Tangent Hypersphere Method for Optimal Design*

(Maximization of Vibration Eigenvalue and Minimization of Weight)

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A new formulation is proposed to determine the minimal objective function under inequality constraint conditions. Taylor series expansion of the objective function with respect to design variables is employed for the approximation up to the quadratic terms by use of the Hessian matrix, while linear approximation is employed for the inequality constraint conditions. The objective function is standardized on the basis of the eigenvalue analysis of the Hessian matrix so as to form the hypersphere. The center of the hypersphere is used directly for the minimization of the objective function in the case that the center lies inside the feasible domain of the design variables. In the case that the center is outside the feasible domain, the minimal value of the objective function is searched for as the hypersphere tangent to the hyperplanes which represent the boundary of the feasible domain. The numerical examples in problem of beam vibration show that the proposed method is straightforward and efficient in searching for minimal objective function.

Key Words: Optimal Design, Eigenvalue Problem, Hypersphere, Sensitivity Analysis, Hessian Matrix, Beam Vibration

1. Introduction

Shape optimization is regarded as an important process to obtain better structures as well as selection of materials. It is well known that various optimization techniques to obtain the optimal shape have been developed so far and used in practice⁽¹⁾⁻⁽³⁾. The first step of these techniques is to define an objective function appropriately and to choose design variables judiciously, and then minimization or maximization of the objective function is carried out under constraint conditions. Most of such techniques make use of only the first-order sensitivities with respect to design variables. Even in the case that the second-order sensitivities are used, they are approximated on the basis of changes of the first-order sensitivities.

Recent development of sensitivity analysis makes it easy to evaluate the second-order sensitivities exactly. The numerical examples of optimization techniques which employ exact evaluation are

increasing^{(4),(5)}, and, in doing so, development of a method which can determine a precise solution of design variables more efficiently is desired. In this paper we propose an optimization technique to obtain the solution of design variables which minimizes the objective function under the inequality constraint conditions based on the concept of a hypersphere tangent to a hyperplane or an intersection of hyperplanes. The hypersphere is composed by use of the second-order sensitivity matrix. This method enables us to search for the minimal objective function algebraically and systematically. The first- and the second-order sensitivities are evaluated by the perturbation technique⁽⁶⁾.

It is assumed that we have a baseline design and can expand the objective function and inequality constraint conditions into Taylor series usings with respect to design variables in the vicinity of the baseline shape. We use the second-order approximation for the objective function and the first-order approximation for the constraint conditions. The coefficient matrix of the second degree, that is, the second-order sensitivity matrix of the objective function with respect to the design variables, is called a Hessian matrix. The

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eigenvalue analysis of the Hessian matrix is carried out in order to obtain the transformation matrix for the linear transformation of the design variables to standardize the Hessian matrix into the identity matrix. Then the objective function in the space of the basic design variables is converted into the hypersphere in the space of the standardized design variables. The boundary hyperplanes of the constraint conditions are preserved as hyperplanes even in the space of the standardized design variables. Using the single-valued correlation between the radius of the hypersphere and the magnitude of the objective function, we can search for the point of design variables which minimizes the objective function inside the feasible domain defined by the constraint conditions. This method does not require any parameter to be decided through a store of experience as is required in the penalty method⁽²⁾ and the multiplier method⁽²⁾, nor any iterative calculation. We can obtain the solution by simple arithmetic calculation in a straightforward manner.

In this paper we show the formulation of the tangent hypersphere method and its algorithm. Simple but typical numerical examples demonstrate the efficiency of the proposed method. Then, comparing the result of the present method with that of the multiplier method as one of the nonlinear programming methods presently in use, we discuss pros and cons of the proposed method. We show the results of optimal design in the vibration problem of an elastic beam, that is, the maximization of the eigenvalue under weight constraints and the minimization of the weight under eigenvalue constraints.

2. Standardization of Objective Function by Use of Hessian Matrix

The extreme value of the objective function can be searched for easily in the case that the function is quadratic with respect to the design variables. The point of design variables which gives the extreme value of an objective function is called extreme point hereafter. If the extreme point falls inside the feasible domain and the Hessian matrix is positive definite, the extreme point can be a solution for the minimal objective function under the constraints.

