Dynamic Singular Stresses in Cracked Glass-Fiber Reinforced Plastics at Low Temperatures*

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In this paper, we consider the dynamic singular stresses in G-10CR glass-epoxy laminates with a crack at low temperatures. Laplace and Fourier transforms are used to reduce the transient problem to the solution of a pair of dual integral equations in the Laplace transform plane. The solution of the dual integral equations is then expressed in terms of a Fredholm integral equation of the second kind. A numerical Laplace inversion routine is used to recover the time dependence of the solution. Numerical results on the dynamic stress intensity factor at low temperatures are obtained and presented in a graphic form.

Key Words: Elasticity, Fracture Mechanics, Composite Materials, Stress Intensity Factor, Reinforced Plastics, Low Temperatures, Integral Transforms, Impact Response

1. Introduction

G-10CR and G-11CR glass-epoxy laminates⁽¹⁾ are used in magnetic fusion energy structures as thermal insulators, electrical insulators and structural support at low temperatures. The study of the mechanical behavior of these composites is very important for superconducting magnet technology. Taking the temperature-dependent properties of the composites into consideration in the thermal stress analysis, we have considered the thermal mechanical response of G-10CR glass-epoxy laminates with a crack perpendicular to the interfaces at low temperatures⁽²⁾. Moreover, the singular stresses of G-10CR glass-epoxy laminates with a T-shaped crack which consists of an interface crack and a crack perpendicular to the interfaces under tension have been treated⁽³⁾.

In this paper, we consider the dynamic singular

stresses near the crack tip of G-10CR glass-epoxy laminates with a central crack and temperature-dependent properties at low temperatures. The analytical method developed herein applies Fourier transform to the space variable and Laplace transform to the time variable. The boundary value crack problem is formulated with stress and displacement conditions. These conditions reduce the elastodynamic problem to the solution of a standard pair of dual integral equations in the Laplace transform plane. The dual integral equations are solved by the integral transform technique and the solution is expressed in terms of a Fredholm integral equation of the second kind⁽⁴⁾ having a kernel with finite integrals⁽⁵⁾. A numerical Laplace inversion technique⁽⁶⁾ is then also used to recover the time dependence of the solution. Numerical results on the dynamic stress intensity factor at low temperatures are obtained and presented in a graphic form.

2. Formulation of Problem

Let the layered composite⁽²⁾ be made of a cracked layer *I* of width $2h_1$ bonded between two layers *II* of width h_2-h_1 , as shown in Fig. 1. The ends of the crack

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of length $2a(a < h_1)$ are situated at equal distance away from the interfaces which are directed normal to the crack plane. The system of rectangular Cartesian coordinates (x, y, z) is introduced in the composite in such a way that the crack is located centrally along the x -axis, and the y -axis is parallel to the interfaces. Layer I is reinforced along the z -axis and has the Young's modulus $E_{II}(\Phi)$, shear modulus $G_{IjI}(\Phi)$ and Poisson's ratio $\nu_{ijI}(\Phi)$ ($i, j = x, y, z$). $\nu_{ijI}(\Phi)$ denotes Poisson's ratio for the transverse strain in the j -direction when stressed in the i -direction. Layer II is reinforced along the y -axis and has the elastic properties $E_{IjI}(\Phi)$, $G_{IjII}(\Phi)$ and $\nu_{ijII}(\Phi)$ ($i, j = x, y, z$). Φ denotes the temperature of the composite and these elastic properties are functions of the temperature. In the following, the subscripts $(i, j) = (x, y, z)$ will be used to refer to the direction of coordinates and the subscripts $\delta = I, II$ indicate layers I and II , respectively.

Let the components of the displacement in the x and y directions be labeled by $u_{i\delta}(x, y)$ ($i = x, y, \delta = I, II$), respectively. Under plane strain, the nonzero stress components are found as

$$\left. \begin{aligned} \frac{\sigma_{x\delta}(x, y)}{G_{xy\delta}(\Phi)} &= c_{11\delta}(\Phi) \frac{\partial u_{x\delta}(x, y)}{\partial x} + c_{12\delta}(\Phi) \frac{\partial u_{y\delta}(x, y)}{\partial y} \\ \frac{\sigma_{y\delta}(x, y)}{G_{xy\delta}(\Phi)} &= c_{12\delta}(\Phi) \frac{\partial u_{x\delta}(x, y)}{\partial x} + c_{22\delta}(\Phi) \frac{\partial u_{y\delta}(x, y)}{\partial y} \\ \frac{\tau_{xy\delta}(x, y)}{G_{xy\delta}(\Phi)} &= \frac{\partial u_{x\delta}(x, y)}{\partial y} + \frac{\partial u_{y\delta}(x, y)}{\partial x} \end{aligned} \right\} \quad (\delta = I, II), \quad (1)$$

where

Fig. 1 Composite model with a crack

$$\left. \begin{aligned} c_{11\delta}(\Phi) &= \{1/E_{z\delta}(\Phi) - \nu_{yz\delta}(\Phi)^2/E_{y\delta}(\Phi)\} \\ &\quad /E_{y\delta}(\Phi)G_{xy\delta}(\Phi)C_{0\delta}(\Phi) \\ c_{22\delta}(\Phi) &= \{1/E_{x\delta}(\Phi) \\ &\quad - \nu_{zx\delta}(\Phi)^2/E_{z\delta}(\Phi)\}/E_{z\delta}(\Phi)G_{xy\delta}(\Phi)C_{0\delta}(\Phi) \\ c_{12\delta}(\Phi) &= \{\nu_{zx\delta}(\Phi)\nu_{yz\delta}(\Phi)/E_{y\delta}(\Phi) \\ &\quad + \nu_{xy\delta}(\Phi)/E_{x\delta}(\Phi)\}/E_{z\delta}(\Phi)G_{xy\delta}(\Phi)C_{0\delta}(\Phi) \\ c_{0\delta}(\Phi) &= \{1 - 2\nu_{xy\delta}(\Phi)\nu_{yz\delta}(\Phi)\nu_{zx\delta}(\Phi) \\ &\quad - E_{x\delta}(\Phi)\nu_{kx\delta}(\Phi)^2/E_{z\delta}(\Phi) - E_{y\delta}(\Phi)\nu_{xy\delta}(\Phi)^2 \\ &\quad /E_{x\delta}(\Phi) - E_{z\delta}(\Phi)\nu_{yz\delta}(\Phi)^2/E_{y\delta}(\Phi)\} \\ &\quad /E_{x\delta}(\Phi)E_{y\delta}(\Phi)E_{z\delta}(\Phi) \end{aligned} \right\} \quad (\delta = I, II). \quad (2)$$

