - \bar{z}_0}{(\bar{z} - z_0)^2} - (1-\nu) \right] \end{aligned} \quad (29)$$

(a) X-directional doublet

(b) Y-directional doublet

(c) Torsional doublet

Fig. 2 Moment doublets

$$\times \frac{z - \bar{z}}{(z - \bar{z}_0)^2} + 2 \frac{(1 - \nu)^2}{3 + \nu} \frac{(z - \bar{z})(z_0 - \bar{z}_0)}{(z - \bar{z}_0)^3} \Big]_A^B \quad (30)$$

$$- [R_y + iR_x]_A^B \Big|_{r=1} = \frac{i}{4\pi} \left[(1 - \nu) \frac{\bar{z} - \bar{z}_0}{(z - \bar{z}_0)^2} - (3 + \nu) \frac{1}{\bar{z} - \bar{z}_0} - (1 - \nu) \frac{\bar{z} - \bar{z}_0}{(\bar{z} - \bar{z}_0)^2} + (3 + \nu) \frac{1}{z - \bar{z}_0} - \frac{(1 - \nu)^2}{3 + \nu} \left\{ \frac{z - \bar{z}}{(z - \bar{z}_0)^2} - 2 \frac{(z - \bar{z})(z_0 - \bar{z}_0)}{(z - \bar{z}_0)^3} \right\} \right]_A^B \quad (31)$$

2.3 Density of bending moment doublets and stress intensity factors

In this paper, we conduct the analysis of the transverse bending of semi-infinite plates or rectangular plates. Since these problems are symmetrical with respect to the crack surface, the bending moment boundary conditions on the imaginary crack surface can be satisfied by distributing the x -directional doublets X alone. In the numerical calculation, letting X_c be the density of bending moment doublet applied to an infinitesimal element ds of the imaginary crack surface and letting X_b , Y_b and T_b be the densities of bending and torsional moment doublets applied to an infinitesimal element ds of the other imaginary boundaries, we have the following expressions for X , Y and T in Eqs.(29) and (30) :

$$X_c = 4\sqrt{a^2 - s^2} f_c ds \quad (32)$$

$$X_b = f_b ds \quad (33)$$

$$Y_b = g_b ds \quad (34)$$

$$T_b = h_b ds \quad (35)$$

where f_c , f_b , g_b and h_b are weighting functions. By these equations, unknown X_c , X_b , Y_b and T_b are transformed into f_c , f_b , g_b and h_b , respectively. s in Eq. (32) denotes the distance between the crack tip at the straight boundary and the point of application of the doublet. The weighting functions f_c , f_b , g_b , and h_b were determined from the boundary conditions of resulting moments. In that case, the weighting functions were approximated by the partly linear continuous functions characterized by the value of the divided points⁽¹²⁾.

Moment doublets were also applied to the dotted line outside of the rectangle ABCD. This is to cancel out the singularity which occurs when a boundary includes a corner and to improve the accuracy⁽¹³⁾.

The stress intensity factor K_I can be obtained by

Fig. 3 Imaginary boundaries where moment doublets are distributed

the equation

$$K_I = f_{ct} \sqrt{\pi a} \quad (36)$$

where f_{ct} denotes the value of the weighting function at the crack tip.

3. Results of Calculation

In the calculation of stress intensity factors, dividing the boundaries of the crack surface, straight edges and side boundaries (we call the straight boundaries AB and CD the straight edges, and call the other straight boundaries, BC and DA, the side boundaries) into n_1 or n_2 elements' we calculate the values of the stress intensity factor for the two kinds of division and predict the exact solution by extrapolation. We determined the dividing numbers n_1 and n_2 such that the error of the extrapolated solution would be smaller than 0.1 %.

Table 1 F_I of a rectangular plate with an edge crack ($F_I = K_I / \sigma_b \sqrt{\pi a}$, $\sigma_b = 6M/2$)

ν	0.0	0.3	0.5
Hasebe et.al. [14]	1.0107	1.0042	1.0019
Present Analysis	1.0108	1.0043	1.0019

Fig. 4 A semi-infinite plate with an edge crack

(a) Single edge-cracked plate

(b) Double edge-cracked plate

Fig. 5 Rectangular plates with single- or double-edge cracks under transverse bending

Table 2 F_I of a rectangular plate with a single-edge crack
 $(F_I = K_I / \sigma_b \sqrt{\pi a}, \sigma_b = 6M/t^2, \nu = 0.3)$

a/W \ L/W	0.2	0.4	0.6	0.8	1.0	1.2	2.0	∞	∞ [2]
0	1.004	1.004	1.004	1.004	1.004	1.004	1.004	1.004	-
0.1	1.080	1.024	1.014	1.011	1.010	1.009	1.009	1.009	1.020*
0.2	1.232	1.081	1.043	1.031	1.027	1.025	1.024	1.024	1.023*
0.3	1.386	1.156	1.090	1.066	1.057	1.053	1.051	1.051	1.045*
0.4	1.529	1.239	1.149	1.115	1.102	1.096	1.093	1.093	1.082*
0.5	1.662	1.327	1.222	1.182	1.165	1.159	1.154	1.154	1.136*
0.6	1.788	1.423	1.315	1.274	1.259	1.252	1.246	1.245	1.217*
0.7	1.915	1.547	1.449	1.416	1.404	1.397	1.392	1.386	1.346*
0.8	2.075	1.755	1.689	1.667	1.659	1.651	1.641	1.635	1.578*
0.9	2.469	2.303	2.272	2.255	2.250	2.240	2.236	2.230	2.130*

