Improvements of $J_2$-Deformation Theory and Their Applications to FEM Analyses of Large Elastic-Plastic Deformation

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Large elastic-plastic deformation in, for example, metal forming processes has been analyzed, so far, primarily on the basis of $J_2$-deformation theory, especially in situation where strain localization, such as shear-band formation in a block or localized necking in a sheet, takes place. In order to improve the $J_2$-deformation theory, three constitutive equaitons, one of which was previously proposed by the author, are introduced. By several FEM analyses of fundamental large elastic-plastic deformatons, such as uniaxial tension and simple shear, it is demonstrated that these three constitutive equaitons are equivalently effective in predicting the strain localization phenomenon even in a quantitative sense.

Key Words: Plasticity, Constitutive Equation, Vertex Hardening, Large Deformation, FEM, Tension, Shear, J 2 D, MG c. e., J 2 G

1. Introduction

Numerical analysis of large plastic deformations in, for example, metal forming processes by means of the finite element method (FEM) has recently been prevalent. In such an analysis, the large deformation version of the classical Prandtl-Reuss' equation called $J_2$-flow theory (J 2 F) is commonly used as the constitutive equation. However, it is now well known that J 2 F is inappropriate for predicting the bifurcation or instability phenomenon, especially severe strain localization such as the shear-band-type bifurcation in a block or the localized necking in a thin sheet, or for precisely predicting the load at the onset of buckling, which is also a bifurcation phenomenon of the compressive type. Therefore, in such cases, the large deformation version of the rate-type of the classical Hencky's total deformation theory ($J_T$-deformation theory; J 2 D) is often used instead of J 2 F. The J 2 D was first introduced by Stören and Rice (1975)** in order to predict the forming limit strain of thin sheets subjected to biaxial loading using the localized necking as the fracture condition. However, the J 2 D gives no definite condition for judging whether plastic loading will continue or not for the subsequent deformation increment at the current plastic state, because its original form (Hencky's total deformation theory) allows no unloading.

To avoid this difficulty of J 2 D, Christoffersen and Hutchinson (1979)** proposed phenomenological plasticity constitutive equations which allow vertex-formation at the loading point on the subsequent loading surface, and which are now called the vertex-hardening plasticity theory, by introducing the plastic potential as the function of the stress increment and not of the stress itself as in the conventional potential theory. The simplest form of this theory is now well known as the $J_T$-corner theory (J 2 C). The J 2 C is sometimes used in the numerical simulation of the shear-band formation (see, e.g., Tvergaard, Needleman and Lo (1981)**, Triantafyllidis, Needleman and Tvergaard (1982)**, and Larsson, Needleman, Tvergaard and Strákers (1982)**). However, even for J 2 C, freedom to choose the factors and functions

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involved remains. No effort has been made to determine the best of these to obtain the quantitative agreement with the corresponding experimental data for, for example, the forming limit strains of the thin sheets.

Independently of Christoffersen and Hutchinson’s theory, the author also proposed a new class of plasticity constitutive equations on the basis of tensor algebra which also can be considered as a kind of vertex-hardening plasticity theory (Gotoh (1981)(10) and (1985)(17)(19)). Its simplest form is now called the MG-constitutive equation (MG c.e.) or J 2G and was used to study the bifurcation phenomenon in the plane strain state (Gotoh (1981)(10) and (1982)(11), to predict the forming limit strains in thin sheets subjected to proportional and non-proportional biaxial loadings (Gotoh (1981)(10) and (1985)(12)(13)), and to simulate numerically the large plastic deformations in the metal forming processes (Gotoh (1988)(14)(15)). The forming limit diagrams of thin sheets predicted by the J 2G are confirmed to agree well with the corresponding experimental data (Gotoh (1984)(16), Gotoh, Satoh and Tanaka (1985)(17) and (1986)(18) and Gotoh, Miura and Tanaka (1987)(19).

In this paper, we propose two kinds of plasticity constitutive equations other than the J 2G, both of which are closely related to the original J 2G and are linear with respect to the stress and strain increment components, unlike the J 2G. We call these three constitutive equations MG, MG-I and MG-II (or J 2G, J 2G-I and J 2G-II) for brevity, respectively. Formally, they are all similar to J 2D and therefore we may say that they represent some improvement over the J 2D. The fundamental large plastic deformations such as uniaxial tension in plane stress, plane strain and axisymmetric states, and simple shear of a plane strain block with embedded edges are numerically analyzed by the finite element method (FEM) to confirm that the three J 2Gs give similar results and are equivalently effective especially for the problem of strain localization. Therefore, J 2G-I and J 2G-II are also useful for the problems which were previously treated on the basis of J 2G. J 2G-I and J 2G-II are more convenient to introduce into the existing computer programs based on J 2 F, because of their linearity. Rate-sensitive plasticity is also considered in a few problems.

[Note: J 2G is an abbreviation of Jf-Gotoh’s corner theory.]