The composite elastic properties $E_{iI}(\Phi)$, $E_{tI}(\Phi)$, $G_{ul}(\Phi)$, $G_{ut}(\Phi)$, $\nu_{ul}(\Phi)$, $\nu_{ut}(\Phi)$ and $\nu_{tt}(\Phi)$ are related to the elastic properties $E_{i\delta}(\Phi)$, $G_{xy\delta}(\Phi)$ and $\nu_{ij\delta}(\Phi)$ ($i, j = x, y, z, \delta = I, II$) of layers I and II by the relations

$$\left. \begin{aligned} E_{zI}(\Phi) &= E_{yII}(\Phi) = E_{tI}(\Phi) \\ E_{xI}(\Phi) &= E_{yI}(\Phi) = E_{xII}(\Phi) = E_{zII}(\Phi) = E_{tI}(\Phi) \\ G_{xyI}(\Phi) &= G_{tt}(\Phi) \\ G_{xyII}(\Phi) &= G_{ul}(\Phi) \\ \nu_{zxI}(\Phi) &= \nu_{yzII}(\Phi) = \nu_{ul}(\Phi) \\ \nu_{yzI}(\Phi) &= \nu_{xyII}(\Phi) = \nu_{ul}(\Phi) \\ \nu_{xyI}(\Phi) &= \nu_{zxII}(\Phi) = \nu_{ut}(\Phi) \end{aligned} \right\}, \quad (3)$$

where the subscripts l and t indicate the directions longitudinal and transverse to the fiber axis, respectively. The composite elastic properties are related to the material properties of the fiber and matrix constituents, as given in Appendix A.

The displacement equations of motion are

$$\left. \begin{aligned} c_{11\delta}(\Phi) \frac{\partial^2 u_{x\delta}(x, y)}{\partial x^2} + \frac{\partial^2 u_{x\delta}(x, y)}{\partial y^2} + \{c_{12\delta}(\Phi) + 1\} \\ \times \frac{\partial^2 u_{y\delta}(x, y)}{\partial x \partial y} = \left\{ \frac{1}{c_{s\delta}(\Phi)^2} \right\} \frac{\partial^2 u_{x\delta}(x, y)}{\partial t^2} \\ c_{22\delta}(\Phi) \frac{\partial^2 u_{y\delta}(x, y)}{\partial y^2} + \frac{\partial^2 u_{y\delta}(x, y)}{\partial x^2} + \{c_{12\delta}(\Phi) + 1\} \\ \times \frac{\partial^2 u_{x\delta}(x, y)}{\partial x \partial y} = \left\{ \frac{1}{c_{s\delta}(\Phi)^2} \right\} \frac{\partial^2 u_{y\delta}(x, y)}{\partial t^2} \end{aligned} \right\} \quad (\delta = I, II), \quad (4)$$

where t denotes the time and $c_{s\delta}(\Phi) = \sqrt{G_{xy\delta}(\Phi)/\rho}$ ($\delta = I, II$), ρ being the mass density of the material. In the isotropic solid, $c_{s\delta}(\Phi)$ ($\delta = I, II$) represent the velocity of the shear wave.

Suppose that the crack is now loaded suddenly by a pair of normal stresses such that the upper and lower crack surface move in opposite directions at temperature Φ . The composite is assumed to be initially at rest. The symmetry and boundary conditions can be written as

$$\left. \begin{aligned} \sigma_{yI}(x, 0, t) &= -\sigma_0 H(t) & (0 \leq x < a) \\ u_{yI}(x, 0, t) &= 0 & (a \leq x \leq h_1) \end{aligned} \right\} \quad (5)$$

$$\tau_{xyI}(x, 0, t) = 0 \quad (0 \leq x \leq h_1) \quad (6)$$

$$\tau_{xyII}(x, 0, t) = 0 \quad (h_1 \leq x \leq h_2) \quad (7)$$

$$u_{yII}(x, 0, t) = 0 \quad (h_1 \leq x \leq h_2) \quad (8)$$

$$\tau_{xyI}(0, y, t) = 0 \quad (0 \leq y < \infty) \quad (9)$$

$$u_{xI}(0, y, t) = 0 \quad (0 \leq y < \infty) \quad (10)$$

$$\sigma_{xI}(h_1, y, t) = \sigma_{xII}(h_1, y, t) \quad (0 \leq y < \infty) \quad (11)$$

$$\tau_{xyI}(h_1, y, t) = \tau_{xyII}(h_1, y, t) \quad (0 \leq y < \infty) \quad (12)$$

$$u_{xI}(h_1, y, t) = u_{xII}(h_1, y, t) \quad (0 \leq y < \infty) \quad (13)$$

$$u_{yI}(h_1, y, t) = u_{yII}(h_1, y, t) \quad (0 \leq y < \infty) \quad (14)$$

$$\sigma_{xII}(h_2, y, t) = 0 \quad (0 \leq y < \infty) \quad (15)$$

$$\tau_{xyII}(h_2, y, t) = 0 \quad (0 \leq y < \infty), \quad (16)$$

where $H(t)$ is the Heaviside unit step function in time and σ_0 is a constant with the dimensions of stress.