On the other hand, if the extreme point exists outside the feasible domain, it is necessary to find the point of design variables which minimizes the objective function inside the feasible domain. The value of the objective function at the minimal point is larger than that at the extreme point in this case. If a single-valued correlation exists between the value of the objective function and the distance from the extreme point to the hyperplane which is the bound-

ary of the inequality constraint condition, the point which minimizes the objective function inside the feasible domain can be easily found. In order to obtain such a single-valued correlation, the Hessian matrix of the objective function is standardized into the identity matrix, namely, the objective function is standardized as a hypersphere, the center of which is at the extreme point. It should be noted that the distance from the extreme point to the hyperplane is equal to the radius of the hypersphere tangent to the hyperplane after the standardization.

In general, the objective function is nonlinear with respect to design variables. In this paper, the objective function is approximated by the second-order Taylor series expansion with respect to N design variables in the form

$$f = \bar{f} + \begin{Bmatrix} f_1 \\ \vdots \\ f_N \end{Bmatrix}^T \begin{Bmatrix} x_1 \\ \vdots \\ x_N \end{Bmatrix} + (1/2) \begin{Bmatrix} x_1 \\ \vdots \\ x_N \end{Bmatrix}^T \begin{bmatrix} f_{11} & \cdots & f_{1N} \\ \vdots & \ddots & \vdots \\ f_{N1} & \cdots & f_{NN} \end{bmatrix} \begin{Bmatrix} x_1 \\ \vdots \\ x_N \end{Bmatrix} \\ = \bar{f} + \{B\}^T \{X\} + (1/2) \{X\}^T [A] \{X\}, \quad (1)$$

where $[A]$ stands for the Hessian matrix, and $\{X\}$ is a vector of N design variables. Hereafter, a lower-case letter indicates a component, and upper-case letter its set. A letter with an upper bar denotes the value at the baseline design, f_n the first-order sensitivity with respect to the design variable x_n and f_{nm} the second-order sensitivity with respect to x_n and x_m . The quadratic function Eq. (1) is standardized by linear transformation of the basic design variables $\{X\}$. By use of the eigenvalue λ_n and modal matrix $[\Phi]$ of the matrix $[A]$, the transformation matrix $[T]$ from the basic design variables $\{X\}$ to the standardized design variables $\{Y\}$ is composed as follows.

$$\begin{Bmatrix} \{X\} \\ [T] \end{Bmatrix} = \begin{Bmatrix} [T] \{Y\} \\ [\Phi] \text{diag}[1/\sqrt{\lambda_n}] [\Phi]^T \end{Bmatrix} \quad (2)$$

Once such a transformation matrix $[T]$ is derived, the transformation enables us to rewrite the objective function into the form

$$f = \bar{f} + \{B\}^T [T] \{Y\} + (1/2) \{Y\}^T [I] \{Y\} \\ = \bar{f} + \{B\}^T \{Y\} + (1/2) \{Y\}^T [I] \{Y\}. \quad (3)$$

This procedure means that the second-order sensitivity matrix is transformed into the identity matrix $[I]$, and the objective function is expressed as a hypersphere in the space of the standardized design variables. The first-order approximation of the inequality constraint conditions can be expressed by use of the first-order sensitivity matrix $[S]$ as in Eq. (4). The transformation is also applied to the constraint conditions so that Eq. (4) is rewritten in the form of Eq. (5),

$$[S] \{X\} \leq \{D\} \quad (4)$$

$$[S']\{Y\} \leq \{D\}, \quad (5)$$

where $[S'] = [S][T]$. In this chapter, the Hessian matrix is assumed to be positive definite. The case where this assumption does not hold is discussed in chapter 4.

3. Tangent Hypersphere Method

It is obvious that the extreme point of the objective function of Eq. (3) can be obtained by $\{Y\} = -[B']^T$. In the case that this extreme point is inside the feasible domain, this point affords the minimal value of the objective function.

In this chapter, we describe how to find the point that affords the minimal objective function inside the feasible domain, when the extreme point lies outside the feasible domain, that is, the center of the hypersphere is outside the feasible domain. Because of the single-valued correlation mentioned before, the magnitude of the standardized objective function at the point near the extreme point is smaller than the value at the point far from the extreme point. This means that the point at which a hypersphere is tangent to a hyperplane is the point which minimizes the objective function under a constraint. The point which minimizes the objective function under the constraint conditions can be found by means of searching for the points of the hypersphere tangent to hyperplanes. We call this search process the tangent hypersphere method, because we utilize the hypersphere tangent to a hyperplane from the outside of the feasible domain. The point which minimizes the objective function under the constraint conditions is designated the minimal point, and the point where the hypersphere is tangent to the hyperplane is called contact point hereafter.

