Impact Response of a Unidirectional Composite Laminate*

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The response of a transversely isotropic half-space and a plate with its symmetry axis parallel to the surface, subjected to a transient surface point load at an arbitrary angle is determined. The problem is formulated in terms of an integral transform method in the frequency domain. To obtain the near-field solution, double infinite integrals involved in the inverse transform are evaluated by extending the Clenshaw-Curtis quadrature scheme to two dimensions. Approximate far-field solutions are obtained by evaluating the double integrals using the residue theorem and the stationary phase method. As a companion problem, the response of a finite simply supported plate subjected to a dynamic surface load is also obtained using classical plate theory (CPT) with appropriate modifications to include the transverse shear effect (TSE) and rotatory inertia. The numerical results obtained by the different approaches are compared.

Key Words: Fiber-reinforced Composites, Non-destructive Testing, Impact, Acoustic Emission, Plate

1. Introduction

Fiber-reinforced composites are being increasingly used as primary structural components in aerospace and other modern applications, due to their high ratio of strength and stiffness to weight. It is well known that composite laminates suffer significant internal damage when they are subjected to rapidly applied surface loads. Calculation of the stress field induced within the laminate by such loads is of considerable interest in damage analysis and in failure prediction of composite structures. In addition, surface displacements generated by surface loads can be used in calibration of Acoustic Emission (AE) experiments and in the simulation of point source/point receiver⁽¹⁾ experiments. The major objective of this paper is to enhance the theoretical understanding of the wave phenomena which occur within unidirectional composite materials when they are subjected to dynamic loads.

Transient wave propagation in infinite isotropic plates subjected to a surface point source has received considerable attention in the past and some of the methods used in these studies can, in principle, be used in the solution of the present problem. References to various techniques used for isotropic plates can be found in Weaver et al.,⁽²⁾. In contrast to the isotropic media, most of the previous studies on anisotropic media have been restricted to the transversely isotropic full-space or half-space. Cameron and Eason⁽³⁾ studied the response of a transversely isotropic full-space to a point load parallel to the axis of symmetry using the integral transform method. Buchwald⁽⁴⁾ discussed Rayleigh wave propagation in a transversely isotropic half-space due to a time harmonic surface normal load perpendicular to the axis of symmetry. Suh et al.⁽⁵⁾ studied the response of a half-space to a normal impact perpendicular to the axis (quasi-isotropic case) of symmetry using the Cagniard-De Hoop method. The corresponding two dimensional problem was considered by Burrige⁽⁶⁾. The impact response of relatively thin composite plates has been studied extensively⁽⁷⁾⁻⁽¹⁰⁾ by using approximate plate theories. Most of these studies are based

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on the well known Timoshenko beam theory as applied to composite plates with certain "average" laminate properties⁽¹¹⁾. An approximate theory has also been developed assuming the composite material to be inextensible in the fiber direction⁽¹²⁾⁻⁽¹⁴⁾. In the present study we present an "exact" solution of the impact response of a composite plate of finite thickness and infinite lateral dimensions, using the classical linear elastodynamic theory.

In most of the previous work on wave motion in anisotropic solids, two types of approaches have been used. In one of these, the concept of the slowness surface has been used to describe the geometric properties of waves propagating in an infinite elastic medium⁽¹⁵⁾⁽¹⁶⁾. In the second method, a potential function approach has been used to study the wave propagation⁽⁴⁾. A modified form of Buchwald's potential function method is used in this paper to study the response of an infinite plate of finite thickness to a surface impact oblique to the symmetry axis.

The symmetry axis of the material is assumed to be parallel to the surface. The spectral response and time history of the response for a plate and a half-space are calculated when a transient force is applied obliquely at a point on the surface. The material is assumed to be dissipative, its dissipative properties modeled by a frequency dependent damping factor⁽¹⁷⁾. For the far field approximation, the material is assumed to be elastic, so that surface and guided wave contributions to the stresses and displacements can be calculated as the sum of residues at the real roots of the Rayleigh-Lamb dispersion equation. In addition, the problem of a finite plate subjected to a transient, normal point force is solved, both by including and neglecting the effect of transverse shear stress and rotatory inertia. In all cases the solution is first obtained in the Fourier transformed (frequency) domain. The time histories are then obtained through inversion by means of the fast Fourier transform (FFT) routine.

