Boundary Element Analysis System Utilizing Data Base for G, H Matrices*

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With the boundary element analysis in composite materials in mind, this study describes the details of a data-base-utilizing method paying attention to the similarity of kernel functions. It is ascertained that the coefficient matrices of a region are usable for composing those of another region under the condition that the shape is similar even if the number of divisions differs. This indicates that the coefficient matrices can be composed without a numerical integration scheme so that a drastic reduction in CPU time will be expected. On the basis of the results, a new analysis system that avails itself of the data base has been constructed. The advantages of this system are verified by means of several examples.

Key Words: Boundary Element Method, Composite Material, Computational Mechanics, Heat Conduction, Data Base, G and H Matrices

1. Introduction

Composite materials have conventionally been dealt with as homogeneous anisotropic materials⁽¹⁾⁻⁽⁷⁾. Such an approach, which is said to be effective in explaining the macroscopic properties, is not necessarily a sufficient means by which to ascertain the localized states. By dealing with various kinds of composite materials as inhomogeneous polyphase bodies, the materials are to be divided into many regions. Since this necessitates many nodal points and elements, it is anticipated that extensive CPU time is required with the usual boundary element method. To analyze the materials as inhomogeneous polyphase bodies, development of a new boundary element analytical method is required. This study attempts to develop a new device by utilizing the data base for G and H matrices, paying special attention to the similarity of the kernel functions so that the numerical integration scheme can be eliminated.

In this report, fundamental investigation of the data-base-utilizing system is made, primarily with respect to a two-dimensional heat conduction problem and the efficiency of the present method is verified through a few analytical examples.

2. Data Base for G and H Matrices

2.1 A boundary integral equation

A steady heat conduction problem is transformed to solve the boundary integral equation as shown below instead of directly solving the Laplacian equation regarding a potential function $t(x, y)$:

$$C(P)t(P) = \int_S \{ -T^*(P, Q)q(Q) + Q^*(P, Q)t(Q) \} ds(Q) \quad (1)$$

where S is a boundary of the body, Q , an integral point moving along S , P , a source point, C , a constant determined by the configuration of the boundary, and q , heat flux. T^* and Q^* are kernel functions and are concurrently functions of the distance r between the two points P and Q . In two-dimensional problems, these are given by the following equations (ν denotes direction cosine):

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$$T^* = \frac{1}{2\pi} \ln \frac{1}{r} \quad (2)$$

$$Q^* = \frac{1}{2\pi} \frac{1}{r} \frac{\partial r}{\partial \nu} \quad (3)$$

By discretizing Eq. (1) with elements (suppose that the total number of the nodes is n), the following algebraic equation is obtained for the i -th source point:

$$h_{i1}t_1 + h_{i2}t_2 + \cdots + h_{in}t_n = g_{i1}q_1 + g_{i2}q_2 + \cdots + g_{in}q_n \quad (4)$$

where $t_j (j=1, 2, \dots, n)$, $q_j (j=1, 2, \dots, n)$ are the nodal values regarding the temperature and the heat flux. Furthermore, h_{ij} and g_{ij} are coefficients of the nodal values. They are the quantities determined by the product of interpolation functions, kernel functions and the Jacobian of the elements. By taking the source points for all of the nodal points, the simultaneous linear equation is finally composed as shown below:

$$Ht = Gq. \quad (5)$$

In the above equation, t and q respectively represent vectors with which nodal values are aligned. Meanwhile G and H indicate matrices with which coefficients of the nodal values are arranged.

In the usual boundary element analysis, the numerical integration to compose the matrices accounts for a large proportion of the time required for analysis. If the time can be reduced, the analytical system derived by the BEM is believed to be more efficient and practical. In this study, a new device to attain the said purpose is introduced; that is, a systematic method for determining the coefficient matrices by the use of data base for the G and H matrices previously constructed. Since the numerical integration is not required in this manner, drastic reduction in CPU time can be expected. Fundamental discussion will hereafter be made with respect to the theoretical subjects in standardizing the matrices into a specific data base.

2.2 Similarity of kernel functions

As determined from Eqs. (2) and (3), the kernel functions have similarity and the coefficient matrices depend on the distance between the source point and the element, their relative position and the element size. As for these coefficients, their values might vary when the source points are changed. Therefore, it is necessary to deal with both the source points and the elements as a pair, and it is convenient to retain a data base with respect to each regional configuration. Here, to what extent the data base is retained utilizing the coefficient matrices is primarily discussed for the case when a specific objective region is discretized.