3. Method of Solution

Define a Laplace transform pair by the equations

$$f^*(p) = \int_0^\infty f(t) \exp(-pt) dt,$$

$$f(t) = \frac{1}{2\pi i} \int_{B_r} f^*(p) \exp(pt) dp, \quad (17)$$

in which the second integral is integrated over the Bromwich path. In the x - y plane, layer I is isotropic and layer II is orthotropic. Applying Laplace and Fourier transforms to Eqs. (4), the displacement field for layer I may be expressed as⁽⁵⁾

$$\begin{aligned} u_{xI}^*(x, y, p) = & -\frac{2}{\pi} \int_0^\infty [sA_1(s, p) \exp(-\gamma_{1I}y) \\ & + \gamma_{2I}A_0(s, p) \exp(-\gamma_{2I}y)] \sin(sx) \\ & - \{\gamma_{1I}A_2(s, p) \sinh(\gamma_{1I}x) \\ & + sA_3(s, p) \sinh(\gamma_{2I}x)\} \cos(sy) ds \end{aligned} \quad (18)$$

$$\begin{aligned} u_{yI}^*(x, y, p) = & -\frac{2}{\pi} \int_0^\infty [\{\gamma_{1I}A_1(s, p) \exp(-\gamma_{1I}y) \\ & + sA_0(s, p) \exp(-\gamma_{2I}y)\} \cos(sx) \\ & + \{sA_2(s, p) \cosh(\gamma_{1I}x) \\ & + \gamma_{2I}A_3(s, p) \cosh(\gamma_{2I}x)\} \sin(sy)] ds, \end{aligned} \quad (19)$$

where

$$\left. \begin{aligned} \gamma_{kI} &= \sqrt{s^2 + \{p/c_{sII}(\Phi)\}^2} \quad (k=1, 2) \\ c_1(\Phi) &= \sqrt{2G_{xyI}(\Phi)\{1-\nu_I(\Phi)\}}/\rho\{1-2\nu_I(\Phi)\} \\ c_2(\Phi) &= c_{sI}(\Phi) \\ \nu_I(\Phi) &= 2G_{xyI}(\Phi)\{\nu_{xyI}(\Phi)E_{yI}(\Phi)/E_{xI}(\Phi) \\ &+ \nu_{zxI}(\Phi)\nu_{yzi}(\Phi)\}/E_{yI}(\Phi) \end{aligned} \right\} \quad (20)$$

Similarly, the displacement field for layer II can be expressed as

$$\begin{aligned} u_{xII}^*(x, y, p) = & \frac{2}{\pi} \int_0^\infty s\{A_4(s, p) \exp(\gamma_{1II}x) \\ & + A_5(s, p) \exp(-\gamma_{1II}x) + A_6(s, p) \exp(\gamma_{2II}x) \\ & + A_7(s, p) \exp(-\gamma_{2II}x)\} \cos(sy) ds \end{aligned} \quad (21)$$

$$\begin{aligned} u_{yII}^*(x, y, p) = & \frac{2}{\pi} \int_0^\infty [\beta_1\{A_4(s, p) \exp(\gamma_{1II}x) \\ & - A_5(s, p) \exp(-\gamma_{1II}x)\} + \beta_2\{A_6(s, p) \exp(\gamma_{2II}x) \\ & - A_7(s, p) \exp(-\gamma_{2II}x)\}] \sin(sy) ds, \end{aligned} \quad (22)$$

when $\gamma_{kII}^2 (k=1, 2)$ are the roots of the characteristic equation:

$$\begin{aligned} c_{11II}(\Phi)\gamma^4 + \{c_{12II}(\Phi)^2 + 2c_{12II}(\Phi) \\ - c_{11II}(\Phi)c_{22II}(\Phi)\}s \\ - \{1 + c_{11II}(\Phi)\{p/c_{sII}(\Phi)\}^2\}\gamma^2 \\ + [c_{22II}(\Phi)s^2 + \{p/c_{sII}(\Phi)\}^2][s^2 + \{p/c_{sII}(\Phi)\}^2] = 0. \end{aligned}$$

$$\beta_k (k=1, 2) \text{ are given by} \quad (23)$$

$$\beta_k = \frac{s^2 + \{p/c_{sII}(\Phi)\}^2 - c_{11II}(\Phi)\gamma_{kII}^2}{\{1 + c_{12II}(\Phi)\}\gamma_{kII}} \quad (k=1, 2). \quad (24)$$

In Eqs. (18), (19), (21) and (22), $A_n(s, p)$ ($n=0\sim 7$) are the unknowns to be solved. Substituting Eqs. (18), (19), (21) and (22) into Eq. (1), one obtains the stress expressions in the Laplace transform plane.

In the Laplace transform plane, Eqs. (5)~(16) become

$$\left. \begin{aligned} \sigma_{yI}^*(x, 0, p) &= -\sigma_0/p & (0 \leq x < a) \\ u_{yI}^*(x, 0, p) &= 0 & (a \leq x \leq h_1) \end{aligned} \right\} \quad (25)$$

$$\tau_{xyI}^*(x, 0, p) = 0 \quad (0 \leq x \leq h_1) \quad (26)$$

$$\tau_{xyII}^*(x, 0, p) = 0 \quad (h_1 \leq x \leq h_2) \quad (27)$$

$$u_{yII}^*(x, 0, p) = 0 \quad (h_1 \leq x \leq h_2) \quad (28)$$

$$\tau_{xyI}^*(0, y, p) = 0 \quad (0 \leq y < \infty) \quad (29)$$

$$u_{xI}^*(0, y, p) = 0 \quad (0 \leq y < \infty) \quad (30)$$

$$\sigma_{xI}^*(h_1, y, p) = \sigma_{xII}^*(h_1, y, p) \quad (0 \leq y < \infty) \quad (31)$$

