Dynamic Thermal Stresses in Composite Hollow Cylinders and Spheres Caused by Sudden Heating* 

Naobumi SUMI** and Yoshimoto ITO***

A numerical method is developed to determine transient, uncoupled dynamic thermal stresses in composite hollow cylinders and spheres subjected to arbitrary prescribed symmetric temperature fields. The numerical procedure employs appropriate characteristic relationships on boundaries and on interfaces between media with different material properties while using more convenient explicit finite difference approximations at all other points. Numerical examples are given for stress wave propagation in two-layered composite hollow cylinders and spheres subjected to sudden heating on the inner boundary, and the results clearly show the propagation and reflection of discontinuities in stresses and velocities.

**Key Words**: Thermal Stress, Wave Propagation, Composite Material, Characteristic Method, Finite Difference Method

1. Introduction

The conventional quasi-static thermoelastic problems in the presence of time-dependent temperature fields rest on the assumption that the inertia terms may be neglected in the equations of equilibrium. On the other hand, in metallic components of high-power microwave pulse generators, and in power-generating equipment such as the fast nuclear reactor, one has encountered extremely higher and higher rates of heating, and it becomes necessary to reexamine the role of inertia. We find that several investigations of the role of inertia in dynamic thermoelastic problems have been conducted. Infinite and semi-infinite bodies subjected to sudden heating have been considered by Danilovskaya(1),(2), Mura(3), Sternberg and Chakravorty(4), and Puri(5). The dynamic thermoelastic problems of an infinite medium with a spherical cavity and a hollow sphere, whose surface temperature is suddenly changed, have also been examined by Tsui and Kraus(6), respectively. Similar investigations of a suddenly heated infinitely long cylinder and infinite medium with a cylindrical hole were reported by Mura(3), and Dhalliwal and Chowdhury(7).

In general, the dynamic thermoelastic problems have been attacked almost exclusively using integral transform techniques. The method usually involves inversion difficulty, and numerical integration is often required to evaluate the resulting inversion integrals. The present study of the dynamic thermal stresses in composite hollow cylinders and spheres caused by sudden heating employs appropriate characteristic relationships on boundaries and interfaces while using more convenient explicit finite difference approximations at all other points. The numerical solutions of the specific examples show excellent agreement with existing solutions obtained by other methods.

2. Governing Equations

We consider a two-layered composite hollow cylinder or sphere of inner radius $R_1$ and outer radius $R_2$...
\( R_2 \) of which \( R_1 < r < R_2 \) for one medium and \( R_2 < r < R_3 \) for another (see Fig. 1). We assume that the body is suddenly subjected to a uniformly applied step change in temperature on its inner boundary while its outer boundary is maintained at a constant temperature.

The unsteady heat conduction equation in cylindrical or spherical coordinates for the cases of a temperature function that is independent of angular position and varies in radial direction only is

\[
\frac{\partial^2 T}{\partial r^2} + \frac{N}{r} \frac{\partial T}{\partial r} = \frac{1}{\partial t} \frac{\partial T}{\partial t},
\]

where \( T(r, t) \) is the temperature; \( r \) is the coordinate; \( t \) is the thermal diffusivity. \( N \) is a constant, with values of one or two, corresponding to the cylindrical or spherical coordinate, respectively.

The initial condition for the temperature is taken as

\[
T(r, 0) = 0, \quad (R_1 < r < R_3),
\]

while the boundary conditions are

\[
\begin{align*}
T(R_1, t) &= T_1, \quad (0 \leq t) \quad (3) \\
T(R_3, t) &= 0, \quad (0 \leq t),
\end{align*}
\]

where \( T_1 \) is a constant temperature suddenly assumed by the inner boundary. If there is no contact resistance at the interface \( r = R_2 \) between two layers, the interface conditions for the temperature are

\[
T_1 = T_2,
\]

\[
\lambda \left( \frac{\partial T}{\partial r} \right)_1 = \lambda \left( \frac{\partial T}{\partial r} \right)_2,
\]

where the subscripts 1 and 2 refer to the inner and outer layers, respectively.