First of all, the similarity of the kernel functions depending on the expansion and contraction of the

region is considered. In the region shown in Fig. 1(a), let the distance between the two points P and Q be r , values of the kernel functions be T^* and Q^* , the Jacobian be J , and the coefficients be g and h . Next, consider a similar region which is expanded (contracted) with the ratio of similarity k as illustrated in Fig. 1(b). In this case, it can be easily shown that the kernel functions T_k^* and Q_k^* and the Jacobian J_k are provided, respectively, as shown below:

$$T_k^* = T^* + \frac{1}{2\pi} \ln \frac{1}{k} \quad (6)$$

$$Q_k^* = \frac{1}{k} Q^* \quad (7)$$

$$J_k = kJ. \quad (8)$$

Accordingly, the coefficient matrices corresponding to a new region are provided as

$$g_k = kg + \frac{kJ}{2\pi} \ln \frac{1}{k} \quad (9)$$

$$h_k = h. \quad (10)$$

When the region is rotated by ϕ as with the case in Fig. 1(c), no change is noted with either the kernel functions or the Jacobian. Thus, the new coefficient matrices are equal to those determined before the rigid body transition.

It is explained as shown above that the coefficient matrices of the similar configuration can be determined by utilizing the coefficient matrices obtained in the original region. On the other hand, when the regional configuration is not similar, the mutual relationship among the coefficient matrices becomes loose and it is difficult to directly utilize the data base. Despite the above problem, the analysis utilizing the data base, as will be described later, is possible if we avail ourselves of the proper techniques, and a high degree of versatility can be maintained.

2.3 Change of coefficient matrices by dividing numbers

In constructing the data-base system as

Fig. 1 Expansion, contraction and rotation of the regional configuration

mentioned above, we must deal with a problematic point; i.e., the dimension of the matrices and their values are changed as the element dividing numbers are changed, even if the configurations of the regions are the same. From these facts, it is evident that the generality of the data base is lost and its effectiveness is remarkably decreased, which is inconvenient in constructing the analytical system utilizing the data base. Therefore, we herewith examine the relationships among the coefficient matrices when the element-dividing numbers are different.

Suppose that a region is divided into two second-order elements, (1) and (2), as illustrated in Fig. 2(a), and that nodal values $t_a \sim t_e$ are set onto the nodes a~e. By taking the second term of the right side of Eq.(1) as an example, the boundary integral in this interval with respect to a source point can be provided in a discretized form, as shown below with the individual coefficients as $h_a \sim h_e$:

$$\int_a^e Q^* t ds = h_a t_a + h_b t_b + h_c t_c + h_d t_d + h_e t_e. \quad (11)$$

In the next stage when the same interval is divided into one second-order element as illustrated in Fig. 2(b), only t_a , t_c and t_e are set as the nodal values and t_b and t_d disappear. However, when the following procedure is taken, the integration of this interval can be expressed by utilizing the results of Eq.(11). In other words, the nodal values t_b and t_d that disappeared can be expressed with relation to Fig. 2(b) as shown in the following:

$$t_b = \phi_1(-1/2)t_a + \phi_2(-1/2)t_c + \phi_3(-1/2)t_e \quad (12 \cdot a)$$

$$t_d = \phi_1(1/2)t_a + \phi_2(1/2)t_c + \phi_3(1/2)t_e, \quad (12 \cdot b)$$

where $\phi_1 \sim \phi_3$ are interpolation functions. By substituting the above equations into Eq.(11), the following can be obtained:

$$\begin{aligned} \int_a^e Q^* t ds = & t_a \{ h_a + \phi_1(-1/2)h_b + \phi_1(1/2)h_d \} \\ & + t_c \{ h_c + \phi_2(-1/2)h_b + \phi_2(1/2)h_d \} \\ & + t_e \{ h_e + \phi_3(-1/2)h_b + \phi_3(1/2)h_d \}. \end{aligned} \quad (13)$$

The above equations reveal that the coefficients of the nodal values t_a , t_c and t_e in Fig. 2(b) can also be obtained without new calculation, as long as the coefficient values $h_a \sim h_e$ are calculated previously.

Fig. 2 Reduction in element-dividing number

The results brought about from the above can be expanded very easily, even when the dividing numbers become generally $1/n$. From these, it is indicated that the coefficient matrices of a given problem which is divided into certain numbers of elements complying with the necessity can at any time be composed without a numerical integration scheme, as long as the coefficient matrices have been memorized as the data base. According to the analytical example to be referred to later, Eq. (13) gives almost the same results as the solutions obtained by the usual solution with the same dividing numbers. Thus, it is concluded that the calculation utilizing the data base has no adverse effect on the accuracy of the solution. Furthermore, it is to be pointed out as being worthy of note that when calculation is made previously as the data base, the order of the numerical integration can be arranged as desired. (In some cases, an analytical integration is possible.) Thus, coefficient matrices, without most integral errors, can be obtained and retained.

