Optimum Shape Design of Structures under Uncertain Loading by Boundary Element Method*

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A structure optimized for a given determinate load is not optimum for other loads. This paper proposes a shape determination method by BEM for structures which are subjected to uncertain loads. In the case that the loads are represented by some random variables which obey certain probabilistic laws, response quantities such as displacement and strain energy can be expressed as random variables by the use of a method of "decomposition into fundamental loads". Then only several runs of analysis can give optimum shapes where the expected values of strain energy densities are uniform on every boundary element.

Key Words: Optimum Design, Probabilistic Method, Boundary Element Method, Shape Determination, Uncertain Loading, Expected Value, Strain Energy Density, Fundamental Loads, Pareto-Optima

1. Introduction

The optimum shape of a structure differs for each loading and supporting condition. The structure which is optimized for a given loading condition is not optimum for other loads, and according to the circumstances, may only be dangerous due to its specialized form for the anticipated loads. The authors consider that loads acting on the structure are not determinate but vary probabilistically and have proposed shape optimization methods in which a shape that makes the expected value of the total potential energy stored in the structure by the uncertain load^{(1),(2)} stationary is sought.

In the structural analysis problem, it is pointed out that the boundary element method is more advantageous in making and inputting data, compared with the finite-element method. In the shape optimization problem⁽³⁾, by the finite-element method, besides there being the troublesome labor of inputting data, it is necessary to refine and regenerate the mesh in order to ensure the accuracy of the analysis in the course of changing the boundary of the object body. The applications of the boundary element method are observed also in the shape optimization problem because BEM needs only partition of the boundary, and the

refinement of the element division can be done with comparative ease. The authors also proposed a method⁽⁴⁾ for the formulation of the shape determination problem based on the inverse variational principle⁽⁵⁾ by the boundary element method.

This paper proposes a method of shape optimization for structures under uncertain loading conditions by the boundary element method. This method is an extended one from the previous paper⁽⁴⁾, which was for determinate loading conditions, and can treat the loading conditions that vary probabilistically. For the formulation of the method, we show a method to represent the displacements and strain energy as random variables in the case that the variation of the loads are expressed in terms of continuous random variables.

2. Boundary Element Approach to Optimum Shape Design Based on the Inverse Variational Principle⁽⁴⁾

2.1 Inverse variational principle⁽⁵⁾

With the inverse variational principle, such a shape and a displacement of a body are considered that makes the total potential energy of the body,

$$F = \int_V \{ (1/2) C_{ijkl} e_{ij} e_{kl} - K_i u_i \} dV - \int_{S_p} P_i u_i dS, \quad (1)$$

stationary under the condition of volume constancy

$$\int_V dV = C, \quad (2)$$

where C_{ijkl} is the tensor of the elastic coefficient, e_{ij} is

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the strain tensor, K_i , P_i and u_i are the vectors of volume force, surface traction and displacement, respectively, and C is the constant for the volume constraint. By introducing a Lagrange multiplier λ , the principle is written as

$$L = F + \lambda \left(\int_V dV - C \right) \quad \dots \rightarrow \text{stationary (w. r. t. } u_i, \lambda \text{ and shape).} \quad (3)$$

The stationary conditions of Eq. (3) are derived by taking variations with respect to every variable, and they can be expressed by the boundary form⁽⁴⁾ as

- (a) $\delta u_i \rightarrow$ equilibrium condition $\Rightarrow$ boundary integral equation
- (b) $\delta \lambda \rightarrow$ volume constancy $\Rightarrow$ zero volume change, ($\Delta V = 0$)
- (c) δ shape $\rightarrow$ condition for shape determination $\Rightarrow$ uniform distribution of strain energy density D on Surface of the body, that is,

$$D = (1/2) C_{ijkl} e_{ij} e_{kl} = -\lambda \text{ (constant) on surface } S. \quad (4)$$

2.2 Method for shape determination

For the boundary element model, the optimality condition of the shape, Eq. (4), means that the strain energy density D_i of each boundary element i is equal to each other; that is,

$$D_i = -\lambda \text{ (constant)} \quad (i=1, \dots, n : n \text{ is number of elements}). \quad (5)$$