2. Proposed Constitutive Equations

2.1 MG constitutive equation (MG or J2G)

The MG constitutive equation (MG or J2G) is expressed as follows (see, e.g., Gotoh (1985)(19):

\[
\dot{\varepsilon}_r = \dot{\varepsilon}_r(\epsilon, \sigma) = \frac{1}{2} \sigma \dot{\epsilon}_r \quad \text{(elasticity)}
\]

\[
\dot{\varepsilon}_v = \dot{\varepsilon}_v = \frac{1}{2} \dot{\sigma}, \quad \dot{\varepsilon}_p = \frac{1}{2} \dot{\sigma} \quad \text{(plasticity)}
\]

where \( \dot{\varepsilon}_r = \dot{\varepsilon} \) (Euler strain rate), \( \dot{\varepsilon}_v = \dot{\varepsilon}_p \) (volumetric strain rate), \( \dot{\sigma} = \dot{\varepsilon}_v \) (velocity gradient tensor), \( \sigma = \dot{\varepsilon}_p \) (Cauchy stress tensor), \( T = \dot{\varepsilon}_v \) (deviator of \( \sigma \)), \( \dot{T} = \dot{T} - \alpha T \) (Jaumann rate), \( G = \text{elastic rigidity modulus}, \) and the subscripts \( e \) and \( p \) denote ‘elasticity’ and ‘plasticity’, respectively. \( \dot{\varepsilon}_p \) is assumed. (Namely incompressible plasticity is assumed.) \( \dot{\sigma}_n = \dot{\sigma}_n / 3 = \text{hydrostatic stress rate}, \) \( \dot{\sigma} = \text{elastic bulk modulus}. \) \( h_0 \) and \( h_6 \) are instantaneous work-and-vertex-hardening rates, respectively. \( h_0 = (1/3) \times \text{slope of the stress-strain curve obtained by uniaxial tension (say)}, \) \( h_6 = h_0 / h_6 = \cos \frac{\theta_i}{1 + \cos \theta_i} \) (see below). \( h_0 = \sqrt{3/2} (T_{ij} T_{ij})^{1/2} \) is the so-called Mises’ equivalent stress. \( P(\Theta) = \text{function of the angle } \Theta \), which is the angle between \( \dot{T} \) in the five dimensional space of the deviatoric stress tensor. \( \cos \Theta = (T_{ij} T_{ij} \Gamma_{ij})^{1/2} \times (\dot{T}_{ij} \dot{T}_{ij} \Gamma_{ij})^{1/2} \).

Furthermore,

\[
\dot{\sigma} = \sqrt{3/2} (\dot{T}_{ij} T_{ij} \Gamma_{ij})^{1/2}, \quad \text{(Note } \dot{\sigma} = \dot{\sigma} \text{ in general)}
\]

\[
\dot{\sigma} = (\dot{\sigma}) = [\sqrt{3/2} (T_{ij} T_{ij})^{1/2}] = \cos (\Theta) \cdot \dot{\Theta}
\]

The function \( P(\Theta) \) is defined as follows:

\[
P(\Theta) = a + b \cdot \cos \Theta
\]

This function specifies the condition of whether plastic deformation will continue or unloading will take place for the subsequent deformation increment during plastic deformation undergoes. In other words, it guarantees a smooth transition from plastic loading state to elastic unloading state. That is,

\[
P(\Theta) = a + b \cdot \cos \Theta > 0
\]

is the condition that the plastic loading state continues. This condition can be written in the following inequality:

\[
0 \leq \Theta \leq \Theta_{\text{max}} : \Theta_{\text{max}} = \cos^{-1} (-a/b).
\]

We are concerned with the work-hardening material here and thus \( (a/b) > 0 \). Therefore, in general, \( \pi/2 \leq \Theta_{\text{max}} < \pi \), in which the equality holds for \( a = 0 \) and thus \( H_{\infty} \rightarrow \infty \), which leads Eq. (1) to the conventional J 2F. Except for in this special case, the plastic loading state will continue even for the subsequent stress increment deviating from the current stress state by more than 90° but less than \( \Theta_{\text{max}} \) given in Eq. (4). The subsequent loading surface has a pointwise vertex at the current loading point or stress state.