3.1 Case of hypersphere tangent to a hyperplane

It is assumed that the inequality conditions are expressed by J hyperplanes. Figure 1 illustrates the hypersphere tangent to a hyperplane from the outside of the feasible domain in a simple case of $N=2$ and $J=3$. The feasible domain is indicated by shading in the figure. At first, such a case in which the center of the hypersphere lies outside the feasible domain, and in which the minimal point should exist as contact point of the hypersphere with a hyperplane is discussed. J hyperplanes with N variables are given by the following formula.

$$\sum_{n=1}^N s'_{jn} y_n - d_j = 0, \quad j=1 \sim J \quad (6)$$

The center of the hypersphere $\{Y^*\}$ is obtained as the aforementioned extreme point, so that the radius of the hypersphere tangent to the J -th hyperplane r_j is given by Eq. (7).

Fig. 1 Inequality constraint conditions and tangent circles

$$r_j = \left(\sum_{n=1}^N s'_{jn} y_n^* - d_j \right) / \sqrt{\sum_{n=1}^N s_{jn}^2}, \quad j=1 \sim J \quad (7)$$

According to Fig. 1, the positive and maximal radius among J radii becomes the radius of the hypersphere which has the minimal value in the feasible domain. Each distance from the center to the hyperplanes in this example can be evaluated by Eq. (7). The distances OA and OB are positive, while OC is negative. This means that the center O lies inside the feasible domain with respect to line 3-3, but not lines 1-1 and 2-2. Therefore, the positive and maximal distance to be searched for should be line OA, and point A is found to be the minimal point. Point A is the contact point of the hypersphere which has the largest radius r_m . This point is obtained as the location of the foot of the perpendicular from the center $\{Y^*\}$ to the hyperplane m in the form

$$y_l = y_l^* - r_m s'_{ml} / \sqrt{\sum_{n=1}^N s_{mn}^2}, \quad l=1 \sim N. \quad (8)$$

If the contact point falls in the feasible domain, the basic design variables are obtained by the inverse transformation of the standardized design variables of the contact point. If the contact point is found to be outside the feasible domain, it is necessary to search for the minimal point as the case in which the hypersphere is tangent to an intersection of several hyperplanes as is discussed in the following section.

3.2 Case of hypersphere tangent to an intersection of hyperplanes

When the contact point is outside the feasible domain, it is implied that the minimal point is decided by a point tangent to several hyperplanes, as is illustrated in Fig. 2, and not by one hyperplane. In the case of two variables as shown in this figure, the hyperplanes are straight lines and the intersection of the hyperplanes is a point. In this example of searching for the minimal point, contact point A, which is determined so as to make the distance from the

Fig. 2 Circle tangent to intersection of two lines

extreme point O positive and maximal, is first obtained. Since contact point A is outside the feasible domain, the point of interest is shifted from point O to point A . Then contact point B , which is determined so as to make the distance from the point A positive and maximal, can be obtained by the aforementioned procedure. When point B is outside the feasible domain again, a similar procedure is repeated to trace points C and D and so forth, as shown in Fig. 2. It is not necessary, however, to reiterate this procedure until the points of interest converge successively to an intersection of two hyperplanes. This is because we can estimate a set of decisive hyperplanes specifying the intersection to which the hypersphere is tangent, after the procedure is repeated J times at most. In this example, after three iterations lines 2-2 and 3-3 can be judged as the decisive equality constraint conditions. Then the minimal point can be determined through the following steps.

<Step 1>

First, we search for point $\{Y'\}$, which is determined so as to minimize the distance from the extreme point O under the decisive equality constraint conditions. If point $\{Y'\}$ satisfies all of the inequality constraint conditions, this $\{Y'\}$ is the minimal point under the constraint conditions. If one or more of the inequality constraint conditions are not satisfied, one must proceed to the next step.

<Step 2>

Choosing the inequality constraint condition which has the positive and maximal distance from $\{Y'\}$, we add it to the set of decisive equality constraint conditions already picked up and then repeat the procedure of Step 1.

Point $\{Y'\}$ in the procedure of Step 1 can be determined by use of the Moore-Penrose generalized inverse matrix⁽⁷⁾ or a technique of structural synthesis based on the concept of minimal design change⁽⁸⁾.