Asterisked : $\nu = 0.25$

Fig. 6 F_I of a rectangular plate with a single-edge crack

Fig. 8 Effect of side boundaries compared with that under tension

Fig. 7 Effect of side boundaries of a rectangular plate with a single-edge crack

3.1 Semi-infinite plates with an edge crack

We conducted the analysis of semi-infinite plates with a crack (Fig. 4) in order to check the accuracy. Table 1 shows the comparison between the present results and Hasegawa's results based on classical theory⁽¹⁴⁾. The present results agree with Hasegawa's results quite well.

3.2 Rectangular plates with a single-edge crack

Table 2 and Fig. 6 show the relation between the stress intensity factor of rectangular plates with a single-edge crack [Fig. 5(a)] and crack depth a/W .

Figure 7 shows the effect of the side boundaries on the stress intensity factors in the case of Fig. 6. From this figure we find that the effect of the side boundaries almost disappears when $L/W > 1.0$.

Figure 8 shows a comparison between the stress intensity factors in Fig. 7 and those of the same form as Fig. 7 under uniform tension⁽¹⁵⁾. From this figure, we find that the effect of plate length on the stress

Table 3 F_I of a rectangular plate with double-edge cracks
 $(F_I = K_I/\sigma_b\sqrt{\pi a}, \sigma_b = 6M/t^2, \nu = 0.3)$

L/W	0.2	0.4	0.6	0.8	1.0	1.2	2.0	∞	∞ [3]
a/W									
0	1.004	1.004	1.004	1.004	1.004	1.004	1.004	1.004	1.004
0.1	1.148	1.070	1.048	1.038	1.032	1.027	1.017	1.009	1.011
0.2	1.337	1.151	1.097	1.074	1.062	1.054	1.037	1.022	1.021
0.3	1.524	1.245	1.155	1.118	1.099	1.089	1.066	1.046	1.042
0.4	1.696	1.342	1.222	1.171	1.147	1.134	1.108	1.083	-
0.5	1.855	1.439	1.297	1.238	1.211	1.196	1.167	1.138	-
0.6	1.994	1.541	1.384	1.326	1.296	1.283	1.251	1.221	1.197
0.7	2.132	1.658	1.507	1.454	1.427	1.414	1.383	1.351	-
0.8	2.276	1.841	1.718	1.677	1.655	1.644	1.614	1.582	1.552
0.9	2.586	2.309	2.234	2.211	2.195	2.188	2.161	2.133	-

Fig. 9 F_I of a rectangular plate with double-edge cracks

intensity factor is less under transverse bending than under uniform tension in rectangular plates having the same form.

3.3 Rectangular plates with double-edge cracks

Table 3 and Fig. 9 show the relation between the stress intensity factor for transverse bending of a rectangular plate [Fig. 5(b)] and the crack depth a/W . The dashed line in Fig. 9 denotes the stress intensity factor for a pair of semi-infinite cracks in which the distance of crack tips is $2(W-a)$. The stress intensity factor for this case can be obtained by the equations

$$F_I = \frac{2/\pi}{\sqrt{(a/W)(1-a/W)}} \quad (37)$$

$$K_I = F_I \cdot \sigma_b \sqrt{\pi a} \quad (38)$$

$$\sigma_b = 6M/t^2 \quad (39)$$

From Fig. 9 we find that the stress intensity factor for an infinite strip with double-edge cracks agrees well with that for a pair of semi-infinite cracks when

Fig. 10 Effect of side boundaries of a rectangular plate with double-edge cracks

$a/W > 0.8$.

Figure 10 shows the effect of the side boundaries on the stress intensity factor in the case of Fig. 9.

As seen by comparing Fig. 7 with Fig. 10, the effect of the side boundaries on the stress intensity factor in the case of a rectangular plate with a single-edge crack is less than that in the case of a rectangular plate with double-edge cracks. This is caused by the difference of bending moments in the crack direction along the straight boundary having no crack for a rectangular plate with a single-edge crack and the corresponding boundary for a rectangular plate with double-edge cracks. That is, the bending moment for the rectangular plate with a single-edge crack is zero for all values of the plate length L , but the one for the rectangular plate with double-edge cracks depends on the plate length L .

4. Conclusions

The stress intensity factors of rectangular plates with single- or double-edge cracks under transverse bending were evaluated by the body-force doublet method.

The effect of side boundaries on the stress intensity factors, which is the effect of plate length L , is less under transverse bending than under uniform tension in rectangular plates of the same form. In a rectangular plate with a single-edge crack, the effect of side boundaries on the stress intensity factor can hardly be seen for $L/W > 1.0$. On the other hand, in a rectangular plate with double-edge cracks, the effect can be seen even for $L/W > 2.0$.

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