The angle \( \Theta_{\text{max}} \), in the expression \( a = h_0 / h_6 = \cos \theta_i / (1 + \cos \theta_i) \) is the half-angle of this vertexed cone.
and thus is given simply by the following equation:

$$\Theta_0 = \pi - \Theta_{\text{max}}. \quad (5)$$

Before yielding, we can assume the yield surface to be smooth everywhere. Therefore, it is natural to think that the vertex will eventually have plastic deformation. We can successfully assume the following expression (see, e.g., Gottoh (1984)\cite{16}, Gottoh et al. (1985)\cite{17}, (1986)\cite{18} and Gottoh et al. (1987)\cite{19}):

$$\Theta_0 = (\pi/2) - \rho \hat{\varepsilon}^u : \varepsilon = \int \hat{\varepsilon}^p dt, \quad (6)$$

where \(\hat{\varepsilon}^p = \sqrt{2/3} (\varepsilon^p \varepsilon^p)^{1/2}\) is equivalent plastic strain rate, and \(\varepsilon = \text{total length of plastic strain trajectory.}\)

\(\rho\) is a newly introduced material constant and just one additional constant in the J2G which governs the evolution rate of the vertex. If we set \(\rho\) identically equal to 0, then \(a = 0, H_0 = \infty\) and thus J2G reduces to J2F. Also, for proportional loading with \(\rho\) identically equal to 0, J2G reduces formally to J2D. (If we set \(H_0 = (1/3)(3/2)\), then J2G coincides totally with J2D, though this choice of \(H_0\) is a special case for J2G.)

In the above and in the following, the term ‘rate’ should be understood as ‘increment’ for rate-independent plasticity.

2.2 J2F flow theory (J2F)

This is the well-known constitutive equation which is expressed as follows:

$$\dot{\varepsilon} = \hat{T} \dot{\varepsilon}, \quad \dot{\varepsilon} = \dot{\varepsilon}^p = (\dot{\varepsilon}^p | \varepsilon^p)^{1/2}, \quad (7)$$

(7)

where from Eq. (2) we find

$$\dot{\lambda} = \dot{\varepsilon} = \dot{\varepsilon}^p = (\varepsilon^p | \varepsilon^p)^{1/2}, \quad \dot{\varepsilon}^p = (\dot{\varepsilon}^p | (\dot{\varepsilon}^p)^{1/2}). \quad (8)$$

which was used in the derivation of Eq. (7). From comparison of Eq. (1) with Eq. (7), it is evident that J2G reduces to J2F when \(\rho\) is identically equal to 0, because \(a = 0, b = 1, H_0 = \infty\) and \(P(\Theta) = \cos \Theta\) hold for \(\rho = 0\).

2.3 J2G-I constitutive equation

As expressed in Eq. (1), J2G (or MG) itself appears sufficiently simple to use in the numerical analysis of deformation. Actually, it has been recently found that J2G can be rather easily integrated into the existing computer program for FEM analysis of large elastic-plastic deformations on the basis of J2F (Gottoh (1989)\cite{19}).

Combining the elastic and plastic parts in Eq. (1) and creating its inverse form, we have the following expression of the deviatoric part:

$$\dot{T} = (1/G^*) \varepsilon - (\hat{P} | \Theta) 3b/2h_0 \dot{\varepsilon}^p \dot{\varepsilon}, \quad (9)$$

$$1/G^* = (1/G^* + 2(\hat{P} | b_0 H_0)) \cos \Theta + (\hat{P} | b_0 H_0)^{1/2}, \quad (10)$$

$$\dot{\varepsilon}^p = \sqrt{2/3} (\dot{\varepsilon} | \dot{\varepsilon})^{1/2}, \quad (11)$$

where \(\dot{\varepsilon}^p = \dot{\varepsilon}^p + \dot{\varepsilon}^p = \text{total deviatoric strain rate. When Eq. (9) is introduced into the formulation of FEM, the term of \(\dot{\varepsilon}^p\) yields the non-linear stiffness equation with respect to the nodal displacement increments. This kind of stiffness equation can be solved by an iteration method for each calculation step (see, e.g., Peirce, Shih and Needleman (1985)\cite{20}).

However, such iteration method necessitates more computation time. Moreover, it requires us to change the computation algorithm if we want to use the existing computer program developed on the basis of J2F.

If we use a very small step (like in the r-min method, see below) for each increment of deformation, then we can use such an approximation that the value of \(\cos \Theta\) at the current step is set equal to that at one step earlier. This approximation makes the formulation easy and simple to a great extent and allows us to use directly the existing program with a few modifications (see (20)).

Here we deduce two variations of J2G. These have simpler forms than that of J2G itself and thus are easier to handle.

Comparing the expressions of \(\dot{\varepsilon}^p\) in Eqs. (1) and (7), we find that the effect of the vertex on the subsequent loading surface appears primarily in the terms of \((\hat{P} | H_0) \dot{T}\). Therefore, we can deduce the following constitutive equation as one of the variations of J2G which can be thought to be linear with respect to the components of \(\dot{T}\) and \(\dot{\varepsilon}\), if we use \(\cos \Theta\) at the preceding step as the current \(\cos \Theta\) with a very small increment per step in the usual incremental method of FEM.

$$\dot{\varepsilon} = \hat{T} \dot{\varepsilon}, \quad \dot{\varepsilon} = \dot{\varepsilon}^p = (\dot{\varepsilon}^p | \dot{\varepsilon}^p)^{1/2}, \quad (12)$$

We call this constitutive equation simply J2G-I (or MG-I). For this J2G-I, the relations (2)~ (6) are also available, as for J2G. That is, the vertex angle \(\Theta\) and its evolution rate, and the condition for unloading during plastic deformation are all the same as those for J2G. The numerical examples given in the previous papers were performed mainly by the use of this J2G-I (Gottoh (1988)\cite{19}, Gottoh and Sawa (1989)\cite{20}).