2. Formulation of the Problem

Let x_1, x_2, x_3 denote a cartesian coordinate system with x_1 -axis parallel to the fibers and x_3 -axis along the thickness direction of the plate or into the half-space (Fig. 1). The thickness of the plate is assumed to be $2h$. We consider time harmonic disturbances first and suppress the time dependence $\exp(-i\omega t)$ from the field variables. Let $[C_{ij}]$ be the complex stiffness matrix for the material, $[\sigma_{ij}]$ the stress tensor and $\{u_i\}$ the displacement vector.

As indicated in the introduction, the dissipative property of the composites will be modeled by a frequency dependent, real damping factor ' p ', resulting

in complex-valued stiffness constants C_{ij} . These can be related to the real stiffness matrix c_{ij} in the following form⁽¹⁷⁾.

$$\begin{aligned} C_{11} &= c_{11}[1 - ip(c_{55}/c_{11})^{1/2}], & C_{22} &= c_{22}[1 - ip(c_{55}/c_{22})^{1/2}] \\ C_{12} + C_{55} &= (c_{12} + c_{55})[1 - ip(c_{55}/(c_{12} + c_{55}))^{1/2}] \\ C_{44} &= c_{44}[1 - ip(c_{55}/c_{44})^{1/2}], & C_{55} &= c_{55}[1 - ip] \end{aligned} \quad (1)$$

The damping factor p may be represented in the following simple form⁽¹⁷⁾,

$$p = p_0 \left[1 + a_0 \frac{(f - f_0)}{f_0} H(f - f_0) \right] \quad (2)$$

where f ($=\omega/2\pi$) is the frequency in MHz, f_0 is a specific frequency, and $H(f)$ is the Heaviside step function. The dimensionless constants p_0 and a_0 determine the degree of decay in the amplitude of the waves with propagation distance. The effect of material dissipation is essentially controlled by p_0 and that of wave scattering by a_0 . It should be noted that attenuation in the medium increases with p .

Following Buchwald⁽⁴⁾, the equations of motion may be expressed in terms of three potential functions Φ_1, Φ_2, Φ_3 as

$$\begin{aligned} a_3 \frac{\partial^2}{\partial x_1^2} \nabla_1^2 \Phi_1 + \left(a_1 \nabla_1^2 + a_5 \frac{\partial^2}{\partial x_1^2} + \omega^2 \right) \nabla_1^2 \Phi_2 &= 0 \\ a_3 \frac{\partial^2}{\partial x_1^2} \nabla_1^2 \Phi_2 + \left(a_5 \nabla_1^2 + a_2 \frac{\partial^2}{\partial x_1^2} + \omega^2 \right) \frac{\partial^2 \Phi_1}{\partial x_1^2} &= 0 \\ \left(a_5 \frac{\partial^2}{\partial x_1^2} + a_4 \nabla_1^2 + \omega^2 \right) \Phi_3 &= 0 \end{aligned} \quad (3)$$

where

$$\begin{aligned} \nabla_1^2 &= \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2, & a_1 &= C_{22}/\rho, & a_2 &= C_{11}/\rho, \\ a_3 &= (C_{12} + C_{55})/\rho, & a_4 &= (C_{22} - C_{23})/2\rho, & a_5 &= C_{55}/\rho \end{aligned}$$

and ρ is the mass density.

The general solution of Eq. (3) for a point load may be expressed in the form

$$\begin{aligned} \Phi_1 &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{ [q_{11} A_1^+ e^{i\xi_1 x_3} + q_{12} A_2^+ e^{i\xi_2 x_3}] \\ &\quad + [q_{11} A_1^- e^{-i\xi_1 (2h - x_3)} \\ &\quad + q_{12} A_2^- e^{-i\xi_2 (2h - x_3)}] \} e^{i\xi_1 x_1 + i\xi_2 x_2} d\xi_1 d\xi_2 \end{aligned} \quad (4)$$

Fig. 1 A schematic diagram of a transient load applied on a composite plate.