$$\tau_{xyI}^*(h_1, y, p) = \tau_{xyII}^*(h_1, y, p) \quad (0 \leq y < \infty) \quad (32)$$

$$u_{xI}^*(h_1, y, p) = u_{xII}^*(h_1, y, p) \quad (0 \leq y < \infty) \quad (33)$$

$$u_{yI}^*(h_1, y, p) = u_{yII}^*(h_1, y, p) \quad (0 \leq y < \infty) \quad (34)$$

$$\sigma_{xII}^*(h_2, y, p) = 0 \quad (0 \leq y < \infty) \quad (35)$$

$$\tau_{xyII}^*(h_2, y, p) = 0 \quad (0 \leq y < \infty). \quad (36)$$

Note that these stress components and displacement components satisfy the symmetry conditions (27)~(30), automatically. Through Eqs. (26) and (31)~(36), the unknowns $A_n(s)$ ($n=1\sim 7$) are related to the unknown $A_0(s)$ by

$$\mathbf{D}\mathbf{A} = \mathbf{R}. \quad (37)$$

In Eq. (37), $\mathbf{D}$ is a square matrix of order 7, and the element in row m and column n is $d_{mn}(s)$ ($m, n=1\sim 7$). $\mathbf{A}$ and $\mathbf{R}$ are the column vectors of order 7, and the elements in row n are $A_n(s)$ and $R_n(s)$ ($n=1\sim 7$), respectively. The elements $d_{mn}(s)$ ($m, n=1\sim 7$) and the elements $R_n(s)$ ($n=1\sim 7$) are given in Appendix B. Using relation (37), the unknowns $A_n(s)$ ($n=1\sim 7$) are

$$A_n(s) = \sum_{m=1}^7 Q_{mn}(s) R_m(s) \quad (n=1\sim 7). \quad (38)$$

In Eq. (38), $Q_{mn}(s)$ ($m, n=1\sim 7$) are given by

$$Q_{mn}(s) = D_{mn}(s)/|\mathbf{D}| \quad (m, n=1\sim 7), \quad (39)$$

where $|\mathbf{D}|$ is the determinant of the square matrix $\mathbf{D}$, and $D_{mn}(s)$ are the cofactors of the elements $d_{mn}(s)$ ($m, n=1\sim 7$).

Making use of the mixed boundary condition (25), we have the dual integral equations for the determination of the unknown function $A_0(s, p)$.

$$\left. \begin{aligned} & \frac{2G_{xyI}(\Phi)}{\pi} \int_0^\infty \frac{(s^2 + \gamma_{2I}^2)^2 - 4s^2 \gamma_{1I} \gamma_{2I}}{2s \gamma_{1I}} A_0(s, p) \\ & \times \cos(sx) ds - \frac{2G_{xyI}(\Phi)}{\pi} \int_0^\infty \left[\left\{ \frac{p}{c_2(\Phi)} \right\}^2 - 2\gamma_{1I}^2 \right] \\ & \times \cosh(\gamma_{1I}x) \sum_{m=1}^7 Q_{m2}(s, p) R_m(s, p) \\ & - 2s \gamma_{2I} \cosh(\gamma_{2I}x) \sum_{m=1}^7 Q_{m4}(s, p) R_m(s, p) \Big] ds = \frac{\sigma_0}{p} \\ & \quad (0 \leq x < a) \\ & \frac{2}{\pi} \int_0^\infty \left\{ \frac{p}{c_2(\Phi)} \right\}^2 \frac{A_0(s, p)}{2s} \cos(sx) ds = 0 \\ & \quad (a \leq x \leq h_1) \end{aligned} \right\} \quad (40)$$

The set of dual integral equations in Eq. (40) may be solved and the result is

$$A_0(s, p) = \frac{\pi a^2 c_2(\Phi)^2 s}{2G_{xyI}(\Phi) \{1 - c_0(\Phi)^2\} p^3} \int_0^1 \sqrt{\xi} \Gamma(\xi, p) \times J_0(sa\xi) d\xi, \quad (41)$$

where $c_0(\Phi) = c_2(\Phi)/c_1(\Phi)$, $J_0(\cdot)$ is the zero-order Bessel function of the first kind and $\Gamma(\xi, p)$ is governed by a Fredholm integral equation of the second kind,

$$\Gamma\{\xi, c_2(\Phi)P/a\} + \int_0^1 \Gamma\{\xi, c_2(\Phi)P/a\} \times \{M_1(\xi, \zeta, P) + M_2(\xi, \zeta, P)\} d\xi = \sigma_0 \sqrt{\xi}, \quad (42)$$

in which $P = pa/c_2(\Phi)$ and the kernel functions $M_1(\xi, \zeta, P)$, $M_2(\xi, \zeta, P)$ are given by

$$M_1(\xi, \zeta, P) = \sqrt{\xi\zeta} \int_0^\infty \left[\frac{\{s^2 + (\gamma_{2I}^{(1)})^2\}^2 - 4s^2 \gamma_{1I}^{(1)} \gamma_{2I}^{(1)}}{2\{1 - c_0(\Phi)^2\} P^2 \gamma_{1I}^{(1)}} - s \right] J_0(s\xi) J_0(s\zeta) ds \quad (43)$$