Equation (12) is rewritten in the following form:

$$\dot{\varepsilon}^p = \hat{P} \dot{\varepsilon}^p, \quad \dot{\varepsilon} = \dot{\varepsilon}^p = (\dot{\varepsilon}^p | \dot{\varepsilon}^p)^{1/2}, \quad (13)$$

which apparently shows how to change the computer program developed for J2F. All that is to be done is to replace \(G\) by \(G^*\) keeping \(K\) unchanged, which is equivalent to the replacement of \(E\) by \(E^*\) and \(\nu\) by \(\nu^*\), where \(E = \text{Young’s modulus}, \nu = \text{Poisson’s ratio}, \) and \(E^* = E \cdot [1/(1-2\nu)+(E/G^*)/(1-2\nu)], \nu^* = (E/G^* - 2(1-2\nu)+[2(1-2\nu)+E(G^*)])]\).

$$\nu^* = (E/G^* - 2(1-2\nu)+[2(1-2\nu)+E(G^*)]) \quad (14)$$
which are derived from the relations between \((G,K)\) and \((E,\nu)\) with the replacement of \(G\) by \(G^*\) in Eq.(10) keeping \(K\) unchanged, as required above.

2.4 \(J_2\)-deformation theory (J2D)

The well-known \(J_2\) D is expressed as follows:
\[
\begin{align*}
\dot{\epsilon}^* &= T/2G, \quad \dot{\epsilon}_s = \dot{\sigma}_n/K, \quad \text{(elasticity)} \\
(\text{rate-type Hencky's equation})
\end{align*}
\]
where \(H^* = (1/3)(\dot{\epsilon}_s/\dot{\sigma}_n)\) = (secant modulus of stress-plastic strain curve) = (instantaneous vertex-hardening rate for \(J_2\) D). \(H^*\) apparently corresponds to \(H_0\) in \(J_2\) G, though it is essentially different from \(H_0\). In addition,
\[
b^* = 1 - a^*, \quad a^* = h_0/H^*, \tag{16}
\]
which apparently correspond to \(a\) and \(b\) in \(J_2 G\) (see Eq.(3)).

Basically, unloading during plastic deformation is rejected for \(J_2\) D, because its original equation (Hencky's total deformation theory) rejects it. However, for convenience, unloading is often considered to occur when \(\langle \lambda \rangle = 0\) or equivalently, \(\cos \theta \leq 0\), \((\pi/2) \leq \theta \leq \pi\) holds, though \(\dot{\epsilon}^* = (1/H^*) \hat{T}\) never becomes 0. Therefore, in this case, the transition from the plastic loading state to the elastic unloading state is assumed to occur discontinuously. Or, in this case, the angle \(\theta_{\text{max}}\) in Eq.(4) is assumed to be equal to \(\pi/2\), and thus no vertex exists in the subsequent loading surface. When we avoid these contradictory circumstances, no unloading is assumed to occur using \(\hat{\lambda}\) itself instead of \(\langle \lambda \rangle\) in Eq.(15), which means that the material is of nonlinear elasticity apart from plasticity.

Recently, in the plastic constitutive equation of the type of Eq.(15), it is sometimes assumed that the meaning of \(H^*\) is not restricted to that mentioned above, and that the term of \(H^*\) expresses the so-called “non-normality” of the plastic response (see, e.g., Chung, Nemat-Nasser and Taylor (1988)\(^{20}\)). In this case, no vertex is assumed, and thus one of the contradictions described above is avoided. However, how to reasonably determine the quantity corresponding to \(H^*\) becomes a new problem, and another difficulty of the discontinuous transition from plastic loading to elastic unloading still remains unsolved. This kind of model is effective as long as we are concerned with the study on the qualitative effect of the term of \(\hat{T}\) on the shear-band formation. From the more practical point of view or in an industrial sense, we have to give more quantitative value to any model, in which sense the “non-normality” model mentioned here seems lose its effectiveness.

Now back to \(J_2\) D. Let us make \(\dot{\epsilon}_s \hat{\epsilon}_s\) in Eq.(15).

Then we have
\[
\dot{\epsilon}_s \hat{\epsilon}_s = (b^*/h_0)^3 (\langle \cos \Theta \rangle)^2 + (1/H^*)^2.
\]

In general, \(0 < h_0 < H^*\) holds and thus \(\dot{\epsilon}_s \hat{\epsilon}_s > 0\) is satisfied for an arbitrary value of \(\Theta\). Therefore, there exists no direction of stress increment such that \(\dot{\epsilon}^* = \rho\) holds at any instant of plastic deformation, which means that unloading never takes place for \(J_2\) D, at least in a logical sense. The subsequent loading surface itself ceases to exist.