$$\begin{aligned}\Phi_2 &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{ [q_{21} A_1^+ e^{i\zeta_1 x_3} + q_{22} A_2^+ e^{i\zeta_2 x_3}] \\ &\quad + [q_{21} A_1^- e^{-i\zeta_1(2h-x_3)} \\ &\quad + q_{22} A_2^- e^{-i\zeta_2(2h-x_3)}] \} e^{i\zeta_1 x_1 + i\zeta_2 x_2} d\zeta_1 d\zeta_2 \\ \Phi_3 &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [A_3^+ e^{i\zeta_3 x_3} \\ &\quad + A_3^- e^{i\zeta_3(2h-x_3)}] e^{i\zeta_1 x_1 + i\zeta_2 x_2} d\zeta_1 d\zeta_2\end{aligned}$$

where $A_j^\pm$ are constants,

$$\begin{aligned}q_{11} &= a_3 b_1, \quad q_{21} = \omega^2 - a_2 \xi_1^2 - a_5 b_1 \\ q_{12} &= a_3 b_2, \quad q_{22} = \omega^2 - a_2 \xi_2^2 - a_5 b_2 \\ b_1 &= -\frac{B}{2A} - \sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A}} \\ b_2 &= -\frac{B}{2A} + \sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A}} \\ A &= a_1 a_5, \quad B = (a_1 a_2 + a_3^2 - a_5^2) \xi_1^2 - \omega^2 (a_1 + a_5) \\ C &= (a_2 \xi_1^2 - \omega^2)(a_5 \xi_1^2 - \omega^2)\end{aligned}\quad (5)$$

and

$$\begin{aligned}\xi_1^2 &= -\xi_2^2 + b_1, \quad \xi_2^2 = -\xi_1^2 + b_2, \\ \xi_3^2 &= -\xi_2^2 + (\omega^2 - a_5 \xi_1^2)/a_4, \quad \text{Im}(\xi_j) \geq 0\end{aligned}\quad (6)$$

It is convenient to rewrite the general solution of Eq. (4) in matrix form through the introduction of the stress-displacement vector $\{S\}$ defined by

$$\begin{aligned}\{u_1, u_2, u_3, \sigma_{13}, \sigma_{23}, \sigma_{33}\} \\ = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{S(x_3, \xi_1, \xi_2, \xi_3, \omega)\} e^{i(\xi_1 x_1 + \xi_2 x_2)} d\zeta_1 d\zeta_2\end{aligned}\quad (7)$$

Then, suppressing the explicit dependence of S on $\xi_1, \xi_2, \xi_3, \omega$,

$$\{S(x_3)\} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} [E] \{C\} \quad (8)$$

where

$$\begin{aligned}[E] \\ = \text{Diag} [e^{i\zeta_1 x_3} e^{i\zeta_2 x_3} e^{i\zeta_3 x_3} e^{i\zeta_1(2h-x_3)} e^{i\zeta_2(2h-x_3)} e^{i\zeta_3(2h-x_3)}]\end{aligned}\quad (9)$$

$$[Q_{11}] = \begin{bmatrix} i\zeta_1 q_{11} & i\zeta_1 q_{12} \\ i\zeta_2 q_{21} & i\zeta_2 q_{22} \end{bmatrix}, \quad (10\cdot a)$$

$$[Q_{12}] = \begin{bmatrix} 0 & i\zeta_1 q_{11} & i\zeta_1 q_{12} & 0 \\ i\zeta_3 & i\zeta_2 q_{21} & i\zeta_2 q_{22} & -i\zeta_3 \end{bmatrix}$$

$$\begin{aligned}[Q_{21}]^T \\ = \begin{bmatrix} i\zeta_1 q_{21} & -\rho a_5 \xi_1 \zeta_1 (q_{11} + q_{21}) & -2\rho a_4 \xi_2 \zeta_1 q_{21} & e \\ i\zeta_2 q_{22} & -\rho a_5 \xi_1 \zeta_2 (q_{12} + q_{22}) & -2\rho a_4 \xi_2 \zeta_2 q_{22} & f \end{bmatrix}\end{aligned}\quad (10\cdot b)$$

$$\begin{aligned}[Q_{22}] = \begin{bmatrix} -i\zeta_2 & -i\zeta_1 q_{21} \\ \rho a_5 \xi_1 \zeta_2 & \rho a_5 \xi_1 \zeta_1 (q_{11} + q_{21}) \\ \rho a_4 (\xi_2^2 - \zeta_3^2) & 2\rho a_4 \xi_2 \zeta_1 q_{21} \\ 2\rho a_4 \xi_2 \zeta_3 & e \\ -i\zeta_2 q_{22} & i\zeta_2 \\ \rho a_5 \xi_1 \zeta_2 (q_{12} + q_{22}) & \rho a_5 \xi_1 \zeta_2 \\ 2\rho a_4 \xi_2 \zeta_2 q_{22} & \rho a_4 (\xi_2^2 - \zeta_3^2) \\ f & -2\rho a_4 \xi_2 \zeta_3 \end{bmatrix} * \\ * \begin{bmatrix} \rho a_5 \xi_1 \zeta_2 (q_{12} + q_{22}) & \rho a_5 \xi_1 \zeta_2 \\ 2\rho a_4 \xi_2 \zeta_2 q_{22} & \rho a_4 (\xi_2^2 - \zeta_3^2) \\ f & -2\rho a_4 \xi_2 \zeta_3 \end{bmatrix}\end{aligned}\quad (10\cdot c)$$