Therefore, with the boundary element approach, firstly, the strain energy density D_i of the i -th element is calculated by the boundary element analysis, and the average value of n D_i 's is defined as $\bar{D}$; that is,

$$\bar{D} = \sum_{i=1}^n D_i / n. \quad (6)$$

Then, each element is moved outward in the normal direction by

$$\delta \theta_i = \beta \ln(D_i / \bar{D}). \quad (7)$$

In the actual procedure, extreme points of boundary elements are adopted as design points. The amount of movement of the point is computed by combining two vectors which are derived from the movements of adjacent elements. In Eq. (7), β is a positive constant and is given suitably according to the representative size of the objective structure.

Next, in order to counteract the volume change ΔV due to the above shape reformation, extreme points are moved proportionally to the distance from a reference point by the amount which is derived from the linear approximation using the Taylor expansion. These two-step shape reformations which use only information on the boundary are repeated until the following criterion is satisfied:

$$\epsilon = \max_i |(D_i - \bar{D}) / \bar{D}| \leq \epsilon_0. \quad (8)$$

The obtained shape fulfils the optimality criteria, Eq.

(5). This method for shape reformation is called the energy-ratio-method.

3. Shape Determination Method Considering Probabilistic Variation of Load

3.1 Problem making expected value of potential energy stationary

If a load acting on a structure can be assumed to vary probabilistically, its effect must be taken into consideration in the formulation of the problem of optimum shape determination. When the load is a random variable, the responses of the structure to the load also become random variables. In place of the problem of section 2, this section considers a problem that makes the expected value of the potential energy stationary under the subsidiary condition of the volume constancy. Let the random variable which represents the variation of the load be denoted by X and the calculation of the expectation value be expressed by the operation $E[]$. Then, the problem is written as

$$\Omega = E[F(X)] + \lambda \left(\int_V dV - C \right) \quad \dots \rightarrow \text{stationary (w. r. t. } u_i(x), \lambda \text{ and shape),} \quad (9)$$

where x is an arbitrary value of X . The stationary conditions of Eq. (9) are

- (a) the equilibrium condition at $X = x$
- (b) equation of volume constancy
- (c) uniform expected value of the strain energy density on the surface of the body; that is,

$$E[D(X)] = -\lambda \text{ (constant) on } S. \quad (10)$$

Corresponding to (a), (b) and (c) in section 2, the above three conditions also can be treated only by the quantities on the boundary.

3.2 In the case of a discrete random variable

Let us consider m loading conditions $(P_i^a, K_i^a)(\alpha = 1, \dots, m)$ which are not active at the same time and whose probabilities of occurrence are each r^a , provided that

$$\sum_{a=1}^m r^a = 1. \quad (11)$$

When the total potential energy for the α -th loading condition is defined as

$$F^a = \int_V \{ (1/2) C_{ijkl} e_{ij}^a e_{kl}^a - K_i^a u_i^a \} dV - \int_{S_p} P_i^a u_i^a dS, \quad (12)$$

the expected value of the potential energy is obtained by

$$F^* = E[F] = \sum_{a=1}^m r^a F^a. \quad (13)$$

Therefore, the problem, Eq. (9), is written as

$$\Omega = \sum_{a=1}^m r^a F^a + \lambda \left(\int_V dV - C \right) \quad \dots \rightarrow \text{stationary (w. r. t. } u_i^a, \lambda \text{ and shape),} \quad (14)$$

where u_i^α is the displacement for the α -th loading condition. In this case, the stationary conditions (a) and (c) reduce to

- (a) equilibrium condition for the α -th loading condition, ($\alpha=1, \dots, m$)
 - (c) equal expected value of the strain energy density on the surface; that is,
- $$\sum_{\alpha=1}^m r^\alpha \left\{ (1/2) C_{ijkl} e_{ij}^\alpha e_{kl}^\alpha \right\} = -\lambda \text{ (constant) on } S. \quad (15)$$

The quantity in the braces $\{ \}$ is the strain energy density for the α -th loading condition on the surface.