3. Composition of the System

3.1 Analytical system

According to the results obtained in the previous section, it is elucidated that the original calculated outcomes can be utilized even if the dividing number is decreased, as long as the configuration of the objective region to be analyzed is a similar figure. This indicates that when the regional configuration is a square or a circle, the results once calculated can be repeatedly utilized since all of them exist as similar figures. It is considered that the filing of the results with respect to these configurations is very useful. Let such files be left as a permanent data base. If other configurations often manifesting themselves as usual are available, add them to the permanent data base. However, it is impossible for all of the configurations to be stored as a data base of this kind due to the restriction of disc space or other such factors. It is a matter of course that the kind of configuration that should be stored depends on the individual case.

Additionally, when the same configuration repeatedly appears in an analysis and it is not stored in the permanent files, a method of utilizing the temporary file as shown in the following becomes effectual. That is to say, the coefficient matrices with respect to the common portion in a place are once calculated by numerical integration and are stored as the temporary file. Then they can be utilized in calculating matrices of common portion in other places without numerical integration. The temporary file might be eliminated after the calculation is completed. Such a file will hereafter be called the temporary data base. The temporary data base has been practically achieved,

and its effect has been verified^{(8),(9)}.

In actual analysis, the regular configurations which are stored as the data base are not always available. However, even with an oblong region, for example, the data base can be fully utilized by analysis with the square coupling method. By skillfully incorporating those two kinds of data bases into the existing analytical system, it is possible to construct a high-performance versatile analytical system. As an example of such a system, a flowchart in the analytical system utilizing a data base is schematically illustrated in Fig. 3. In this system, utilization of the data base is optional, and such a problem which requires both utilization of the data base and prosecution of the numerical integration can also be confronted. In the conventional procedure, a route (shown by a broken line) is depicted in the figure. This route is also included in the system in question, and it is concluded that the versatility of the analytical system is not impaired at all. In the case of a problem with which matrices can be composed only with a data base, needless to say, large-scale shortening of the CPU time is expected.

3.2 Data base in a square region

Here, let a data base in a square region actually be compiled and incorporated into the existing

Fig. 3 An example of versatile system utilizing the data base

system⁽¹⁰⁾. As shown in Fig. 4, the individual sides of the square are divided into 12 second-order elements. Thus, the total numbers of elements and nodes become 48 and 96, respectively. The data base constructed from the present model can be utilized in the analysis of the squares whose dividing numbers per side are less than 12.

If the boundary condition is given optionally with respect to each element, the entire number of the components of the G and H matrices that are to be memorized is given by the product of the numbers of source points, numbers of elements and numbers of nodes in one element. Accordingly, the number in the present case is $96 \times 48 \times 3 = 13\,824$. However, the value of the components has the same repetition, and careful observation will ascertain that the number of the components to be actually memorized can be greatly reduced as compared with this. Strict investigation will ensure that a memory of almost 1/8 will be adequate and the capacity of approximately 13.8 kB will be sufficient to memorize the data with double precision.

The system thus obtained has been incorporated in a 32-bit minicomputer (Micro VAX 3400). The calculations hereafter made are all performed with this minicomputer.

4. Analytical Examples

4.1 Accuracy of the solutions

First, the accuracy of the solutions obtained by utilizing the data base is discussed. As an analytical model, the square illustrated in Fig. 5 is taken up, and the boundary condition as shown in the figure is given. Comparison of accuracy of the solutions is made in accordance with the following: (a) The said region is, from the beginning, divided into 6 elements, 4 elements and 3 elements per side, respectively, and the conventional numerical integration scheme is applied to each case. (b) The matrices are fabricated utiliz-

Fig. 4 Square region for the data base

Table 2 Comparison of CPU time in Fig. 6

Squares	nodes	Time for Input [msec]		Time for Matrix [msec]	
		Conven-tional	Data Base	Conven-tional	Data Base
1	8	250	700	520	20
2	16	320	740	1040	40
3	24	320	770	1560	50
4	32	370	760	2080	70
5	40	390	890	2610	100
6	48	390	840	3060	120
7	56	450	900	3650	150
8	64	460	880	4140	170
9	72	470	900	4670	180
10	80	490	1040	5210	210