$$e = \rho[(a_5 - a_3)\xi_1^2 q_{11} - (a_1 - 2a_4)\xi_2^2 q_{21} - a_1 \xi_1^2 q_{21}] \quad (11\cdot a)$$

$$f = \rho[(a_5 - a_3)\xi_1^2 q_{12} - (a_1 - 2a_4)\xi_2^2 q_{22} - a_1 \xi_2^2 q_{22}] \quad (11\cdot b)$$

The components of the unknown six-vector $\{C\}$ are related to the constants $A_j^\pm$ introduced earlier in the form,

$$\{C\} = \{A_1^+, A_1^-, A_2^+, A_2^-, A_3^+, A_3^-\}^T \quad (12)$$

These constants must be determined from the boundary conditions. In the present case, we assume that the plate is subjected to a point impact at the surface. The corresponding 'stress-displacement' vectors $\{S(x_3)\}$ on the upper and the lower surfaces are given by

$$\begin{aligned}\{S(0)\} &= [U_0 \ V_0 \ W_0 \ F(\omega) \cos \phi \sin \theta \\ &\quad F(\omega) \sin \phi \sin \theta \ F(\omega) \cos \theta]\end{aligned}\quad (13\cdot a)$$

$$\{S(2h)\} = [U_1 \ V_1 \ W_1 \ 0 \ 0 \ 0] \quad (13\cdot b)$$

where U_0, V_0, W_0 as well as U_1, V_1 and W_1 are the displacement components in the x_1, x_2 and x_3 directions at the top and bottom surfaces, respectively, of the plate. The quantity $F(\omega)$ is the fourier transform of the applied load $f(t)$; ϕ and θ are the angles made by the applied load with the x_1 and x_3 axes, respectively (Fig. 1). Using Eq. (8) to calculate $\{S(0)\}$ and $\{S(2h)\}$, the following system of linear equations is obtained

$$[B]\{C\} = \{G\} \quad (14)$$

where $[B]$ is a 6×6 matrix, whose elements depend upon the material properties and thickness of the plate. It can expressed in the partitioned form⁽¹⁷⁾,

$$[B] = \begin{bmatrix} Q_{21} & Q_{22} E_1 \\ Q_{21} E_2 & Q_{22} E_3 \end{bmatrix} \quad (15)$$

where

$$[E_1] = \text{Diag} [1 \ e^{i\zeta_1 h} \ e^{i\zeta_2 h} \ e^{i\zeta_3 h}] \quad (16\cdot a)$$

$$[E_2] = \text{Diag} [e^{i\zeta_1 h} \ e^{i\zeta_2 h}] \quad (16\cdot b)$$

$$[E_3] = \text{Diag} [e^{i\zeta_3 h} \ 1 \ 1 \ 1] \quad (16\cdot c)$$

In Eq. (14) the vector $\{G\}$ is given by

$$\begin{aligned}\{G\} &= \{F(\omega) \cos \phi \sin \theta \ F(\omega) \sin \phi \cos \theta \\ &\quad F(\omega) \sin \theta \ 0 \ 0 \ 0\}\end{aligned}\quad (17)$$

For the half-space problem, since there is no source at infinity the elements A_1^-, A_2^-, A_3^- must be zero, and only the top surface boundary conditions need to be considered. Then the matrix $[B]$ in Eq. (14) reduces to a 3×3 matrix. After obtaining the coefficient vector $\{C\}$, the displacements and stresses at an arbitrary point can be obtained from the potentials in Eq. (4) as double infinite integrals.

3. Evaluation of the Double Integral

The most difficult part in solving the problem is the evaluation of the associated double integrals in Eq. (4). A recent comparative study by Dravinski and Mossesian⁽¹⁸⁾ have shown that a modified version of the Clenshaw-Curtis method (MCC) is the most efficient of all the existing methods for the numerical evaluation of these integrals. But the MCC method has so far been used only for single integrals. In this section, we discuss the extension of the MCC method to double integrals.

