SH Surface Waves in Functionally Gradient Piezoelectric Plates*

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The numerical method which has been proposed by the present authors for wave propagation analysis in functionally gradient piezoelectric material (FGPM) plates is used to investigate SH waves in FGPM plates. The material properties of the plate change gradually with the thickness of the plate. To evaluate the properties of the energy transfer between the electrical and mechanical energies during acoustic wave propagation in dispersive media, an electromechanical coupling constant is defined on the basis of group velocities of the acoustic waves. Numerical examples are given for a hypothetical FGPM plate and a homogeneous plate. It is found that there are SH surface waves propagating on the softer surface of the FGPM plate. Furthermore, the electromechanical coupling constant based on group velocities is found to be more appropriate than that based on phase velocities for waves in dispersive media.

Key Words: SH Surface Wave, Functionally Gradient Material, Piezoelectric Plate, New Electromechanical Coupling Constant

1. Introduction

Recently, various techniques have been developed to produce functionally gradient material (FGM) plates, the material properties of which have a gradient in the thickness direction⁽¹⁾. The present authors investigated the waves in functionally gradient piezoelectric material (FGPM) plates⁽²⁾ and clarified that SH waves propagate as surface waves on the softer surface of the FGPM plate.

In this paper, SH waves in a functionally gradient piezoelectric material plate made of a hexagonal system of crystallization are investigated by means of the numerical method proposed previously by the present authors. It is indicated that an electromechanical coupling constant based on group velocities should be used for waves in dispersive media, instead of a constant based on phase velocities.

2. Formulation

Consider a FGPM plate with thickness H and a coordinate system as shown in Fig.1. The plate is divided into N plate elements in the thickness direction. The thickness of the n th element is denoted by h_n . The material properties of the FGPM plate are assumed to be independent of the y direction, hence a

Fig. 1 A FGAM plate and an n th isolated plate element

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two-dimensional problem is considered. The displacement v in the y direction can be separated from the other displacements coinciding the z axis with an axis of polarization. With each element, the constitutive relationships are given in the matrix form as follows:

$$\boldsymbol{\sigma} = \mathbf{c}(z)\boldsymbol{\varepsilon} - \mathbf{e}(z)\mathbf{E} \quad (1)$$

$$\mathbf{D} = \mathbf{g}(z)\mathbf{E} + \mathbf{e}(z)\boldsymbol{\varepsilon}, \quad (2)$$

where $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\mathbf{D}$ and $\mathbf{E}$ are the stress, strain, electric displacement and electric field vectors, respectively, while $\mathbf{c}(z)$, $\mathbf{e}(z)$ and $\mathbf{g}(z)$ are the elastic, piezoelectric and dielectric constants and are functions of z for FGPM plates.

$$\begin{aligned} \boldsymbol{\sigma}^T &= \{\sigma_{yz}, \sigma_{xy}\}, \quad \boldsymbol{\varepsilon}^T = \{\varepsilon_{yz}, \varepsilon_{xy}\}, \\ \mathbf{D}^T &= \{D_x, D_z\}, \quad \mathbf{E}^T = \{E_x, E_z\}, \end{aligned} \quad (3)$$

where the superscript T stands for the transposed matrix. The relationship between the displacement v and strain vector $\boldsymbol{\varepsilon}$ is given by

$$\boldsymbol{\varepsilon} = \mathbf{L}_d v, \quad (4)$$

where

$$\mathbf{L}_d^T = \left\{ \frac{\partial}{\partial z}, \frac{\partial}{\partial x} \right\}. \quad (5)$$

The Maxwell equations in the quasi-static approximation are

$$\mathbf{E} = -\text{grad } \psi = -\mathbf{L}_\psi \psi \quad (6)$$

$$\text{div } \mathbf{D} = 0, \quad (7)$$

where ψ is the electrostatic potential, while $\mathbf{L}_\psi$ is given by

$$\mathbf{L}_\psi^T = \left\{ \frac{\partial}{\partial z}, \frac{\partial}{\partial x} \right\}. \quad (8)$$

The equation of motion is given by

$$\rho(z)\ddot{v} - \mathbf{L}_d^T \boldsymbol{\sigma} = 0, \quad (9)$$

where a dot denotes differentiation with respect to time t , while $\rho(z)$ is the density of the material and a function of z .

3. Analysis

We approximate the displacement v and the electrostatic potential ψ in an element as

$$v(x, z, t) = \mathbf{N}(z)\mathbf{d} \exp i(kx - \omega t) \quad (10)$$

$$\psi(x, z, t) = \mathbf{N}(z)\boldsymbol{\phi} \exp i(kx - \omega t), \quad (11)$$

where $\mathbf{d}$ and $\boldsymbol{\phi}$ are the nodal displacement and electrostatic potential vectors, respectively, and k and ω are the wave number and angular frequency, respectively, while i is an imaginary unit and $\mathbf{N}$ is the shape function given by

$$\mathbf{N} = [(1 - 3\bar{z} + 2\bar{z}^2), 4(\bar{z} - \bar{z}^2), (2\bar{z}^2 - \bar{z})], \quad (12)$$

in which $\bar{z} = z/h_n$.