2.5 \(J_2 G^{II}\) constitutive equation

From the comparison of Eqs.(1), (12) and (15), we can readily deduce the following other variation of the original \(J_2 G\), which differs from \(J_2 G^{I}\) only by the factor \(b\) in the second term of \(\dot{\epsilon}^*\), and is derived from \(J_2\) D by the replacement of \((1/H^*)\) by \((\langle P(\Theta)\rangle/h_0)\) and \(b^*\) by \(b^*\) in the expression of \(\dot{\epsilon}^*\):
\[
\begin{align*}
\dot{\epsilon}^* &= T/2G, \quad \dot{\epsilon}_s = \dot{\sigma}_n/K, \quad \text{(elasticity)} \\
\dot{\epsilon}_p &= (\langle P(\Theta)\rangle/h_0^*) \hat{T} + b^* \langle \lambda \rangle \hat{T} \tag{17}
\end{align*}
\]
This is rewritten in the following form:
\[
\begin{align*}
\dot{\epsilon}^{* * } &= T/2G^*, \quad \dot{\epsilon}_s = \dot{\sigma}_n/K, \\
\dot{\epsilon}_p &= b^* \langle \lambda \rangle \hat{T} + b^* (\langle \cos \Theta \rangle/2h_0) \hat{T} \tag{18}
\end{align*}
\]
which correspond to the expression (13) for \(J_2 G^{I}\). \(G^*\) is defined in Eq.(10). The asterisk marks of \(\dot{\epsilon}^{* * }\) and \(\dot{\epsilon}_p^{* * }\) in Eqs.(13) and (18) denote ‘fictitious elastic and plastic parts’.

From Eq.(18), we can easily find that the existing computer program developed for \(J_2 F\) is readily converted to that for \(J_2 G^{II}\) merely by the replacement of \(E\) by \(E^*\) and \(\nu\) by \(\nu^*\), just as for \(J_2 G^{I}\), together with the additional change of the term \(\hat{\lambda}\) being multiplied by the factor \(b\). Modification of the computer program for \(J_2 G^{I}\) to that for \(J_2 G^{II}\) is quite simple.

Almost all of the statements made about \(J_2 G^{I}\) in section 2.3, especially on the vertex angle \(\theta_0\) and its evolution rate and the loading-unloading condition, are also true for \(J_2 G^{II}\). Moreover, for both \(J_2 G^{I}\) and \(J_2 G^{II}\), it is true, as for \(J_2 G\), that the constant \(\rho\) leads to \(J_2 F\) when it is set identically equal to 0. \(J_2 G^{II}\), like \(J_2 G\) itself, takes the same form as that of \(J_2 D\) when the angle \(\Theta\) is identically equal to 0, i.e., for a proportional loading, though the difference in the meanings of the ‘instantaneous vertex-hardening rate’ should be noted again. The method of generalizing \(J_2 G\) to the case where the initial anisotropy and the subsequent anisotropy are taken into consideration, which was described in the previous papers (see Gotoh (1981)\(^{40}\)(1985)\(^{41}\)), is also available for \(J_2 G^{I}\) and \(J_2 G^{II}\), because the primary parts of the characteristics of the original \(J_2 G\) are maintained in both of them, especially in \(J_2 G^{II}\).

2.6 A simple comparison of \(J_2 G\), \(J_2 G^{I}\) and \(J_2 G^{II}\)

When we compare \(J_2 G\), \(J_2 G^{I}\) and \(J_2 G^{II}\) given by Eqs.(1), (12) and (17), respectively, we find...
that the first terms of $\dot{e}^p$, which give the primary effects of the vertex and involve $T$, are common, and that the factors multiplied by $T$ in the second terms of $\dot{e}^p$, such as $b\langle P(\Theta)\rangle$, $\langle \cos \Theta \rangle$, and $b\langle \cos \Theta \rangle$, respectively, are different because the function $P(\Theta)$ is given by Eq. (3) and $\dot{\lambda}$ is given by Eq. (8). Then, let us compare these three quantities numerically as a merely partial comparison of the three constitutive equations, reminding ourselves that the main terms of the vertex effect are common to them all.

When the material constant $\rho$ is chosen as 0.25, we obtain Table 1 for a rather large value of plastic equivalent strain $\varepsilon = 0.5$, and Table 2 for a large one $\varepsilon = 1.2$. [Note that the value of the constant $\rho$ is usually determined by an experiment. For example, for a thin sheet, it was determined from the forming limit strain under equi-biaxial tension (see, e.g., Gotoh (1981)\(^{(11)}\) (1984)\(^{(16)}\) and (1987)\(^{(17)}\). Of course, there are other methods which would be more appropriate under other circumstances.]