A typical wave number integral involved in the present analysis is of the form

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\xi_1, \xi_2, \omega) e^{i\xi_1 x_1 + i\xi_2 x_2} d\xi_1 d\xi_2 \quad (18)$$

The direct numerical evaluation of this double infinite integral will require very long CPU time and, in addition, may cause precision problems⁽¹⁸⁾. To minimize the CPU time requirement we change the limits of the integral by the transformation

$$\begin{aligned} \xi_1 &= \xi \cos \psi & \xi_2 &= \xi \sin \psi \\ x &= R \cos \alpha & y &= R \sin \alpha \end{aligned}$$

Then,

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^{\infty} F(\xi, \psi, \omega) e^{i\xi R \cos(\psi - \alpha)} d\xi d\psi \\ &= \int_0^{2\pi} G(\psi, \omega) d\psi \end{aligned} \quad (19)$$

where

$$G(\psi, \omega) = \int_0^{\infty} F(\xi, \psi, \omega) e^{i\xi R \cos(\psi - \alpha)} d\xi \quad (20)$$

The integral in (20) can be obtained by using the MCC method. Integral in Eq. (19) can then be evaluated by treating $G(\psi, \omega)$ as a new integrand. At first sight, this appears to be a routine exercise. This is especially true for viscoelastic materials, since there is no singular point on the path of integration. But the $[B]$ matrix in Eq. (14) become singular for $\alpha = m\pi/2$, $m=0, 1, 2, 3, 4$, that is, when the waves propagate along or at right angles to the axis of symmetry. For these special situations, the problem becomes two-dimensional and we reformulate them as two dimensional cases. Furthermore, when α is very close to $m\pi/2$, the integrands in Eqs. (18) and (19) exhibit numerical instability. In such situations, we interpolate the integrand over a very small interval (of the order of 10^{-3} radian) between their two and three dimensional limits.

4. Far-field Solution

For large source to station distances and in the absence of material dissipation, the wave number integral can be evaluated analytically by means of asymptotic methods. To this end, write Eq. (18) by separating the numerator and denominator of $F(\xi_1, \xi_2, \omega)$ as

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{N(\xi_1, \xi_2, \omega)}{H(\xi_1, \xi_2, \omega)} e^{i\xi_1 x_1 + i\xi_2 x_2} d\xi_1 d\xi_2 \quad (21)$$

where the singularities of the integrand are at the zeros of $H(\xi_1, \xi_2, \omega)$, which correspond to the roots of the Rayleigh-Love dispersion equation. For Large $R = \sqrt{(x_1^2 + x_2^2)}$, we use a method derived from those developed by Lighthill⁽¹⁹⁾ and Buchwald⁽¹¹⁾ for wave number integrals in anisotropic media. We rotate to a new co-ordinate system x'_1, x'_2 , where the x'_2 -direction is parallel to the vector R , with components (x_1, x_2) . This is best done by expressing the integrand in a

form that is invariant under the rotation of axes. The integral then reduces to

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{N(\xi'_1, \xi'_2, \omega)}{H(\xi'_1, \xi'_2, \omega)} e^{i\xi'_1 x'_1} d\xi'_1 d\xi'_2 \quad (22)$$

We now carry out the integration with respect to ξ'_2 , using a contour in the complex ξ'_2 -plane consisting of the real axis together with a semicircle in the upper half-plane. Applying Jordan's lemma we find that only contribution to I are from the poles of the integrand ($H=0$). We can ignore complex values of ξ'_1, ξ'_2 , which satisfy this equation, as these values give contributions to I which are exponentially small for large R . We are left with contributions from the real roots of the curve $H(\xi'_1, \xi'_2, \omega)=0$, in the ξ'_1, ξ'_2 plane, corresponding for fixed ξ'_1 to poles on the real axis of the complex ξ'_2 -plane. It can further be shown that the contributions from the poles that do not satisfy the radiation condition can be ignored. Thus, the integral in Eq. (22) reduces to

$$I = 2\pi i \int_{-\infty}^{\infty} \frac{N(\xi'_1, \xi'_2, \omega)}{H_2(\xi'_1, \xi'_2, \omega)} e^{i\xi'_1 x'_1} d\xi'_1 \quad (23)$$

where $H_2 = \partial H / \partial \xi'_2$ and ξ'_1, ξ'_2 are evaluated on that part of the curve $H(\xi'_1, \xi'_2, \omega)=0$ for which the radiation condition is satisfied. Since H is a homogeneous function of ξ_1, ξ_2 and ω , the radiation condition reduces simply to $\xi'_2 > 0$ in the special rotated coordinate, and $\xi_1 x_1 + \xi_2 x_2 > 0$ in the original coordinate before rotation.