The functional of the present coupled problem can be given by

$$\begin{aligned} L &= \frac{1}{2} \int_0^{h_n} \boldsymbol{\varepsilon}^+ \boldsymbol{\sigma} dz - \frac{1}{2} \int_0^{h_n} \mathbf{E}^+ \mathbf{D} dz \\ &\quad + \frac{1}{2} \int_0^{h_n} \rho_n v^+ \ddot{v} dz, \end{aligned} \quad (13)$$

where a superscript $+$ indicates the complex conjugate of the transposed matrix.

Using Eqs. (1), (2), (4), (6), (10) and (11), applying the variational principle to Eq. (13) and assembling the matrices of adjacence elements, we obtain the following characteristic equation.

$$\mathbf{A}\mathbf{b} - \omega^2 \mathbf{M}\mathbf{b} = 0 \quad (14)$$

$$\mathbf{A} = \mathbf{K} + \mathbf{C}\mathbf{G}^{-1}\mathbf{C}^+, \quad (15)$$

where $\mathbf{K}$, $\mathbf{C}$, $\mathbf{G}$ and $\mathbf{M}$ are the rigidity, electromechanical coupling, capacitance and mass matrices, respectively, and

$$\mathbf{b}^T = \{\mathbf{d}^T, \boldsymbol{\phi}^T\}. \quad (16)$$

We can obtain eigenvalues and the corresponding eigenvectors by solving equation (14). The Rayleigh quotient for the m th eigenvector $\boldsymbol{\Psi}_m$ can be written as

$$\omega_m^2 = \frac{\boldsymbol{\Psi}_m^L \mathbf{A} \boldsymbol{\Psi}_m^R}{\mathbf{M}_m}, \quad (17)$$

where superscripts L and R stand for the left and right eigenvectors, respectively, and $\boldsymbol{\Psi}_m^L = (\boldsymbol{\Psi}_m^R)^+$, because $\mathbf{A}$ is a Hermitian matrix. From Eq. (16), the left and right eigenvectors can be expressed by

$$\boldsymbol{\Psi}_m^L = \{\mathbf{A}_m^L \boldsymbol{\Phi}_m^L\}, \quad (\boldsymbol{\Psi}_m^R)^T = \{(\mathbf{A}_m^R)^T (\boldsymbol{\Phi}_m^R)^T\}, \quad (18)$$

where $\mathbf{A}_m$ and $\boldsymbol{\Phi}_m$ are the m th displacement and electrostatic potential eigenvectors. The equivalent mass of the m th mode is given by

$$\mathbf{M}_m = \boldsymbol{\Psi}_m^L \mathbf{M} \boldsymbol{\Psi}_m^R = \mathbf{A}_m^L \mathbf{M} \mathbf{A}_m^R. \quad (19)$$

From the eigenvalues, the phase velocity c_{pm} is given by

$$c_{pm} = \frac{\omega_m}{k}. \quad (20)$$

Taking the derivative of Eq. (17) with respect to k using the orthogonality condition of the left and right eigenvectors, and then using Eqs. (15) and (17), we can express the group velocity of the m th mode c_{gm} as

$$c_{gm} = \frac{d\omega_m}{dk} = \frac{\boldsymbol{\Psi}_m^L \mathbf{A}_k \boldsymbol{\Psi}_m^R}{2\omega_m \mathbf{M}_m}. \quad (21)$$

Using Eqs. (14), (15), (16) and (18), we rewrite Eq. (21) as follows.

$$c_{gm} = \frac{\mathbf{A}_m^L \mathbf{M}_{,k} \mathbf{A}_m^R + 2\text{Re}(\mathbf{A}_m^L \mathbf{C}_{,k} \boldsymbol{\Phi}_m^R) - \boldsymbol{\Phi}_m^L \mathbf{G}_{,k} \boldsymbol{\Phi}_m^R}{2\omega_m \mathbf{M}_m}, \quad (22)$$

where $\mathbf{M}_{,k} = \partial \mathbf{M}(k)/\partial k$, $\mathbf{C}_{,k} = \partial \mathbf{C}(k)/\partial k$ and $\mathbf{G}_{,k} = \partial \mathbf{G}(k)/\partial k$.