From these two tables, we find that the three quantities give substantially different values for a large strain greater than 1, other than that, they are very similar. As for the three constitutive equations, the numeral comparisons here are just partial as emphasized above, and moreover, plastic deformation begins with zero strain ($\varepsilon = 0$) together with no vertex ($\Theta = \pi/2$) at which they are all coincident with $J_2$ F and thus with each other. Therefore, we can say that $J_2$ G, $J_2$ G-I and $J_2$ G-II are very close to each other even in a quantitative sense, throughout the whole deformation history, at least in a range of strain less than 1. This point is confirmed by some numerical examples of large plastic deformation given in the following section.

3. Examples of FEM Analysis of Large Plastic Deformations

In this section, we present several numerical examples of large plastic deformations by the incremental elastic-plastic FEM primarily on the basis of the constitutive equations $J_2$ G-I and $J_2$ G-II newly proposed in this paper. Where necessary, the results are compared with those based on $J_2$ F and $J_2$ G.

The base computer program for the FEM analysis used here is the “GOLDA” which was previously developed by one of the authors (Gotoh (1988)\(^{(14)}\)\(^{(15)}\)). For two examples of its applications, see Gotoh, Amaki and Tanaka (1988)\(^{(16)}\) and Gotoh (1988)\(^{(17)}\). The basic formulation for FEM analysis of large plastic deformation used in it is also found in the previous papers (Gotoh (1988)\(^{(14)}\)\(^{(15)}\)\(^{(25)}\)). The so-called ‘crossed triangles’ linear finite elements are used (that is, the material is divided into many rectangular regions first, and then each of them is subdivided into four triangles). The increment of deformation per each step of calculation is determined by the so-called $r$-min method (Yamada, Yoshimura and Sakurai (1968)\(^{(36)}\), which confines the number of the newly yielded elements per step to one for comparatively early stages of deformation, and by the method for limiting the maximum plastic strain increment $\Delta \varepsilon$ within the prescribed value (= 0.002 here).

The electronic computer used here is the FACOM M-780/20 installed in the Nagoya University Computation Center.

The work-hardening property of the material is expressed by the following power law:

$$\dot{\varepsilon} = \varepsilon^m, \quad \text{(for elastoplasticity)}, \quad \dot{\varepsilon} = \varepsilon^m, \quad \text{(for elastoplasticity)}$$

The initial yield stress is denoted by $\sigma_0$. When the equivalent stress $\sigma$ is less than $\sigma_0$, the element is elastic. The initial yield strain $\varepsilon_0$ is related to $\sigma_0$ by the equation $\sigma_0 = E\varepsilon_0 = \varepsilon_0$. For the elastic elements, including the elements in the unloading state, the relation $\sigma = E\varepsilon$ is applied.

The values of the material constants adopted here are listed below.

- $E = 200$ GPa, $\nu = 1/3$
- $\sigma_0 = E/500$, $\varepsilon_0 = 0.002$
- $n = 0.0625$ or 0.15 (work-hardening exponent)
- $\rho = 0$ (for $J_2$ F), 0.1 or 0.2 (for $J_2$ G, $J_2$ G-I and $J_2$ G-II)
- $m = 0.2$ (strain-rate sensitivity exponent)
- $V_0 = 1.0$ or 100 mm/sec (= half of tensile speed)

Of course, $\sigma_0$ and $V_0$ assume the values above only when the rate-sensitivity of the material is considered.

The problems numerically solved here are the

<table>
<thead>
<tr>
<th>Table 1</th>
<th>A comparison of $\langle \cos \Theta \rangle$, $b\langle \cos \Theta \rangle$ and $b\langle P(\Theta)\rangle$</th>
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<tr>
<td>$\Theta$</td>
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<tr>
<td>$\langle \cos \Theta \rangle$</td>
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<tr>
<td>$b\langle \cos \Theta \rangle$</td>
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<td>$b\langle P(\Theta)\rangle$</td>
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<th>Table 2</th>
<th>A comparison of $\langle \cos \Theta \rangle$, $b\langle \cos \Theta \rangle$ and $b\langle P(\Theta)\rangle$</th>
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<td>0</td>
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<tr>
<td>$\langle \cos \Theta \rangle$</td>
<td>1.000</td>
</tr>
<tr>
<td>$b\langle \cos \Theta \rangle$</td>
<td>0.750</td>
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<tr>
<td>$b\langle P(\Theta)\rangle$</td>
<td>0.750</td>
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uniaxial tension in plane stress, plane strain and axisymmetric states, and the simple shear of a plane strain block, in both of which the edges are assumed to be embedded in the tools.

3.1 Uniaxial tension with embedded edges

The parallel specimens with the aspect-ratio of 3 (= length/breadth in the initial shapes) are subjected to uniaxial tension. Their edges are embedded in the tool, and thus they never shrink laterally at the edges.