In the boundary element model, let the strain energy density of the i -th element when the α -th load set is active be denoted by D_i^α . The optimality condition is represented by

$$D_i^* = E[D_i] = \sum_{\alpha=1}^m r^\alpha D_i^\alpha = -\lambda \text{ (constant)} \\ (i=1, \dots, n), \quad (16)$$

instead of by Eq.(15). Then, the following shape determination scheme is adopted.

- [1] Perform boundary element analyses for m loading conditions and calculate the strain energy density D_i^α of the i -th element ($i=1, \dots, n$) for the α -th loading condition ($\alpha=1, \dots, m$).
- [2] Compute the expected value of the strain energy density D_i^* for the i -th element ($i=1, \dots, n$).
- [3] Reform the shape by a similar method to that in section 2.2 using D_i^* instead of D_i .

Now, let consider a multiobjective problem which aims to make m potential energies F^α ($\alpha=1, \dots, m$) for m loading conditions stationary; that is,

$$[F^1, F^2, \dots, F^m] \dots \text{stationary}. \quad (17)$$

Then, Eq.(14) has the form which gives a Pareto-optima of the multiobjective problem (17) by making a weighted sum F^* of m potentials F^α with weights r^α stationary.

3.3 In the case of a continuous random variable

When the random variable which represents the variation of the load is a continuous random variable, the response quantities such as displacement and stress also must be obtained as continuous random variables. If the range of the load is narrow, the continuous random variable can be substituted by several discrete random variables which correspond to several sets of loads. Then, the optimization problem under uncertain conditions can be solved approximately by the method of section 3.2. However, when the range of the variation of the load is wide, or when there are many parameters which represent the variations of the loads, the number of combinations which must be considered becomes enormous. As a result, the number of structural analyses which are needed for one cycle of shape evaluation increases. Consequently, it becomes impossible to solve the present

probabilistic shape determination problem approximately through the substitution for a continuous random variable by discrete variables.

This paper expresses response quantities such as displacements and strain energies as random variables by the use of a method of "decomposition into fundamental loads". First, a load P which varies with the variation of a random variable X can be expressed in terms of several determinate loads P_j ($j=1, \dots, m$) with probabilistic coefficients $C_j(X)$ as

$$P(X) = \sum_{j=1}^m C_j(X) P_j, \quad (18)$$

where loads are not mere scalar variables, but are dealt with as vectors which are composed of force components at every node as in the FEM or BEM. For example, the load P whose direction ϕ varies as in Fig.1 (a) can be decomposed as

$$P(\phi) = P_1 \cos(\phi) + P_2 \sin(\phi), \quad (19)$$

where P_1 and P_2 are force vectors when a horizontal and vertical load of a unit magnitude are each alone applied to the same point, respectively, as shown in Fig. 1 (b). Next, when two loads, Q_1 and Q_2 , whose magnitudes q_1 and q_2 are variable, are applied to a structure as shown in Fig. 2 (a), the force vector $P(q_1, q_2)$ is decomposed as

$$P(q_1, q_2) = q_1 P_1 + q_2 P_2, \quad (20)$$

where P_1 and P_2 are loads which act to each point from each direction with a unit magnitude, as shown in Fig. 2 (b). From the principle of superposition in linear elasticity, the structural response to the load $P(X)$ can be obtained from the responses to the loads P_j and as a result, the probabilistic characters of the responses will be derived. This paper calls these P_j ($j=1, \dots, m$) fundamental loads.

As in the finite-element approach by the displacement method, the total potential energy of the system is defined by the use of the nodal force vector, then the potential energy and its sensitivity to the shape change are expressed in terms of $C_j(X)P_j$ and the

Fig. 1 Load whose direction varies

Fig. 2 Two loads whose magnitudes vary

effects of probabilistic variation can be assessed only by the coefficients C_j 's⁽²⁾.

In the boundary element method, it is impossible to express the total potential energy of the system explicitly in terms of the design variables. Therefore, instead of treating the sensitivity of the potential energy, the strain energy density on the surface of the body was used for the evaluation of the shape optimality as Eq. (5) in section 2.2. Similarly, in the case of uncertain loading conditions, the probabilistic nature is considered through the variation of stress on the surface of the body, not by treating directly the variation of the loads.