$$\begin{aligned} M_2(\xi, \zeta, P) = & \frac{\sqrt{\xi\zeta}}{2\{1 - c_0(\Phi)^2\} P^2} \int_0^\infty \{[P^2 - 2(\gamma_{1I}^{(1)})^2] \\ & \times \left\{ \sum_{m=2}^5 z_{m-1}(s, P) Q_{m2}(s, p) \right. \\ & \times \exp(-\gamma_{1I}^{(1)} h_1/a) I_0(\gamma_{1I}^{(1)} \xi) I_0(\gamma_{1I}^{(1)} \zeta) \\ & + \sum_{m=2}^5 z_{m+3}(s, P) Q_{m2}(s, p) \\ & \times \exp(-\gamma_{2I}^{(1)} h_1/a) I_0(\gamma_{2I}^{(1)} \xi) I_0(\gamma_{2I}^{(1)} \zeta) \Big\} \\ & + 2s \gamma_{2I}^{(1)} \left\{ \sum_{m=2}^5 z_{m-1}(s, P) Q_{m4}(s, p) \right. \\ & \times \exp(-\gamma_{1I}^{(1)} h_1/a) I_0(\gamma_{1I}^{(1)} \xi) I_0(\gamma_{2I}^{(1)} \zeta) \\ & + \sum_{m=2}^5 z_{m+3}(s, P) Q_{m4}(s, p) \\ & \times \exp(-\gamma_{2I}^{(1)} h_1/a) I_0(\gamma_{2I}^{(1)} \xi) I_0(\gamma_{2I}^{(1)} \zeta) \Big\} \Big] ds. \end{aligned} \quad (44)$$

In Eqs. (43) and (44), $I_0(\cdot)$ denotes the zero-order modified Bessel function of the first kind and $\gamma_{kI}^{(1)} (k=1, 2)$ are

$$\left. \begin{aligned} \gamma_{1I}^{(1)} &= \sqrt{s^2 + \{c_0(\Phi)P\}^2} \\ \gamma_{2I}^{(1)} &= \sqrt{s^2 + P^2} \end{aligned} \right\}. \quad (45)$$

The functions $z_k(s, P)$ ($k=1\sim 8$) are given by

$$\left. \begin{aligned} z_1(s, P) &= 4(\gamma_{1I}^{(1)})^3 - 4\{P c_0(\Phi)\}^2 \gamma_{1I}^{(1)} \\ & \quad + P^4 \{2c_0(\Phi)^2 - 1\} / \gamma_{1I}^{(1)} \\ z_2(s, P) &= 4s(\gamma_{1I}^{(1)})^2 - 2sP^2 \\ z_3(s, P) &= -2(\gamma_{1I}^{(1)})^2 + P^2 \\ z_4(s, P) &= -2s\gamma_{1I}^{(1)} - sP^2 / \gamma_{1I}^{(1)} \\ z_5(s, P) &= -4(\gamma_{2I}^{(1)})^3 + 4P^2 \gamma_{2I}^{(1)} \\ z_6(s, P) &= -4s(\gamma_{2I}^{(1)})^2 + 2sP^2 \\ z_7(s, P) &= 2(\gamma_{2I}^{(1)})^2 - 2P^2 \\ z_8(s, P) &= 2s\gamma_{2I}^{(1)} \end{aligned} \right\} \quad (46)$$

We note that the kernel function $M_1(\xi, \zeta, P)$, Eq. (43), is a semi-infinite integral which has a rather slow rate of convergence. To simplify the numerical analysis, we evaluate the kernel. The kernel $M_1(\xi, \zeta, P)$ for $\xi < \zeta$ can be written as⁽⁵⁾

$$\begin{aligned} M_1(\xi, \zeta, P) &= \frac{\sqrt{\xi\zeta} P^2}{\{1 - c_0(\Phi)^2\} \pi} \int_0^{c_0(\Phi)} \frac{\{\eta^2 - (\gamma_{2I}^{(2)})^2\}^2 + 4\eta^2 \gamma_{1I}^{(2)} \gamma_{2I}^{(2)}}{\gamma_{1I}^{(2)}} \\ & \times I_0(\eta P \xi) K_0(\eta P \zeta) d\eta \\ & + \frac{\sqrt{\xi\zeta} P^2}{\{1 - c_0(\Phi)^2\} \pi} \int_{c_0(\Phi)}^1 \frac{4\eta^2 \gamma_{2I}^{(2)} I_0(\eta P \xi) K_0(\eta P \zeta) d\eta}{\gamma_{1I}^{(2)}} \end{aligned} \quad (\xi < \eta) \quad (47)$$

in which $K_0(\cdot)$ is the zero-order modified Bessel function of the second kind and $\gamma_{kI}^{(2)} (k=1, 2)$ are

$$\left. \begin{aligned} \gamma_{1I}^{(2)} &= \sqrt{c_0(\Phi)^2 - \eta^2} \\ \gamma_{2I}^{(2)} &= \sqrt{1 - \eta^2} \end{aligned} \right\}. \quad (48)$$

The kernel $M_1(\xi, \zeta, P)$ is symmetric in ξ and ζ . Thus, the value of $M_1(\xi, \zeta, P)$ for $\xi > \zeta$ is obtained by interchanging ξ and ζ in Eq. (47).

The dynamic stress intensity factor may be determined by obtaining the asymptotic stress near the crack border in the Laplace transform plane and performing a Laplace inversion. The dynamic stress intensity factor $K_I^*(P)$ in the Laplace transform plane is

$$\begin{aligned} K_I^*(P) &= \lim_{x \rightarrow a^+} \sqrt{2\pi(x-a)} \sigma_y^*(x, 0, c_2(\Phi)P/a) \\ &= \frac{\sqrt{\pi a} \Gamma\{1, c_2(\Phi)P/a\}}{P}. \end{aligned} \quad (49)$$

Then the dynamic stress intensity factor $K_I(t)$ is

$$\begin{aligned} K_I(t) &= \frac{\sqrt{\pi a}}{2\pi i} \int_{Br} \frac{\Gamma\{1, c_2(\Phi)P/a\}}{P} \\ & \times \exp\{c_2(\Phi)tP/a\} dP. \end{aligned} \quad (50)$$