In the following, the figures of the deformed shapes of the specimens are illustrated only for one quadrant because of symmetry. The mark - is drawn at the center of the element if it is plastic; this is also done for the simple shear problem in the next section. Therefore, the plastic regions appear dark or gray in the illustrations.

Figure 1 (a) illustrates various stages of deformation in the case of the plane stress state in which J2G-II (MG-II) is used. In this figure, u/L denotes elongation/initial length which represents the stage of deformation. As seen in the figure, all elements are in the plastic state at u/L=0.05 close to the maximum load point. This is true not for the plane strain case in which the edge-central portions remain elastic throughout the whole deformation history. Here n=0.0625 is adopted and at around u/L=0.1, the shear-band-type strain localization, which can be thought of as the so-called localized necking band, clearly takes place. As long as we use 0.1 for the value of ρ, which seems rather low, the deformation patterns are similar for all constitutive equations considered here.

Figure 1 (b) illustrates an example for the case of n=0.15 at u/L=0.1. In this case, no localized necking is seen because of the larger value of n. Such necking appeared at around u/L=0.2. The localized necking simulated here corresponds to that reported in the previous paper (Gotoh, Amaki and Tanaka (1987).)

Figure 2 illustrates the relationship between the tensile load and u/L, and that between the necking-down at the center and u/L for the plane stress state. In this figure, F/(A0×SIGY) in the ordinates denotes “tensile load/(σ0×initial cross-sectional area)” and V/W denotes “necking-down at the center/initial breadth”. In the abscissa, u/L stands for u/L in Fig. 1. From this figure, we find that, as long as we use ρ =0.1 and n=0.0625, the tensile load is almost the same for all J2G and J2F, whereas the development of necking is almost the same for all J2G with a substantially higher value than that due to J2F. The steep drop of the tensile load after its peak point is of course due to the development of the localized necking.

Figure 3 illustrates the deformation stages corresponding to Fig. 1 in the case of the plane strain, (a), and the axisymmetrical state, (b), where the value of ρ is taken as 0.2. From the comparison of (a) with Fig.1 (a), we find that the shear-band-type strain localization occurs at lesser development of the diffuse-type necking in the plane stress state than in the plane strain state. In Fig.3 (a), we can see the clear development of the so-called shear-band, which looks far narrower than the corresponding band in the plane stress in Fig.1 (a) and appears at a far later stage of deformation than the latter does. To catch this kind of narrow shear band, we have to choose

![Fig. 1 Various deformation stages under uniaxial tension in the plane stress state (MG=J2G)](image_url)
carefully an appropriate element subdivision pattern.  

Figure 3 (b) shows that the development of the diffuse-type necking itself is very slow for the case of the axisymmetric state, and that even at \( u/L = 0.2 \), no shear-band-type strain localization takes place even for the J2G-type constitutive equation. It is not clear at present whether the shear band will evolve at a later stage of deformation or never evolve at all. 

Figure 4 illustrates a comparison of the deformed shapes around the necked regions at \( u/L = 0.2 \) in the plane strain state for J2F, J2G-I with \( \rho = 0.1 \) and J2G-II with \( \rho = 0.2 \). As clearly seen in this figure, J2F yields no shear band, whereas J2G-I and J2G-II do, though a different \( \rho \) of somewhat lower value is used for each J2G (MG). This example shows simply that J2G-I and J2G-II can be used for such strain localization problems equivalently with J2G, which has often been used to predict, for example, the forming limit strains of thin sheets. 

The material of rate-sensitive plasticity is also analyzed in the plane stress state. In such a case, usually no unloading is assumed to occur during plastic (or more exactly, viscoplastic) deformation. Here we assume that unloading may take place when the condition (4) is violated, which is one of the new aspects of our analysis to be noted. By making this assumption, we can simulate such sharp localized necking as that in Fig. 1(a) of a rate-insensitive plastic material along which severe viscoplastic flow concentrates. Figure 5, in which various stages of deformation are illustrated using J2G-II (MG-II) with \( \rho = 0.1 \), and \( n = 0.0625 \), \( m = 0.2 \), verifies this point. Except for the \( m \)-value, the calculation condition is the same as that of Fig. 1(a). The half of the tensile speed \( V_t \) is set at 1 mm/sec, though \( V_t \) has little effect on the deformation pattern. Comparing Fig. 5 with Fig. 1(a), we find that strain localization is delayed by the rate sensitivity \( (m=0) \), as has often been noted in the literature, and that the shear-band-type localized necking takes place even in the visco-

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**Fig. 2** Variations of tensile load and necking-down at the center with deformation under uniaxial tension in the plane stress state (MG=J2G) 

**Fig. 3** Various deformation stages under uniaxial tension in the plane strain state, and an example in the axisymmetrical state (MG=J2G)
plastic material. Figure 5 verifies the common statement that the strain-rate sensitivity stabilizes inelastic flow and thus increases the forming limit strain. Figure 6 illustrates the variations of the tensile force and the necking-down at the center with deformation ($u/L$) for two tensile speeds, i.e., 1 and 100 mm/sec. This figure simply shows that higher tensile speed gives higher tensile load, as was already well known, and that the tensile speed influences the deformation pattern very little, as mentioned above.