Table 1 Numerical results in Fig. 5

nodes	Numerical Integration [°C]	Use Database [°C]
1/4	82.1122557733947	82.1122557736659
1/3	82.1195155404765	82.1195155405037
1/2	82.1237334915118	82.1237334903213
Numerical Integration of 1/1 nodes		82.1255197604179

Fig. 5 Analytical model for verifying accuracy

ing the data base with the dividing numbers as 1/2, 1/3 and 1/4 by means of Eq. (13). In Table 1, the solutions of the temperature at the point O are provided as an example. With respect to each dividing number, the solutions between these two cases coincide with each other. Thus, it is explained that the accuracy of the solutions obtained by utilizing the data base is never inferior to the ones obtained by the conventional method.

4.2 Analytical examples utilizing the data base

Examples of the analysis using the system in Fig. 3 are provided here. At first, a steady thermal conduction problem of strips as shown in Fig. 6 is analyzed. This is an example that makes use of only the perma-

Fig. 6 Analytical model utilizing the data base

Fig. 7 Comparison of CPU time in Fig. 6

nent data base and requires no numerical integration. Ten kinds of strips with heights that are multiple lengths ranging from 1 to 10 times the width are put to trial. The strips are composed of 2 types of materials whose thermal conductance λ is different from each other and they are placed alternately. The present data-base-using method is compared with the conventional method in solving this problem. In both methods, the regional coupling method is employed for solving the problems of multiconnected bodies. Since only the squares are connected with each other, no numerical integration is required. Additionally, all of the squares are divided into one element per side.

The CPU time required for the calculation is shown in Table 2. Although it takes some time to read the data in the case of the data-base-using method, the time required for composing the matrices is greatly decreased as compared with the conventional case.

Its decreasing rate is almost 1/25-1/30. In Fig. 7, these results are provided in the form of a graph, from which it is noted that the CPU time of the matrices' composition in the case of the conventional method is somewhat linearly increased in the direction of the regional connection numbers.

The model illustrated in Fig. 8 is an example utilizing the temporary data base. These are ten kinds of strips with the same dimensions as the ones in Fig. 6 and include in each square an inclusion which is round in shape. The inclusions are aligned with the same size at an equal interval. The CPU time for solving these problems is compared in accordance with the following three measures: (1) a measure to utilize the temporary data base of a square, including in itself a round foreign material (Fig. 8(a)). (2) an ordinary measure depending on the conventional numerical integration with a regional coupling method (Fig. 8(a)), and (3) an ordinary measure depending on the conventional numerical integration without dividing into square regions (Fig. 8(b)). The square portions are divided into the same numbers of elements as in the case in Fig. 6, whereas the round portions are divided into 4 elements and 8 nodes.

Illustrated in Fig. 9 is the CPU time for composing matrices using the three types of measures mentioned above. With the method utilizing the data base in (1), the CPU time is hardly increased although the strips are elongated. On the contrary, with both methods (2) and (3), the increase is great.

Fig. 8 Strips with round inclusions

However, although the increase is linear with the solving method in (2), this increase is drastic with the solving method in (3) in comparison with the method in (2). From this, it is understood that it is more advantageous to employ the regional coupling method than to solve directly the whole region at one time. This advantage can be applied to a solver to solve simultaneous equations. That is to say, if the coefficient matrices become of the band type, the said type of solver is available; otherwise a solver of full matrices must be utilized. The influence of this difference is illustrated in Fig. 10. As shown here, the larger the connection becomes, the higher the advantage in utilizing the band-type solver becomes at a rapid pace. This indicates that it is advantageous, according to circumstances, to solve the homogeneous problems by means of dividing into some regions.

5. Conclusions

This study aims to develop a boundary element analytical method efficient enough to eliminate the numerical integration by standardizing the G and H matrices into a specific data base, paying attention to the similarity of the kernel functions as a method of dealing with the composite materials as in-

Fig. 9 Comparison of matrices' composing time in Fig. 8

Fig. 10 Comparison of simultaneous solvers

homogeneous polyphase bodies. In this report, fundamental discussion for constructing the data base and several examples of the analysis have been described. From this, it is explained that the numerical data of the G and H matrices in the original region can be utilized even if the element division is different, so long as the regional configuration is similar. Based on the above investigation, it is furthermore ascertained that a system retaining a high degree of versatility can be constructed by making use of two types of data bases, that is to say, a permanent type and a temporary type. By analyzing two-dimensional thermal conduction problems of strips, the effects of the method have been verified.

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