4. Numerical Results and Discussion

The Fredholm integral equation of the second kind (42) can be solved by standard numerical techniques. Once this is done, $K_I(t)$ can be found from Eq. (50) by using a numerical Laplace inversion scheme⁽⁶⁾. First, we examine the effect of the geometric parameter h_2/h_1 on the time dependence of the dynamic stress intensity factor $K_I(t)$. For comparison with the results for the composite material obtained by Nozaki et al⁽⁶⁾, we performed calculations for the case where layer I is isotropic and layer II is nearly isotropic. Neglecting

the temperature-dependent properties of the composites, the elastic constant of layers *I* and *II* are given by

$$\left. \begin{aligned} \nu_{ij\delta} &= \nu(i, j=x, y, z, \delta=I, II) \\ \frac{E_{xI}}{G_{xyI}} &= \frac{E_{yI}}{G_{xyI}} = 2(1+\nu) \\ \frac{E_{yII}}{E_{xII}} &= 1.01 \\ \frac{E_{xII}}{G_{xyII}} &= 2(1+\nu) \end{aligned} \right\} \quad (51)$$

Figure 2 gives a plot of the normalized dynamic stress intensity factor $K_I(t)/\sigma_0(\pi a)^{1/2}$ versus normalized time $c_2 t/a$ for $\nu=0.29$, $G_{xyII}/G_{xyI}=5.0$ and $a/h_1=0.7$, while h_2/h_1 takes the values of 1.5, 2.0, 3.0 and ∞ . As the h_2/h_1 ratio increases, the peak value of $K_I(t)/\sigma_0(\pi a)^{1/2}$ is seen to decrease and occur at an earlier time. The value for the case of $h_2/h_1 \rightarrow \infty$ corresponds to the result for the composite material⁽⁵⁾ which is identified by the dotted curve in Fig. 2.

Next, we study the effect of the temperature on the time dependence of the dynamic stress intensity factor $K_I(t)$ of the glass-fiber reinforced plastic with a crack. The elastic properties of the fiber and matrix are approximated by the following exponential functions⁽²⁾:

E glass fiber

$$E_f(\Phi) = 83.76 \exp(-2.405 \times 10^{-4} \Phi) \quad [\text{GPa}] \quad (52)$$

$$\nu_f(\Phi) = 0.251 \exp(6.038 \times 10^{-4} \Phi) \quad (53)$$

$$G_f(\Phi) = E_f(\Phi)/2\{1 + \nu_f(\Phi)\} \quad [\text{GPa}] \quad (54)$$

Bisphenol A epoxy

$$E_m(\Phi) = 9.338 \exp(-3.443 \times 10^{-3} \Phi) \quad [\text{GPa}] \quad (55)$$

$$\nu_m(\Phi) = 0.326 \exp(-1.419 \times 10^{-3} \Phi) \quad (56)$$

$$G_m(\Phi) = E_m(\Phi)/2\{1 + \nu_m(\Phi)\} \quad [\text{GPa}] \quad (57)$$

where $E_f(\Phi)$, $E_m(\Phi)$ are Young's moduli, $G_f(\Phi)$, $G_m(\Phi)$ are shear moduli, $\nu_f(\Phi)$, $\nu_m(\Phi)$ are Poisson's ratios, and the subscripts *f* and *m* denote the fiber and the matrix. The elastic properties of layers *I* and *II*

are determined by Eqs. (3) and (52)~(59) with the volume fraction of the fiber $V_f=0.5$. Figure 3 exhibits the variation of the normalized dynamic stress intensity factor $K_I(t)/\sigma_0(\pi a)^{1/2}$ with normalized time $c_2(\Phi)t/a$ for $h_2/h_1=2.0$, $\Phi=4$ [K] and $a/h_1=0.1, 0.8, 0.9$. Note that the dynamic stress intensity factor rises sharply at first, reaches a maximum value and then decays with oscillating amplitude. As $t \rightarrow \infty$, the dynamic stress intensity factor approaches the corresponding static solution⁽²⁾. As the a/h_1 ratio increases, the peak value of $K_I(t)/\sigma_0(\pi a)^{1/2}$ decreases and occurs at an earlier time. Figure 4 and 5 show the results for the temperatures $\Phi=77$ [K] and 293 [K], respectively. The effect of a/h_1 on the time dependence of the dynamic stress intensity factor is more pronounced than that for the case of $\Phi=4$ [K]. Table 1 shows the peak value of normalized dynamic stress intensity factor $K_{I \max}(t)/\sigma_0(\pi a)^{1/2}$, the normalized time $c_2(\Phi)t_{\max}/a$ at which the peak value of the dynamic stress intensity factor occurs and the corresponding static

Fig. 3 Dynamic stress intensity factor versus time for $\Phi=4$ [K]

Fig. 2 Dynamic stress intensity factor versus time for $a/h_1=0.7$

Fig. 4 Dynamic stress intensity factor versus time for $\Phi=77$ [K]

Fig. 5 Dynamic stress intensity factor versus time for $\Phi=293$ [K]

Table 1 The values of $K_{I\max}/\sigma_0(\pi a)^{1/2}$, $c_2(\Phi)t_{\max}/a$ and $K_{Is}/\sigma_0(\pi a)^{1/2}$

Φ [K]	a/h_1	$K_{I\max}/\sigma_0(\pi a)^{1/2}$	$c_2(\Phi)t_{\max}/a$	$K_{Is}/\sigma_0(\pi a)^{1/2}$
4	0.1	1.253	2.78	1.0003
	0.8	1.236	2.74	1.0008
	0.9	1.207	2.70	0.9822
77	0.1	1.252	2.80	1.0002
	0.6	1.245	2.72	0.9948
	0.8	1.217	2.64	0.9749
	0.9	1.175	2.56	0.9443
293	0.1	1.252	2.82	0.9992
	0.6	1.240	2.74	0.9625
	0.8	1.174	2.58	0.9096
	0.9	1.099	2.48	0.8505

solution $K_{Is}/\sigma_0(\pi a)^{1/2(2)}$.