In this paper, the problem is formulated by using constant boundary elements⁽⁶⁾, and the vector in which stress components in one boundary element are arranged is denoted by $\mathbf{s}$. When the fundamental load P_j gives the stress $\mathbf{s}_j$ to an element, the stress of the element, $\mathbf{s}(X)$, which is caused by the variable load $P(X)$, is represented as

$$\mathbf{s}(X) = \sum_{j=1}^m C_j(X) \mathbf{s}_j \quad (21)$$

by the principle of superposition. The strain energy density of a boundary element, D , is obtained by a quadratic form of the stress as

$$D = (1/2) \mathbf{s}^T \mathbf{H} \mathbf{s}, \quad (22)$$

where $\mathbf{H}$ is a matrix constituted by elastic material constants. As a random variable, it is written by

$$D(X) = (1/2) \mathbf{s}(X)^T \mathbf{H} \mathbf{s}(X). \quad (23)$$

Substituting Eq. (21) into Eq. (23),

$$\begin{aligned} D(X) &= (1/2) \left[\sum_{j=1}^m C_j(X) \mathbf{s}_j \right]^T \mathbf{H} \left[\sum_{k=1}^m C_k(X) \mathbf{s}_k \right] \\ &= (1/2) \left[\sum_{j=1}^m \sum_{k=1}^m C_j(X) C_k(X) \mathbf{s}_j^T \mathbf{H} \mathbf{s}_k \right]. \end{aligned} \quad (24)$$

Therefore, the expected value of the strain energy density for the element is calculated by

$$\begin{aligned} D^* &= E[D(X)] = \int_a^b D(x) f(x) dx \\ &= (1/2) \left[\sum_{j=1}^m \sum_{k=1}^m \left\{ \int_a^b C_j(x) C_k(x) f(x) dx \right\} \mathbf{s}_j^T \mathbf{H} \mathbf{s}_k \right], \end{aligned} \quad (25)$$

where a and b are the upper and lower limits of the range of the random variable X , respectively, and $f(x)$ is the probability density function of X . Equation (25) shows that the expected value of the strain energy density can be calculated by only m times of boundary element analyses for obtaining the stress $\mathbf{s}_j$ due to each fundamental load.

By the use of this D^* in the optimization algorithm, the present problem can be solved for the case of continuous random variables.

4. Examples of Optimum Design

4.1 Column subjected to uniform compressive load

In this section, the optimum shape of a column which is subjected to a uniform distributed load is

determined. The side surfaces of the column are reformable in the horizontal direction.

4.1.1 Deterministic design Figure 3 shows the results of the optimization when the direction of the load is determinate. Figure 3 (a) is the optimum shape when the load acts from the vertical direction, and the shape drawn by a solid line in Fig. 3 (b) is the optimum one for the load acting at an angle of 60° from the horizontal line. The broken line in Fig. 3 (b) shows the initial square column. The marks $\bigcirc$ and $\triangle$ in Fig. 3 represent boundary nodes (midpoints of boundary elements). Moreover, the curves drawn in the right- and left-hand sides of the column figure show the distributions of the strain energy density on each side surface. The broken curves are those when the structure is subjected to the load from the vertical direction, and the solid curves are those caused by the load from the direction at 60°. It is observed from the figures that each structure becomes optimum by having a shape whose central line is parallel with the direction of the objective load. For the other load, however, high strain energy densities are induced in both Figs. 3 (a) and (b).

4.1.2 Design considering discrete variation When the direction of action of the load is limited to either the vertical one or the 60° one, and if the relative frequencies of occurrence for each load are known, the load is regarded as that of discrete variation. That is, we consider two uniform distributed loads, P_1 and P_2 , and apply the method of section 3.2; P_1 is the load from the vertical direction and P_2 is that from the 60° direction. The probabilities of action for P_1 and P_2 are r^1 and r^2 , respectively; $r^1 + r^2 = 1$. Figure

(a) Optimum shape for vertical load, (1 : 0)

(b) Optimum shape for 60° load, (0 : 1)