The equations that govern the propagation of one-dimensional, thermoelastic dilatational waves, including cylindrical and spherical waves, are

\[
\rho \frac{\partial u}{\partial t} = \frac{\partial \sigma_r}{\partial r} + N \frac{\sigma_r - \sigma_\theta}{r}
\]

\[
\frac{\partial \sigma_r}{\partial t} = E \left\{ \frac{1 - (N - 1)\nu^*}{r} \frac{\partial v}{\partial r} + N \nu^* \frac{v}{r} \right\}
\]

\[
- \alpha^* \frac{\partial E^*}{\partial t}
\]

\[
\frac{\partial \sigma_\theta}{\partial t} = E \left\{ \frac{1}{r} \frac{\partial \sigma_\theta}{\partial r} \right\} - \alpha^* \frac{\partial E^*}{\partial t},
\]

which \( v = \partial u/\partial t \), and

\[
E = \frac{E^*}{(1 + \nu^*)(1 - N\nu^*)}.
\]

In the above equations, \( u \) is the displacement; \( v \) is the velocity; \( \sigma_r \) and \( \sigma_\theta \) are the normal stresses; \( \rho \) is density; \( E^* \), \( \nu^* \) and \( \alpha^* \) are generalized thermoelastic constants to be defined later.

For cylindrical waves \((N = 1), E^*, \nu^*, \alpha^* \) and the wave propagation velocity \( c \) assume two sets of values corresponding to the geometry of the problem

\( (1) \) For plane stress:

\[
E^* = E, \quad \nu^* = \nu, \quad \alpha^* = \alpha, \quad c^2 = E/\rho(1 - \nu^2).
\]

\( (2) \) For plane strain:

\[
E^* = E/(1 - \nu^2), \quad \nu^* = \nu/(1 - \nu), \quad \alpha^* = (1 + \nu)\alpha, \quad c^2 = (1 - \nu)E/\rho(1 + \nu)(1 - 2\nu).
\]

For spherical waves \((N = 2), \) we have

\[
E^* = E, \quad \nu^* = \nu, \quad \alpha^* = \alpha, \quad c^2 = (1 - \nu)E/\rho(1 + \nu)(1 - 2\nu)
\]

where \( E \) is the modulus of elasticity; \( \nu \) is Poisson's ratio; \( \alpha \) is the coefficient of linear thermal expansion.

We stipulate that the medium be initially at rest and that the surfaces of the medium be free of stress. Then the initial and the boundary conditions for the stress distribution are

\[
\begin{align*}
\sigma_r(r, 0) &= 0, \quad (R_1 < r < R_3) \quad (9) \\
\sigma_r(R_1, t) &= 0, \quad (0 \leq t) \quad (10) \\
\sigma_r(R_3, t) &= 0, \quad (0 \leq t).
\end{align*}
\]

For the points on the interface \( r = R_2 \) between two layers, the values of \( \sigma_r \) and \( v \) must be the same to satisfy the conditions of continuity and equilibrium. Therefore, the interface conditions for the stresses and velocities are

\[
\begin{align*}
\sigma_r &= \sigma_r \quad (9) \\
v &= v \quad (10)
\end{align*}
\]

### 3. Characteristic Equations

Equations \((5) - (7)\) represent a hyperbolic system that permits the propagation of abrupt wavefronts. These equations contain derivatives with respect to both space and time variables. For the system of Eqs. \((5) - (7), \) the characteristic lines in the space-time plane along which the stresses and velocities are related by total derivatives are \( \alpha^* \)

\[
I^* : \frac{dr}{dt} = + c
\]

\[
I^- : \frac{dr}{dt} = - c
\]

\[
II : \frac{dr}{dt} = 0
\]