Using the method of stationary phase, the chief contributions to the integrals in Eq. (23) can be shown to come from the neighborhoods of those points on the curve $H=0$, where $\xi'_2 > 0$ and where the normal is parallel to the ξ'_2 -direction. Let there be n such points, with coordinates (ξ'_{1p}, ξ'_{2p}) , at which ξ'_2 can be expanded in the Taylor series

$$\xi'_2 = \xi'_{2p} + 1/2 k_p (\xi'_1 - \xi'_{1p})^2 + \dots \quad (24)$$

where k_p is the curvature.

After some simplification and rotation of the axes back to the original axis, we can write

$$I = \pm \frac{(2\pi)^{3/2}}{R^{1/2}} \sum_{p=1}^n A_p \frac{N(\xi_1, \xi_2, \omega)}{|\text{grad } H|_p |k_p|^{1/2}} \exp[i(\xi_1 x_1 + \xi_2 x_2)] \quad (25)$$

where $|\text{grad } H| = \sqrt{(H_1^2 + H_2^2)}$. The curvature k_p can be expressed in terms of derivatives of H (indicated by suffices) as follows:

$$k_p = \frac{H_1^2 H_{22} - 2H_1 H_2 H_{12} + H_2^2 H_{11}}{(H_1^2 + H_2^2)^{3/2}} \quad (26 \cdot a)$$

and

$$A_p = \exp[1/4\pi i(2 + \text{sgn } k_p)] \quad (26 \cdot b)$$

The sign in expression (25) is positive if the function H is increasing and negative if H is decreasing with distance from the origin at the point (ξ_1, ξ_2) .

If the integral, I , in (18) represents waves from a source situated at the origin, then the waves observed

at a distant point represented by the position vector $\mathbf{R}$ in (x_1, x_2) space correspond to those points on the curve $H=0$, where the normal is parallel to $\mathbf{R}$. In case of vanishing curvature ($k_p=0$), higher order terms in the Taylor series expansion are also considered⁽¹⁾.

For the half-space problem, $F(\xi_1, \xi_2, \omega)$ can easily be separated into $N(\xi_1, \xi_2, \omega)$ and $H(\xi_1, \xi_2, \omega)$. But for the plate problem, this will require the analytical inversion of a fully populated 6×6 matrix $[B]$ in equation (14). To avoid this, we divide the plate problem into its symmetric and antisymmetric parts, thereby reducing the size of the matrix to 3×3 , and add their individual contributions to obtain the total solution.

5. Formulation of the Plate Bending Problem

The basic concepts of the plate bending solution presented here are similar to those of Chow⁽⁷⁾ and is based on the static solution as presented by Whitney and Pagano⁽¹⁹⁾. To keep presentation consistent, a solution in terms of the Fourier transform is obtained instead of the Laplace transform as used by Chow⁽⁷⁾.

The classical laminated plate equation for a homogeneous symmetric plate (with dissipative materials) in the frequency domain can be written as

$$D_{11} \frac{\partial^4 u_3}{\partial x_1^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 u_3}{\partial x_1^2 \partial x_2^2} + D_{22} \frac{\partial^4 u_3}{\partial x_2^4} - 2\rho\omega^2 h u_3 = F(\omega) \delta(x) \delta(y) \quad (27)$$

where $D_{ij} = 2/3 C_{ij} h^3$ and the constants C_{ij} have been modified to satisfy the plate theory assumptions⁽⁷⁾. Solution to this equation for a simply supported rectangular ($a \times b$) plate with support along and across the fiber directions can be written as

$$u_3 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{mn} \sin(m\pi x_1/a)}{D_{11}(m\pi/a)^4 + 2(D_{12} + 2D_{66})(m\pi/a)^2} * \frac{\sin(n\pi x_2/b)}{(n\pi/b)^2 + D_{22}(n\pi/b)^4 - 2\rho h \omega^2} \quad (28)$$