It is possible to judge which inequality constraint condition is satisfied in the stage of Step 1 and to decide which condition should be added to the set of

the decisive equality constraint conditions in the stage of Step 2 by means of evaluating the distance r_j from $\{Y'\}$ to each hyperplane.

4. Case of Hessian Matrix with Zero Eigenvalues

The method described in chapter 2 is valid for determination of the transformation matrix $[T]$ in the form of Eq. (2) in the case that the Hessian matrix is positive definite. In the case that the Hessian matrix is not positive definite, having some zero eigenvalues, we must devise a suitable transformation matrix in another way. Equation (9) is an objective function f with the Hessian matrix $[A]$ expressed as Eq. (10).

$$f = (x_1 + x_2)^2 \quad (9)$$

$$[A] = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \quad (10)$$

The eigenvalues of the matrix $[A]$ are $\lambda_1=2$ and $\lambda_2=0$. When we introduce $z=x_1+x_2$, f becomes the quadratic function with respect to a single variable z , and the extreme point is given as $z=x_1+x_2=0$. Namely, this function has the indefinite extreme point with the relation $x_1+x_2=0$. This means that not every design variable is independent when positive definiteness does not hold. If we can properly choose design variables independent of each other, positive definiteness can be assured. In the case that it is impossible to choose independent design variables, we must generate an approximate matrix $[A']$ by modifying some components of matrix $[A]$. Matrix $[A']$ can be formed by applying Greenstadt's method⁽⁹⁾. When matrix $[A]$ is expressed as Eq. (11) with eigenvalues λ_n and eigenvectors $\{\phi_n\}$ of matrix $[A]$,

$$[A] = \sum_{n=1}^N \lambda_n \{\phi_n\} \{\phi_n\}^T, \quad (11)$$

we can replace the zero eigenvalue with small value ε to modify the basic matrix $[A]$ into modified matrix $[A']$. The value of ε should be set considering computer digit accuracy. Every component of basic matrix $[A]$ is almost equal to the corresponding component of modified matrix $[A']$. The transformation matrix, determined on the basis of the eigenvalue analysis of modified matrix $[A']$, can be used for the tangent hypersphere method.

5. Numerical Examples of Tangent Hypersphere Method

5.1 [Example 1] Contact point found on intersection of hyperplanes

An example of the quadratic objective function of Eq. (12) under linear inequality constraint conditions of Eq. (13) is discussed, while the Hessian matrix is positive definite and the extreme point is outside the feasible domain. Figure 3 shows the constraint conditions given by Eq. (13) and the ellipse of Eq. (12)

tangent to the constraint conditions in the space of X variables.

$$f = \begin{Bmatrix} -\sqrt{2} \\ -\sqrt{2} \end{Bmatrix}^T \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^T \begin{bmatrix} 2.5 & -1.5 \\ -1.5 & 2.5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \quad (12)$$

$$\begin{cases} -x_1 - x_2 \leq -2 \\ -0.25x_1 - x_2 \leq -1 \\ -x_1 + x_2 \leq -2 \end{cases} \quad (13)$$

Transforming the X variables into Y variables, we can find the contact point of the hypersphere tangent to the hyperplanes in Y variables with minimal radius under the decisive equality conditions by the technique of minimal design change⁽⁸⁾. The point is obtained as $y_1=3.4$ and $y_2=-0.6$. When these are inversely transformed into the basic X variables, the minimal point Q is obtained as $x_1=2.4$ and $x_2=0.4$, resulting in the minimal value of the objective function equal to 7.96.

5.2 [Example 2] Hessian matrix with zero eigenvalue

The following objective function and constraint conditions are taken as an example of the singular Hessian matrix with a zero eigenvalue.

$$f = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}^T \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^T \begin{bmatrix} 0.5 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \quad (14)$$

$$\begin{cases} x_1 - x_2 \leq 6 \\ -x_1 \leq -2 \end{cases} \quad (15)$$

The eigenvalues of this Hessian matrix are obtained as $\lambda_1=2.5$ and $\lambda_2=0.0$. Replacing 0.0 with $\varepsilon=10^{-10}$ by the procedure described in chapter 4, we can apply the tangent hypersphere method to this example. The objective function composed of modified matrix $[A']$ has the extreme point of $y_1^*=-2 \times 10^4$ and $y_2^*=1 \times 10^4$. As this point is outside the feasible domain, the hypersphere with the center $\{Y^*\}$ tangent to the the feasible domain and with the minimal radius must be searched for. The contact point of hypersphere $\{Y\}$ is inversely transformed into the point in X variables, and then the solution of the minimized objective function is obtained as $x_1=2.0$, $x_2=-1.25$. The minimal value of the objective function is equal to

Fig. 3 Objective function and constraint conditions

0.875 at point Q shown in Fig. 4. The curved lines in this figure depict the contour lines of the objective function corresponding to various $\{X\}$.