The corresponding characteristic equations along \( I^*, \) \( I^- \), and \( II \) are

\[
I^* : \frac{d\sigma_r}{c} = \frac{pc\nu^*}{(1 - (N - 1)\nu^*)} \frac{dr}{c}
\]

\[
I^- : \frac{d\sigma_r}{c} = \frac{pc\nu^*}{(1 - (N - 1)\nu^*)} \frac{dr}{c}
\]

\[
II : \nu^* \frac{d\sigma_r}{c} = \frac{pc\nu^*}{(1 - (N - 1)\nu^*)} \frac{dr}{c}
\]

### 4. Numerical Procedure

The present study of the dynamic thermal stresses in composite hollow cylinders and spheres caused
by sudden heating on the inner boundary employs a technique which combines some of the advantages of both the characteristic and finite difference methods. The method employs appropriate characteristic relationships on boundaries and interfaces while using more convenient explicit finite difference approximations at all other points.

A typical two-layer finite difference and characteristic mesh in the space-time plane is shown in Fig. 1, where layer 1 is of one medium and layer 2 is of another. At a typical interior point B, the governing equations (5)–(7) can be written in the explicit finite difference forms by replacing the derivatives with respect to space and time with central differences. Then, the three quantities \(\sigma_{\text{ep}}, \sigma_{\text{ev}}\) and \(\nu_{\text{p}}\) at point D can be calculated if all stresses and velocities at three neighboring points A, C and E are known.

The conditions along the boundaries of the medium may be obtained from the characteristic equations (15)–(17). For example, if the values of either \(\sigma_{\text{e}}\) or \(\nu_{\text{e}}\) are prescribed at point d on the inner boundary \(r=R_0\), then the remaining two variables may be determined from two equations along I (from points c to d) and II (from points e to d) characteristics. On the outer boundary \(r=R_s\), the relationships along I (from points f to k) and II (from points j to k) are used to find the appropriate quantities at point k.

The conditions on the interface between two layers may be obtained as follows. We assume that the II characteristic passing through points p and m on the interface \(r=R_2\) separates the first layer from the second, as shown in Fig. 1. To satisfy the conditions of compatibility and equilibrium, the values of \(\sigma_r\) and \(\nu\) at the interface must be the same for layers 1 and 2. In general, the values of \(\sigma_r\) on either side of the interface will be different. Therefore, two equations along I (from points n to p) and II (from points m to p) for layer 1, and two equations along I (from points f to p) and II (from points m to p) for layer 2 may be solved simultaneously to yield the current four values of \(\sigma_{\text{ep}}, (\sigma_{\text{ev}})_1, (\sigma_{\text{ev}})_2, \nu_{\text{p}}\) at point p.

In the theoretical analysis of the unsteady heat conduction equation, it is conventional to use the Laplace transform method, but this method usually involves inversion difficulty. Therefore, in this paper, to cover the widest range of problems, the heat conduction equation (11) is developed in the finite difference form by replacing the derivatives with respect to space and time with central differences and then replacing the values of the dependent variable at point B by the average value over points A and C. Then the temperature at point D can be calculated if all the temperatures at three neighboring points A, C and E are known.

5. Numerical Results

In this paper, a set of unified equations is presented, which can be applicable to the dynamic thermoelastic problems of composite hollow cylinders and spheres by setting \(N=1\) and 2, respectively. The specific numerical solutions of an infinite medium with a spherical cavity and a hollow sphere, whose surface temperature is suddenly changed, show excellent agreement with existing solutions obtained by Sternberg and Chakravorty and Tsui and Kraus, respectively. However, in this paper, numerical results are given only for cylindrical waves (plane strain).

In presenting the results, we define dimensionless space and time variables by means of

\[
\xi = rR_0, \quad \eta = xtR_0^2.
\]

(18)

Dimensionless temperature and components of stress as well as displacement are defined through

\[
\bar{T} = \frac{T}{T_1}, \quad \bar{\sigma} = \frac{1 - (N - 1)\nu^*}{\alpha^*E^*T_1} \sigma, \quad \bar{\nu} = \frac{1 - (N - 1)\nu^*}{(1 + \nu^*)\alpha^*T_1x_1} \nu,
\]

(19)

where the subscript 1 on the generalized thermoelastic constants \(E^*, \nu^*, \alpha^* \) and \(x_1\) indicates reference properties of inner layer 1.