Fig. 3 Optimum design for determinate load

4 shows the optimum shape for the case of $(r^1 : r^2) = (0.5 : 0.5)$. The shapes shown in Fig. 3 (a) and (b) correspond to $(r^1 : r^2) = (1 : 0)$ and $(0 : 1)$, respectively. In Fig. 4, the curves drawn on both sides of the column figure are distributions of strain energy density when either of P_1 and P_2 is active. For both loads, no high strain energy density arises. Figure 5 shows the optimum shape obtained by a deterministic design for the load from the direction at 75° , which is the mean of 90° and 60° , and draws the distributions of strain energy density which will be induced by the loads from the 90° and 60° directions, respectively. The values of the strain energy density in Fig. 5 are larger than those in Fig. 4. Moreover, from the problem statement, there is no possibility that this load from 70° is actually active.

In Fig. 6, the changes of the expected value of the total potential energy, $E[F_0]$, in several cases of $(r^1 : r^2)$ are plotted with iteration numbers. The subscript $\circ$ means that the value is evaluated at the equilibrium state. The expected value of the total potential energy at the equilibrium state can be calculated by the inner product of the force vector and the displacement vector for the elements to which the outer load are applied as

$$F_0^a = -(1/2) \mathbf{p}^a \mathbf{u}^a \quad (26)$$

$$E[F_0] = \sum_{a=1}^m r^a F_0^a. \quad (27)$$

It is observed from the figure that the expected values of the potential energy converge to their maximums stably for every set of $(r^1 : r^2)$. The expected value of the total potential energy at the stationary point in the case of $(r^1 : r^2) = (0 : 1)$ is comparatively high (its absolute value is small) compared with those in other

Fig. 4 Optimum design considering two loads from different directions, $(0.5 : 0.5)$

Fig. 5 Optimum design for determinate load from mean direction

cases. The reason for this is that as this is the deterministic design for a 60° load, the obtained shape is in the state of uniform compression, as shown in Fig. 3 (b) and has no bending component as in other cases.

In Fig. 7, the potential energies for each load at convergent state, F_0^1 and F_0^2 , are plotted as the abscissa and ordinate for several sets of $(r^1 : r^2)$. Figure 7 shows that the obtained solutions are Pareto-optima of a multicriteria problem provided that r^a is the weight for the potential energy due to each loading condition.

4.1.3 Design considering continuous variation

This section considers the case where the direction of the load varies uniformly between 60° and 90° . We can decompose the objective varying load according to Eq. (19) by adopting two uniform distributed loads acting on the upper surface as the fundamental loads; P_1 is the load acting leftward in the horizontal direction and P_2 is that acting downward in the vertical direction. As the probability density function is defined by

$$f(\phi) = 6/\pi \quad \pi/3 \leq \phi \leq \pi/2, \quad (28)$$

the coefficients in Eq.(25) are easily calculated as

$$A_{11} = (1/2) - 3\sqrt{3}/(4\pi), \quad A_{22} = (1/2) + 3\sqrt{3}/(4\pi) \quad (29)$$

$$A_{12} = A_{21} = 3/(4\pi),$$

where A_{jk} is defined as

$$A_{jk} = \int_{\pi/3}^{\pi/2} C_j(\phi) C_k(\phi) f(\phi) d\phi. \quad (30)$$

Using these A_{jk} , we can judge the optimality of the shape in each iteration by only two analyses of the boundary element method for the loads P_1 and P_2 . The obtained shape is shown in Fig. 8. The curves drawn in both sides of the column figure are distributions of the strain energy density in the special cases; the broken

Fig. 6 Changes of expected value of total potential energy for problem of section 4.1.2

Fig. 7 Pareto-optimality

line is for the case where the column is subjected to vertical load, and the solid line is for the case with 60° load. It is observed from the figure that the strain energy density is kept low for both extreme loads. Figure 9 shows the states of the expected value of the total potential energy, $E[F_0]$ and the degree of optimality, that is, ε defined by Eq.(8) until convergence. The expected value of the total potential energy is calculated by

$$E[F_0] = -(1/2) \times \left[\sum_{j=1}^m \sum_{k=1}^n \left\{ \int_a^b C_j(x) C_k(x) f(x) dx \right\} p_j^T u_k \right], \quad (31)$$

where p_j and u_j are the force vector and the displacement vector for the j -th loading condition on the surface where the outer loads are applied. It can be said that the process to convergence is stable in this case as well.