For a concentrated load applied at (α, β)

$$a_{mn} = \frac{4F(\omega)}{ab} \sin(m\pi\alpha/a) \sin(n\pi\beta/b) \quad (29)$$

In Eq. (27), the effects of transverse shear stress and rotatory inertia have been neglected. If we include these effects, the equations of motion change to the coupled set

$$-2\rho h \omega^2 u_3^b - 2\rho h u_3^s - L_2 u_3^s = F(\omega) \delta(x) \delta(y) \quad (30 \cdot a)$$

$$(L_1 + \phi \omega^2 J_2) u_3^b + L_2 u_3^s = 0 \quad (30 \cdot b)$$

where

$$\begin{aligned} L_1 &= D_{11} \frac{\partial^4}{\partial x_1^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4}{\partial x_1^2 \partial x_2^2} + D_{22} \frac{\partial^4}{\partial x_2^4} \\ L_2 &= D_{44} \frac{\partial^2}{\partial x_1^2} + D_{55} \frac{\partial^2}{\partial x_2^2} \\ \phi &= 2/3 \rho h^3 \\ D_{44} &= 5/3 h C_{44} \\ D_{55} &= 5/3 h C_{55} \\ u_3 &= u_3^b + u_3^s \end{aligned} \quad (31)$$

u_3^b is the bending component of total displacements, and u_3^s is the shear component of total displacements. The solution to the coupled Eq. (30) can be written as

$$(u_3^b, u_3^s) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn}^b, A_{mn}^s) \sin(m\pi x_1/a) \times \sin(n\pi x_2/b) \quad (32)$$

where

$$\begin{aligned} A_{mn}^b &= \frac{a_{mn} E_{mn}}{2\rho h [\omega^4 F_{mn} - (b_{mn} + E_{mn} + E_{mn} F_{mn}) \omega^2 + E_{mn} b_{mn}]} \\ A_{mn}^s &= \frac{b_{mn} - \omega^2 F_{mn}}{E_{mn}} \\ b_{mn} &= \pi^4 / (2\rho h) [D_{11} m^4 / a^4 + 2(D_{12} + 2D_{66}) m^2 n^2 / (a_2 b_2) + D_{22} n^4 / b^4] \\ E_{mn} &= \pi^2 / (2\rho h) [D_{44} m^2 / a^2 + D_{55} n^2 / b^2] \\ F_{mn} &= h^2 \pi^2 / 3 (m^2 / a^2 + n^2 / b^2) \end{aligned} \quad (33)$$

Once u_3 is calculated, the stresses and displacements at any point can be obtained by using the well known stress-strain and strain-displacement relations⁽¹⁹⁾.

Fig. 2 The source function (top row) and its Fourier transform (bottom row)

Fig. 3 Surface displacement of a 1 mm thick plate at $P(10, 10, 0)$ due to a normal load ($\tau=4 \mu\text{sec}$, $A=1 \text{ KN/mm}^2$) at the origin $0(0, 0, 0)$. Left column shows the spectra and right column the time histories. Top row shows the "exact" displacements and second row shows its far-field approximation for an infinite plate. Third row shows the displacement in a 50 mm by 50 mm plate for the same source and station by the classical plate theory and bottom row shows the displacement by modified plate theory that includes the transverse shear effect and rotatory inertia.

6. Numerical Results

The methods discussed above have been implemented in a FORTRAN program. Results for a few sample problems are presented here. The loading function is assumed to be of the form

$$f(t) = \begin{cases} 0 & t \leq T \\ A \sin(2\pi(t-T)/\tau) & T \leq t \leq T + \tau \\ 0 & t \geq T + \tau \end{cases} \quad (34)$$

In Eq. (34), A defines the peak value of the load, τ is the duration of the load and T is the time of load

application. The sharpness of the load can be increased by decreasing τ but keeping A constant. The delay, T , is introduced to examine the nature and extent of numerically generated noise before the arrival of the source generated disturbance at the receiver.