The electromechanical coupling constant of the m th mode is given by

$$K_{pm}^2 = \frac{2|c_{pm}^2(\text{open}) - c_{pm}^2(\text{short})|}{c_{pm}^2(\text{open})}, \quad (23)$$

where $c_{pm}(\text{open})$ and $c_{pm}(\text{short})$ are the phase velocities of the m th mode obtained using the electrically opened and shorted boundary conditions, respectively. Equation (23) based on the phase velocity can be used for evaluation of the transfer between the electrical and mechanical energies during acoustic wave propagation in infinite nondispersive media, because energy in this media propagates at the phase velocity. However, Eq. (23) cannot be used for the wave propaga-

tion in dispersive media with boundary, because the energy in this media propagates at the group velocity. Therefore, we propose an electromechanical coupling constant based on the group velocity as follows.

$$K_{gm}^2 = \frac{|c_{gm}^2(open) - c_{gm}^2(short)|}{c_{gm}^2(open)} \quad (24)$$

4. Numerical Results

Numerical calculations are performed for a homogeneous piezoelectric plate and a FGPM plate. The piezoelectric material is made of a hexagonal system of crystallization with the following material properties.

$$\begin{aligned} \mathbf{c} &= \begin{bmatrix} 9.070 & 5.100 & 5.100 & 0.000 & 0.000 & 0.000 \\ 5.100 & 9.380 & 5.100 & 0.000 & 0.000 & 0.000 \\ 5.100 & 5.100 & 9.070 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 1.504 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.968 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 1.504 \end{bmatrix} \\ &\times 10^{11} (\text{N/m}^2) \\ \mathbf{e} &= \begin{bmatrix} 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.21 \\ -0.244 & 0.44 & -0.244 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & -0.21 & 0.00 & 0.00 \end{bmatrix} \\ &\times (\text{C/m}^2) \\ \mathbf{g} &= \begin{bmatrix} 9.02 & 0.00 & 0.00 \\ 0.00 & 9.53 & 0.00 \\ 0.00 & 0.00 & 9.02 \end{bmatrix} \times \varepsilon_0, \rho_0 = 4.82 \text{ kg/m}^3 \end{aligned} \quad (25)$$

The material properties on the upper surface ($z=H$) of the FGPM plate are the same as those of the

Fig. 2 Wave modes of the FGPM plate: $kH=25.0$

Fig. 3 Wave modes of the homogeneous plate: $kH=25.0$

abovementioned piezoelectric materials.

$$c_{44}^U = c_{44}, e_{16}^U = e_{16}, g_{11}^U = g_{11}, \rho^U = \rho_0 \quad (26)$$

The material properties on the lower surface ($z=0$) of the FGPN plate are assumed to be

$$c_{44}^L = 12c_{44}, e_{16}^L = 0.1e_{16}, g_{11}^L = 0.5g_{11}, \rho^L = 1.8\rho_0. \quad (27)$$

In the thickness direction, the material properties vary as

$$\begin{aligned} c_{44}(z) &= c_{44}^U + (c_{44}^L - c_{44}^U)(1 - z/H)^2 \\ e_{16}(z) &= e_{16}^U + (e_{16}^L - e_{16}^U)(1 - z/H)^2 \\ g_{11}(z) &= g_{11}^U + (g_{11}^L - g_{11}^U)(1 - z/H)^2 \\ \rho(z) &= \rho^U + (\rho^L - \rho^U)(1 - z/H)^2. \end{aligned} \quad (28)$$

Figures 2 and 3 show the modes of waves. It can be seen from these figures that the lower modes of the FGPM plate are SH surface waves. This is because the upper surface of the FGPM plate is softer than the lower surface and apt to deform.

Figures 4 and 5 show the frequency spectra. Figures 6 and 7 show the electromechanical coupling constants based on the phase velocity. Figures 8 and 9 show the electromechanical coupling constants based on the group velocity. From these figures, it can be seen that the electromechanical coupling constants of

Fig. 4 Frequency spectrum of the FGPM plate

Fig. 5 Frequency spectrum of the homogeneous plate

Fig. 6 Electromechanical coupling constants of the FGPM plate based on the phase velocity

Fig. 8 Electromechanical coupling constants of the FGPM plate based on the group velocity

Fig. 7 Electromechanical coupling constants of the homogeneous plate based on the phase velocity

Fig. 9 Electromechanical coupling constants of the homogeneous plate based on the group velocity

the FGPM plate are larger than those of the homogeneous plate for a large wave number and that K_{gm}^2 differs very much from K_{pm}^2 .

5. Conclusion

The SH waves in the FGPM plate are investigated using the numerical method proposed by the present authors. The following results are obtained; (1) The SH wave in the FGPM plate gathers on the softer surface. This tendency is marked in the SH waves with lower modes. (2) The electromechanical coupling constant based on the group velocity should be used to evaluate the energy transfer in dispersive

media.

References

- (1) Watanabe, R. and Kawasaki, A., Overall View of the P/M Fabrication of Functionally Gradient Materials, Proc. First Int. Symp. on Functionally Gradient Materials, Sendai, (1990), p. 107.
- (2) Liu, G. -R. and Tani, J., Characteristics of Wave Propagation in Functionally Gradient Piezoelectric Material Plates and Its Response Analysis (part 1: Theory, part 2: Calculation Results), Trans. Jpn. Soc. Mech. Eng., (in Japanese), Ser. A, Vol. 541, (1991), p. 180.