5. Conclusions

In the present paper, we considered the impact response of G-10CR glass-epoxy laminates with a crack at low temperatures. The dynamic stress intensity factor of a crack which is perpendicular to the interfaces of a three-layered composite model is obtained. The time dependence of the local stress field is found to depend on the temperature and the crack length. As the temperature decreases, the influence of the crack length on the time dependence of the dynamic stress intensity factor also decreases.

Appendix A

The composite elastic properties are related to the material properties of the fiber and matrix constituents as⁽⁷⁾

$$E_t(\Phi) = E_m(\Phi) V_m + E_f(\Phi) V_f + \frac{4\{\nu_f(\Phi) - \nu_m(\Phi)\}^2 V_m V_f}{\frac{V_m}{k_2(\Phi)} + \frac{V_f}{k_1(\Phi)} + \frac{1}{G_m(\Phi)}}$$

$$G_{tt}(\Phi) = G_{tt}(\Phi) = G_m(\Phi)$$

$$\times \frac{G_m(\Phi) V_m + G_f(\Phi)(1 + V_f)}{G_m(\Phi)(1 + V_f) + G_f(\Phi) V_m}$$

$$\nu_{tt}(\Phi) = \frac{E_t(\Phi)}{E_t(\Phi)} \nu_{tt}(\Phi) = \nu_m(\Phi) V_m + \nu_f(\Phi) V_f + \frac{\{\nu_f(\Phi) - \nu_m(\Phi)\}\{1/k_1(\Phi) - 1/k_2(\Phi)\} V_m V_f}{\frac{V_m}{k_2(\Phi)} + \frac{V_f}{k_1(\Phi)} + \frac{1}{G_m(\Phi)}}$$

$$G_{tt}(\Phi) = G_m(\Phi)$$

$$\times \frac{\{1 + s_1(\Phi) V_f^3\}\{s_2(\Phi) + t_1(\Phi) V_f\} - 3 V_f V_m^2 t_1^2(\Phi)}{\{1 + s_1(\Phi) V_f^3\}\{s_2(\Phi) - V_f\} - 3 V_f V_m^2 t_1^2(\Phi)}$$

$$E_t(\Phi) = \frac{4 k_3 G_{tt}(\Phi)}{k_3(\Phi) + s_3(\Phi) G_{tt}(\Phi)}$$

$$\nu_{tt}(\Phi) = \frac{k_3(\Phi) - s_3(\Phi) G_{tt}(\Phi)}{k_3(\Phi) + s_3(\Phi) G_{tt}(\Phi)} \quad (58)$$

where V_f and $V_m = 1 - V_f$ denote the volume fraction of the fiber and matrix, respectively. $E_f(\Phi)$, $E_m(\Phi)$ are Young's moduli, $G_f(\Phi)$, $G_m(\Phi)$ are shear moduli, $\nu_f(\Phi)$, $\nu_m(\Phi)$ are Poisson's ratios, and the subscripts f and m denote the fiber and matrix, $k_k(\Phi)$, $s_k(\Phi)$ ($k = 1, 2, 3$) and $t_k(\Phi)$ ($k = 1, 2$) are given by

$$k_1(\Phi) = \frac{G_m(\Phi)}{1 - 2\nu_m(\Phi)}$$

$$k_2(\Phi) = \frac{G_f(\Phi)}{1 - 2\nu_f(\Phi)}$$

$$k_3(\Phi) = \frac{k_1(\Phi)\{k_2(\Phi) + G_m(\Phi)\} V_m + k_2(\Phi)\{k_1(\Phi) + G_m(\Phi)\} V_f}{\{k_2(\Phi) + G_m(\Phi)\} V_m + \{k_1(\Phi) + G_m(\Phi)\} V_f}$$

$$s_1(\Phi) = \frac{G_m(\Phi) t_1(\Phi) - G_f(\Phi) t_2(\Phi)}{G_m(\Phi) - G_f(\Phi) t_2(\Phi)}$$

$$s_2(\Phi) = \frac{G_f(\Phi) + G_m(\Phi) t_1(\Phi)}{G_f(\Phi) - G_m(\Phi)}$$

$$s_3(\Phi) = 1 + \frac{4 k_3(\Phi) \nu_{tt}^2(\Phi)}{E_t(\Phi)}$$

$$t_1(\Phi) = \frac{1}{3 - 4\nu_m(\Phi)}$$

$$t_2(\Phi) = \frac{1}{3 - 4\nu_f(\Phi)} \quad (59)$$

Appendix B

The elements $d_{mn}(s)$ ($m, n = 1 \sim 7$) of matrix D are given by

$$d_{11}(s, p) = -s\gamma_{1I}/(s^2 + \gamma_{2I}^2)$$

$$d_{1k}(s, p) = 0 \quad (k = 2 \sim 7)$$

$$d_{21}(s, p) = 0$$

$$d_{22}(s, p) = -(s^2 + \gamma_{2I}^2) \cosh(\gamma_{1I} h_1)$$

$$d_{23}(s, p) = -2s\gamma_{2I} \cosh(\gamma_{2I} h_1)$$

$$d_{24}(s, p) = G_0(\Phi) s\{c_{11II}(\Phi) \gamma_{1II} + c_{12II}(\Phi) \beta_1\} \exp(\gamma_{1I} h_1)$$

$$d_{25}(s, p) = -G_0(\Phi) s\{c_{11II}(\Phi) \gamma_{1II} + c_{12II}(\Phi) \beta_1\} \exp(-\gamma_{1I} h_1)$$

$$d_{26}(s, p) = G_0(\Phi) s\{c_{11II}(\Phi) \gamma_{2II} + c_{12II}(\Phi) \beta_2\} \exp(\gamma_{2I} h_1)$$