In all the results presented here, the peak value of the load, A , is taken as 1 KN/mm^2 , the duration of the load, τ , is taken as $4 \mu\text{sec}$ and the time of loading T is taken as $5 \mu\text{sec}$. The source $f(t)$ and its Fourier transform $F(\omega)$ are plotted in Fig. 2. The material properties of a typical Graphite/Epoxy composite are

Fig. 4 Surface displacement in a 10 mm thick plate at $P(10, 10, 0)$ due to a normal impact ($\tau = 4 \mu\text{sec}$, $A = 1 \text{ KN/mm}^2$) at the origin $O(0, 0, 0)$. Left column shows the spectra and right column shows the time histories. Top row shows the "exact" displacements for an infinite plate. Middle row shows the displacement of a 50 mm by 50 mm simply supported plate by classical plate theory. Bottom row shows the displacements for the same simply supported plate by modified plate theory that includes the transverse shear effect and rotatory inertia.

taken to be those used by Mal et al.,⁽¹⁶⁾

$$c_{11}=160.73 \text{ GPa}, c_{12}=6.44 \text{ GPa}, c_{22}=13.92 \text{ GPa}, \\ c_{23}=6.92 \text{ GPa}, c_{55}=7.07 \text{ GPa}, \rho=1.578 \text{ gm/cc}, \\ \rho_0=0.01, a_0=0.45 \text{ and } f_0=1.0 \text{ MHz}$$

Figure 3 shows the surface displacements at point $P(10, 10, 0)$ due to a normal load at the origin $O(0, 0, 0)$ on a 1 mm thick plate. The left column shows the spectra while the right column shows the time histories. The top row shows the "exact" displacement and the second row shows its far-field approximation. A number of pulses due to the body waves appearing in the exact solution does not show up in the approximate solution. This means that at about fourteen times the thickness of the plate, the displacement in the plate still includes significant contributions from the body waves. However, agreement between the approximate and exact solution is excellent for the guided wave pulses, considering the fact that in the far-field approximation, the material is considered to be elastic, while in the exact solution they are viscoelastic. The third row of Fig. 3 shows the surface displacements of a 50 mm by 50 mm simply supported plate calculated by the classical plate theory and the bottom row shows the displacement for the same plate obtained by a modified classical plate theory including the effect of transverse shear and

rotatory inertia. Theoretically, these two approaches should give the same displacements as the "exact" solution before reflected waves from the boundary of the finite plate arrive at the field point. The time of arrival of the first reflected wave in the present case is approximately $9.3 \mu\text{sec}$ and the results presented in the bottom row shows good agreement up to this time. However, there is some disagreement between the results in the third row and the "exact" solution before the arrival of the first reflected wave.

The classical plate theory is based on the Kirchhoff hypothesis, which neglects transverse shear deformation effects. This is a reasonable assumption for homogeneous isotropic thin plates. However, these effects are significant in the case of laminated composites, due to their relatively low transverse shear modulus. Thus, the effects of transverse shear deformation should be included in the approximate solution for the plate. It should also be noted that although time histories by different approaches compare well with each other, their spectra show substantial dissimilarities (Fig. 3).

Figure 4 shows the surface displacement at the point $P(10, 10, 0)$ due to a normal impact at the origin $O(0, 0, 0)$ on a 10 mm thick composite plate. The top row shows the displacement for an infinite plate, while

Fig. 5 Surface displacements at $P(10, 10, 0)$ of a half-space due to an impact load ($\tau=4 \mu\text{sec}$, $A=1 \text{ KN/mm}^2$) applied at the origin $O(0, 0, 0)$. Top row shows the "exact" solution and bottom row shows the far-field approximations.

the middle and bottom rows show the displacement for a 50 mm by 50 mm, simply supported plate calculated by means of the classical plate theory and modified classical plate theory including the effect of transverse shear and rotatory inertia. Theoretically upto 9.3 μ sec, the two solutions are supposed to give the same results. Although the shape of the displacement time history in the first and third row are similar, their magnitudes differ; the difference is more pronounced in the results presented in the first and second rows of Fig. 4.

Figure 5 shows the surface displacements in a half-space at a point $P(10, 10, 0)$ between "exact" solution (top row) and the far-field approximation (bottom row) for a normal impact at the origin $O(0, 0, 0)$. Again, the far-field approximation gives reasonably good agreement with the "exact" solution in the Rayleigh wave regime. The first pulse in the "exact" solution is due to the longitudinal wave motion. Since the far-field approximation considers only surface wave contributions, the surface displacement due to the longitudinal wave is absent in the approximate solution. The distinctive feature of the far-field solution is the presence of multiple Rayleigh waves. For wave propagation along and across the fibers there is only one Rayleigh wave, as in isotropic materials. But for waves propagating along other directions, three distinct Rayleigh waves have been found to exist⁽⁴⁾.

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