6. Property of Tangent Hypersphere Method and Comparison with Multiplier Method

The tangent hypersphere method is based on the generation of the transformation matrix by use of eigenvalue analysis of the Hessian matrix. In the case of many design variables, central processing unit (CPU) time is inevitably increased due to the eigenvalue analysis. The merit of the simplicity of calculation for the minimal point once the eigenvalue analysis is carried out compensates for the deficiency of extended CPU time. Moreover, it is not necessary to set the initial values of design variables inside the feasible domain as in the gradient projection method⁽¹⁰⁾. An advantage of the present method is that the number of calculations required to obtain the minimal point can be estimated.

In order to examine the convergence of the solutions by the present method and by the other methods, the multiplier method is picked up, and the result for the same example is shown in Fig. 5.

$$f = \sum_{n=1}^3 x_n^2 \quad (16)$$

Fig. 4 Objective function composed of non-positive-definite Hessian matrix and constraint conditions

Fig. 5 Oscillation of solution by multiplier method

$$\left. \begin{aligned} -x_1 - x_2 - x_3 &\leq -10 \\ -2x_1 - x_2 - x_3 &\leq -1 \\ -x_1 - 2x_2 - x_3 &\leq -2 \\ -x_1 - x_2 - x_3 &\leq -3 \end{aligned} \right\} \quad (17)$$

In this example the minimal point given by the multiplier method oscillates with regularity, so that the minimal point cannot be determined. In this case at the 14-th iteration x_1 , x_2 and x_3 take values of 3.2504, 3.2504 and 3.4997, respectively. On the other hand, the correct values of x_1 , x_2 and x_3 equal to 3.333 are obtained exactly by the present method, when the perpendicular foot whose length from the extreme point is positive and maximal among four perpendiculars is chosen. This is an example that the rigorous result is obtained with less calculation by the present method than the multiplier method.

7. Shape Optimization in Linear Eigenvalue Problem of Beam Vibration

As an example of application of the tangent hypersphere method, the shape optimization in a linear, undamped eigenvalue problem of beam vibration is carried out by the finite element method. Suppose that an elastic beam is simply supported at two points as shown in Fig. 6. The moment of inertia of the beam cross section is taken as uniform for the initial design, that is, the first baseline design. The total length of the beam L , the uniform cross-sectional area A and the uniform moment of inertia I are set equal to 1 m, $113.04 \times 10^{-6} \text{ m}^2$ and $4637 \times 10^{-12} \text{ m}^4$, respectively. The Young's modulus and mass density ρ of the material are 205.8 GPa and 7800 kg/m^3 . This beam model is divided into ten elements of equal length.

7.1 Maximization of first-order eigenvalue by change of moment of inertia

The design variables are assigned only to the moment of inertia at each nodal point and the mass of the element is kept constant in this example. The

Fig. 6 Convergence of eigenvalue

stiffness matrix is generated by use of a linear beam element, under the assumption that the moment of inertia is distributed linearly along the beam axis. The first- and the second-order eigenvalues $\lambda(1)$ and $\lambda(2)$ of the initial design are equal to $0.215 \times 10^6/\text{sec}^2$ and $2.119 \times 10^6/\text{sec}^2$, respectively, and the difference between them is $1.904 \times 10^6/\text{sec}^2$. The task in this section is to maximize the first-order eigenvalue under the inequality constraint condition that the eigenvalue difference is kept less than $2 \times 10^6/\text{sec}^2$. Inequality constraint conditions are added such that any moment of inertia should be between 50% and 200% of the initial value. The moments of inertia at the points P and Q are assumed not to change. The objective function $\lambda(1)$ and the inequality conditions are given as follows, where the number of the design variables N is 9 and that of the constraint conditions J is 19.

$$\lambda(1) = \bar{\lambda}(1) + \sum_{n=1}^N \lambda(1)_n x_n + (1/2) \sum_{n=1}^N \sum_{m=1}^N \lambda(1)_{nm} x_n x_m \quad (18)$$

$$\left. \begin{aligned} 0.5 \bar{I}_n &\leq \bar{I}_n (1 + x_n) \leq 2 \bar{I}_n \\ \lambda(2) - \lambda(1) &\leq 2 \times 10^6 \end{aligned} \right\} \quad (19)$$

As the second-order approximation is taken for the objective function $\lambda(1)$, the structural modification should be reiterated to overcome the deficiency of the approximation.