Numerical calculations are carried out for various two-layered hollow cylinders, and the physical thermoelastic constants are listed in Table 1. The single-layered material I is included for comparison.
In all calculations, Poisson's ratio $\nu$ is assumed to be $\nu=1/3$, and the inertia parameter $\gamma$ defined by

$$\gamma = \xi / \alpha \cdot R_1$$

(21)
is selected for an unrealistically large value of $\gamma=1/5$, following the lead of previous investigators in order to bring out the nature of the inertia effect. The stability of numerical calculations is insured by setting $\Delta \gamma = \Delta t / (\xi \Delta \eta)$, and the results are made to converge by reducing the mesh size appropriately. A mesh size of $\Delta \eta = 0.001$ is used in the solutions of the majority of these examples. Hence the mesh size in the radial direction $\Delta \xi = \Delta \eta / \gamma$ is different in each layer with different material properties of a multilayered body.

For a single-layered material I of radius ratio $R_2/R_1=5$, the variations of $\sigma_\alpha$ for various values of $\xi$ are shown in Fig. 2. The stress $\sigma_\alpha$ at the surfaces $\xi = 1, 5$ is continuous and is seen to undergo a pronounced oscillation. The stress at the inner points $\xi = 2, 3, 4$ displays jump discontinuity at the wave-fronts. The propagation and reflection of thermal stress waves are clearly shown in the figure. It should be noted that this is a consequence of using the characteristic relationships on the inner and outer boundaries. For the composite material II of radius ratios $R_2/R_1=3$, and $R_2/R_1=5$, the variations of $\sigma_\alpha$ at $\xi = 1, 2, 3$ for the inner layer and $\xi = 3, 4, 5$ for the outer layer are shown in Figs. 3 and 4, respectively.

For composite materials I~V of radius ratios $R_2/R_1=1.5$, and $R_2/R_1=2$, the variations $\sigma_\alpha$ at $\xi = 1$ are shown in Fig. 5. The variations of $\sigma_\alpha$ at $\xi = 1.5$ on either side of the interface are shown in Figs. 6~8. The circumferential stress on either side of the interface for materials II and III are not equal because of differing elastic modulus $E$ or coefficient of thermal expansion $\alpha$ in the two layers.

Table 1 Physical thermoelastic constants

<table>
<thead>
<tr>
<th>Material</th>
<th>$E_2/E_1$</th>
<th>$\alpha_2/\alpha_1$</th>
<th>$\lambda_2/\lambda_1$</th>
<th>$\kappa_2/\kappa_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>III</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>IV</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>V</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

Fig. 3 Variations $\sigma_\alpha$ at $\xi = 1, 2, 3$ for inner layer of material II ($R_2/R_1=3, R_2/R_1=5, \gamma=1/5, \nu=1/3$)

Fig. 2 Variations $\sigma_\alpha$ for various values of $\xi$ for material I ($R_2/R_1=5, \gamma=1/5, \nu=1/3$)

Fig. 4 Variations $\sigma_\alpha$ at $\xi = 3, 4, 5$ for outer layer of material II ($R_2/R_1=3, R_2/R_1=5, \gamma=1/5, \nu=1/3$)
6. Conclusions

In this paper, the method of characteristic difference has been applied successfully to the propagation and reflection of thermal stress waves in two-layered composite hollow cylinders and spheres subjected to sudden change in temperature on their inner surface. The method employs characteristic relationships on boundaries and on interfaces while using explicit finite difference approximations at all other points. For problems involving multiple wave reflections, the method is believed to be advantageous, because it is not necessary to know the location of wavefronts, and reflections are performed automatically. Therefore, the concepts involved may easily be extended to as many layers as desired.

References