3.2 Shear of a plane strain block with embedded edges

A plane strain block is embedded in the tool at its edges. The upper edge is allowed to slide horizontally, while the lower edge is fixed. The initial aspect ratio of this block (= initial height/initial breadth) is equal to 2.

Figure 7 illustrates an example of the deformation process, in which $U_d/W$ denotes "slide distance of the upper edge/initial breadth" representing each
stage of deformation, and J2G-II is used with $\rho$ of 0.1 and $n$ of 0.0625. The plastic region first appears around both edges, and then spreads longitudinally almost along the center line. Subsequently, almost the whole region becomes plastic. At about $U_r/W=1.7$, the unloading region begins to appear around the center of the body. Up to this point, the deformation pattern is almost indifferent to the constitutive equation used in the calculation. However, after this point, it becomes prominently dependent on the constitutive equation. At about $U_r/W=1.7$, the so-called shear band occurs along both edges at least for J2G-type constitutive equations, though the value of $U_r/W$ at which shear bands appear is dependent on the value of $\rho$ for J2G and, as should be noted, on the pattern of the initial element subdivision. As seen in Fig. 7, the elements around the edges are very elongated and distorted at later stages of deformation. A remeshing process is needed for a finer analysis.

Figure 8 illustrates a comparison of deformation patterns at $U_r/W=2.0$ for J2G-II and J2F. In this figure, the dashed line is for J2F. The severe deformation along the upper edge predominates more for J2G-II than for J2F, and thus the lower part of the body is apt to be delayed in catching up with the upper edge portion in the case of J2G-II. This tendency is more remarkable for J2G-I, showing that J2G-I induces slightly severer shear flow along the upper edge than does J2G-II. The result by J2G is similar to that by J2G-II.

Figure 9 illustrates the variations of the resultant forces along the edges with deformation for J2F, J2G-I and J2G-II. The tensile resultant force takes its peak at about $U_r/W=1.3$ and then decreases with deformation rapidly in the order of J2G-I, J2G-II and J2F. The resultant shearing force increases monotonically with deformation for J2F, whereas it drops at the latest stages of deformation for J2G-I and J2G-II more rapidly in this order. These observations all reflect the tendency of shear flow concentration along the edges depending on the constitutive equation used in the calibration.

4. Conclusion

In order to facilitate introduction of the J2G (MG) constitutive equation proposed previously by the author to the existing computer program for the finite-element analysis of large elastic-plastic deformation on the basis of J2F, two simplified variations J2G-I (MG-I) and J2G-II (MG-II) of J2G are proposed. These three constitutive equations are formally similar to the J2-deformation theory (J2D) introduced first by Stören and Rice. In FEM formulation, J2G-I and J2G-II give a linear stiffness equation with respect to the nodal displacement increments, whereas the original J2G gives a nonlinear

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Fig. 7 Various deformation stages under shearing in the plane strain state (MG=J2G)
one. J2D involves some faults concerning the loading–unloading condition during plastic deformation, while all J2Gs avoid them, like the J5-corner theory of Christoffersen and Hutchinson. They include just one new material constant to be determined experimentally.

Fundamental large plastic deformations such as uniaxial tension in plane stress, plane strain and axisymmetric states, and shearing of a plane strain block, are numerically analyzed by the incremental elastic-plastic FEM by introducing all J2Gs and J2F by means of the computer program developed previously by Gotoh. In the analyses of uniaxial tension, the shear-band-type localized necking is clearly reproduced for plane stress state, even for the material of strain-rate-sensitive plasticity; the sharp shear band is reproduced for J2Gs but not for J2F for the plane strain state; the deformation patterns obtained by all J2Gs are similar to each other but not to that obtained by J2F for all states; and no localized shear band appears for the axisymmetrical state for all constitutive equations used here. In the analyses of shearing, the shear band or severe shear flow along the edges takes place at least for all J2Gs, showing the drop in the resultant shear force at the later stage of deformation which seems to be due to this concentrated shear flow. The deformed shapes of the material obtained by J2Gs are also different from that by J2F. It is finally concluded that J2G-I and J2G-II, especially J2G-II, can replace J2G in the problems of strain localization such as the problem of the forming limit strain in thin sheets and shear-band formation in a block, or of bifurcation in a compressive field, i.e., buckling.

Fig. 8 A comparison of deformed shapes under shearing in the plane strain state (MG=J2 G)

Fig. 9 A comparison of variations of resultant forces with deformation under shearing in the plane strain state (MG=J2 G)

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