Figure 6 shows the convergence of the eigenvalue change with respect to the structural modification, and Fig. 7 illustrates the resultant distribution of the moment of inertia along the beam length after eight iterations. At each iteration the solution of the tangent hypersphere method is obtained at Step 2 mentioned in section 3.2. It is obvious that the value of the first-order eigenvalue becomes maximal by making all the moments of inertia equal to $2 \bar{I}$, unless the constraint condition for the eigenvalue difference is imposed. Instead of such a simple solution, it is shown in Fig. 7 that the moment of inertia is decreased on the right side of the beam under the constraint condition.

Fig. 7 Ratio of optimal moment of inertia to initial moment distributed along beam axis

7.2 Minimization of weight by change of cross-sectional area

In this section, the design variables are assigned to the nodal cross-sectional area whose distribution is assumed to be linear in an element. Under the constraint condition such that the first-order eigenvalue should be less than the prescribed value, the distribution of the cross-sectional area which minimizes the weight of beam is searched for. When the siding constraint conditions such that all cross-sectional areas should be larger than 50% and less than 200% of the initial value are added, this problem is dual with the one dealt with in section 7.1. The objective function of the total weight of the beam W is expressed by use of the element length l ,

$$W = \rho g l \left\{ (A_1 + A_{11})/2 + \sum_{n=2}^{10} A_n \right\}, \quad (20)$$

where g denotes the acceleration of gravity, and the cross-sectional area at the n -th node is written as $A_n = \bar{A}_n(1 + x_n)$. The objective function W does not include the quadratic term of $\{X\}$ as given in Eq. (21). The constraint conditions of this problem are given by Eq. (22).

$$W = \bar{W} + \sum_{n=1}^N W_n x_n \quad (21)$$

$$\left. \begin{aligned} 0.5\bar{A}_n \leq \bar{A}_n(1 + x_n) \leq 2\bar{A}_n \\ \lambda(1) \leq 1.5 \times 10^5 \end{aligned} \right\} \quad (22)$$

In the case that an objective function includes no quadratic term, it is necessary to modify the objective function slightly when the tangent hypersphere method is employed. When the components of the modified Hessian matrix are too small, the modified objective function gives a remarkably large absolute value for the extreme point $\{X^*\}$ and might render the cross-sectional area negative. Therefore, the Hessian matrix must be modified so that each value of the extreme point $\{X^*\}$ is larger than -1 to render the cross-sectional area positive.

Figure 8 shows the distribution of cross-sectional area under the constraint condition that the first-order eigenvalue is lowered considerably below the initial eigenvalue. Figures 9 and 10 show the iteration history of the eigenvalue and weight, and it can be said that they converge after five structural modifications. The solution at the first structural modification is obtained as the point tangent to a hyperplane discussed in section 3.1. The solution at the second structural modification is found as the point tangent to the intersection of hyperplanes at Step 1 discussed in section 3.2. At the third to fifth modifications the solutions are obtained only at Step 2. In general, the weight should be increased in order to decrease the eigenvalue. This example shows that the beam weight is decreased successfully to 77.3% of

Fig. 8 Ratio of optimal cross-sectional area to initial area distributed along beam axis

Fig. 9 Convergence of eigenvalue

Fig. 10 Ratio of optimal weight to initial weight after convergence

the initial weight and the first-order eigenvalue is lowered at the same time, even though weight is generally increased in order to lower the eigenvalue.

8. Conclusion

The tangent hypersphere method is proposed to search for the design variables which minimize an objective function under inequality constraint conditions. The Hessian matrix is constituted by the second-order sensitivities of the objective function. The objective function is standardized into a hypersphere based on the result of the eigenvalue analysis of the Hessian matrix. The minimal value of the objective function is obtained as the hypersphere which is tangent to hyperplanes representing inequality constraint conditions. The process and efficiency of the present method are shown by simple but typical numerical examples, and discussed in comparison with the multiplier method. In addition, the present method is applied to the maximization of the eigenvalue and minimization of the weight in regard to a linear, undamped eigenvalue problem of elastic beam vibration.

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