4.2 Hole in plate under biaxial tension

The second example is a design of a hole in a plate which is pulled by two uniform distributed loads Q_1 and Q_2 ; Q_1 is applied from the horizontal direction and Q_2 from the vertical direction. As shown in Fig. 10, the shape of the hole is circular at first and is reformed in the normal direction of the design surface. From the symmetrical property of the problem, only the first quadrant is analyzed with the element division shown in Fig. 10.

4.2.1 Deterministic design The right figure in Fig.11 is the optimized shape for the condition that the

Fig. 8 Optimum design for load varying uniformly between 60° and 90°

Fig. 9 Process to convergence for problem of section 4.1.3

ratio in the magnitudes of both loads is $q_1 : q_2 = 2 : 1$ and both loads are active at the same time. The left figure in Fig.11 (and in Figs.13 and 16) is an ellipse with the major and minor axes of the ratio 2 : 1 which is well known as the analytical optimum shape of the hole in an infinite plate under the same biaxial loading as this problem. The obtained shape is for the hole in the finite plate, but it is close to the analytical solution of a hole in the infinite plate. In Fig. 12, the distributions of strain energy density when Q_1 and Q_2 are applied singly to the obtained shape are plotted as the number of element being the abscissa, which was defined in Fig.10; the broken line is that by the horizontal load Q_1 and the solid line is by the vertical load Q_2 .

4.2.2 Discrete variation In this section, we consider the case where either of Q_1 or Q_2 is active separately with the likelihood of $r^1 : r^2 = 0.5 : 0.5$ and the ratio $q_1 : q_2 = 2 : 1$ still holds. The figure shown at the right in Fig. 13 is the optimum shape obtained by applying the method of section 3.2. As the loads are both uniaxial, though from different directions, the shape is reformed so as to have large cross sections in

Fig. 10 Plate under biaxial loading and model of hole

Fig. 11 Optimum design when two loads of $q_1 : q_2 = 2 : 1$ are applied at the same time

Fig. 12 Distributions of strain energy density in the shape of Fig.11

Fig. 13 Optimum design for two loads acting separately with the likelihood of $(r^1: r^2) = (0.5: 0.5)$

Fig. 14 Distributions of strain energy density in the shape of Fig.13

the directions perpendicular to each load and becomes a rectangle, each side of which is parallel to each load. Figure 14 shows the distributions of the strain energy density by the two loads. For both loads, the strain energy densities become lower than those in Fig.12.

4.2.3 Continuous variation This section considers the case where the magnitudes q_1 and q_2 of the two loads Q_1 and Q_2 vary independently obeying the probability density functions shown in Fig.15, but the ratio of their mean values is $\bar{q}_1: \bar{q}_2 = 2: 1$. In this case, the uniform distributed loads of unit magnitude and in each direction are adopted as the fundamental loads and in Eq. (20) is applied. The calculation of the expectation value is carried out by

$$E[D(q_1, q_2)] = \iint D(q_1, q_2) f_1(q_1) f_2(q_2) dq_1 dq_2. \quad (32)$$

The shape at the right in Fig.16 is obtained by the method of section 3.3.

The optimization of the structure is dependent on the structural analysis. Therefore, in order to ensure sufficient accuracy of the solution, attention for obtaining good accuracy in the boundary element method is also necessary in modeling, such as the treatment of corner points.

5. Conclusions

In this paper, an optimization method for the structural shape under uncertain loading conditions was formulated by the boundary element method. The object of the optimization was the uniformity of the expected value of the strain energy density of each element provided that the probabilistic laws of the

Fig. 15 Probability density functions of random variables Q_1 and Q_2

Fig. 16 Optimum design when magnitudes of two loads vary independently of each other with $\bar{q}_1: \bar{q}_2 = 2: 1$

variation of the loads were known. The obtained shapes were those which took averagely variable loads into consideration and no high strain energy density arose for a load in the given range of variation.

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