$$d_{27}(s, p) = -G_0(\Phi) s\{c_{11II}(\Phi) \gamma_{2II} + c_{12II}(\Phi) \beta_2\} \exp(-\gamma_{2I} h_1)$$

$$\begin{aligned}
d_{31}(s, p) &= 0 \\
d_{32}(s, p) &= 2s\gamma_{1I} \sinh(\gamma_{1I}h_1) \\
d_{33}(s, p) &= (s^2 + \gamma_{1I}) \sinh(\gamma_{1I}h_1) \\
d_{34}(s, p) &= -G_0(\Phi)(s^2 - \beta_1\gamma_{1II}) \exp(\gamma_{1II}h_1) \\
d_{35}(s, p) &= -G_0(\Phi)(s^2 - \beta_1\gamma_{1II}) \exp(-\gamma_{1II}h_1) \\
d_{36}(s, p) &= -G_0(\Phi)(s^2 - \beta_2\gamma_{2II}) \exp(\gamma_{2II}h_1) \\
d_{37}(s, p) &= -G_0(\Phi)(s^2 - \beta_2\gamma_{2II}) \exp(-\gamma_{2II}h_1) \\
d_{41}(s, p) &= 0 \\
d_{42}(s, p) &= -\gamma_{1I} \sinh(\gamma_{1I}h_1) \\
d_{43}(s, p) &= -s \sinh(\gamma_{2I}h_1) \\
d_{44}(s, p) &= s \exp(\gamma_{1II}h_1) \\
d_{45}(s, p) &= s \exp(-\gamma_{1II}h_1) \\
d_{46}(s, p) &= s \exp(\gamma_{2II}h_1) \\
d_{47}(s, p) &= s \exp(-\gamma_{2II}h_1) \\
d_{51}(s, p) &= 0 \\
d_{52}(s, p) &= s \cosh(\gamma_{1I}h_1) \\
d_{53}(s, p) &= \gamma_{2I} \cosh(\gamma_{2I}h_1) \\
d_{54}(s, p) &= \beta_1 \exp(\gamma_{1II}h_1) \\
d_{55}(s, p) &= -\beta_1 \exp(-\gamma_{1II}h_1) \\
d_{56}(s, p) &= \beta_2 \exp(\gamma_{2II}h_1) \\
d_{57}(s, p) &= -\beta_2 \exp(-\gamma_{2II}h_1) \\
d_{6k}(s, p) &= 0 \quad (k=1\sim3) \\
d_{64}(s, p) &= \{c_{11II}(\Phi)\gamma_{1II} + c_{12II}(\Phi)\beta_1\} \exp(\gamma_{1II}h_2) \\
d_{65}(s, p) &= -\{c_{11II}(\Phi)\gamma_{1II} + c_{12II}(\Phi)\beta_1\} \exp(-\gamma_{1II}h_2) \\
d_{66}(s, p) &= \{c_{11II}(\Phi)\gamma_{1II} + c_{12II}(\Phi)\beta_2\} \exp(\gamma_{2II}h_2) \\
d_{67}(s, p) &= -\{c_{11II}(\Phi)\gamma_{1II} + c_{12II}(\Phi)\beta_2\} \exp(-\gamma_{2II}h_2) \\
d_{7k}(s, p) &= 0 \quad (k=1\sim3) \\
d_{74}(s, p) &= (s^2 - \beta_1\gamma_{1II}) \exp(\gamma_{1II}h_2) \\
d_{75}(s, p) &= (s^2 - \beta_1\gamma_{1II}) \exp(-\gamma_{1II}h_2) \\
d_{76}(s, p) &= (s^2 - \beta_2\gamma_{2II}) \exp(\gamma_{2II}h_2) \\
d_{77}(s, p) &= (s^2 - \beta_2\gamma_{2II}) \exp(-\gamma_{2II}h_2)
\end{aligned} \tag{60}$$

where $G_0(\Phi) = G_{xyII}(\Phi)/G_{xyI}(\Phi)$. The elements $R_n(s)$ ($n=1\sim7$) of column vector $\mathbf{R}$ are given by

$$\begin{aligned}
R_1(s, p) &= A_0(s, p) \\
R_2(s, p) &= \frac{2}{\pi} \int_0^\infty \left[\frac{[p/c_2(\Phi)]^2 - 2\gamma_{1I}^2}{\eta(\gamma_{1I}^2 + s^2)} \right. \\
&\quad \left. + \frac{2\eta\gamma_{2I}^2}{\gamma_{2I}^2 + s^2} \right] A_0(\eta, p) \cos(\eta h_1) d\eta \\
R_3(s, p) &= \frac{2}{\pi} \int_0^\infty s \left(\frac{\gamma_{2I}^2 + \eta^2}{\gamma_{2I}^2 + s^2} - \frac{\gamma_{1I}^2 + \eta^2}{\gamma_{1I}^2 + s^2} \right) \\
&\quad \times A_0(\eta, p) \sin(\eta h_1) d\eta \\
R_4(s, p) &= -\frac{2}{\pi} \int_0^\infty \left\{ \frac{\gamma_{2I}^2 + \eta^2}{2(\gamma_{1I}^2 + s^2)} + \frac{\gamma_{2I}^2}{\gamma_{2I}^2 + s^2} \right\} \\
&\quad \times A_0(\eta, p) \sin(\eta h_1) d\eta \\
R_5(s, p) &= \frac{2}{\pi} \int_0^\infty s \left\{ \frac{\gamma_{2I}^2 + \eta^2}{2\eta(\gamma_{1I}^2 + s^2)} - \frac{\eta}{\gamma_{2I}^2 + s^2} \right\} \\
&\quad \times A_0(\eta, p) \cos(\eta h_1) d\eta \\
R_6(s, p) &= R_7(s, p) = 0
\end{aligned} \tag{61}$$

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