The State of the Art in Computational Dynamic Fracture Mechanics*

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The state of the art in computational mechanics of dynamic fracture is reviewed. The concept of dynamic fracture mechanics is essentially based on asymptotic behaviors of a dynamically propagating crack and of a stationary crack subjected to dynamic loading. First, the asymptotic solutions of near-tip field of an elastodynamically propagating crack under steady-state conditions and under transient conditions are summarized. Next, the features of the dynamic J integral pertaining to nonlinear dynamic fracture mechanics as well as to elastodynamic fracture mechanics are described. The dynamic J integral may play important roles in both theoretical and computational aspects of dynamic fracture mechanics. Special attention is focused on various computational models of dynamic crack propagation, which are used in the finite difference method, finite-element method and boundary-element method. Various techniques to overcome inherent difficulties in the discretization of fast-moving surface problems associated with crack-tip singularities are thoroughly reviewed and summarized.

Key Words : Fracture Mechanics, Computational Mechanics, Brittle Fracture, Finite-Element Method, Finite Difference Method, Boundary-Element Method, Dynamic Fracture, Fast Fracture Mechanics, Impact Fracture Mechanics, Dynamic Crack Propagation, Numerical Model

1. Introduction

Dynamic fracture mechanics can be broadly classified into (i) *impact fracture mechanics* and (ii) *fast fracture mechanics*. Impact fracture mechanics is concerned with bodies with stationary cracks that are subjected to impact or dynamic loadings. In such impact fracture phenomena, the effects of material inertia and stress wave interaction play significant roles. Contrary to this, fast fracture mechanics is concerned mainly with various motions of fast propagating crack tips such as fracture initiation, propagation, acceleration and deceleration, arrest, kinking and curving, and branching. In the vicinity of the fast propagating crack tip, a material point is

subjected to a high loading rate due to the passage of a concentrated stress region around the crack tip. Thus, in the aforementioned crack-tip motions, the effects of crack velocity play significant roles.

Due to the highly transient features in dynamic fracture as described above, analytical solutions in dynamic fracture mechanics are limited to the cases of simple loadings and of unbounded plane bodies. Usually, the analytical solutions become invalid when the stress waves reflected from the boundaries of a finite body arrive at the crack tip. Thus, it often becomes indispensable to use computational methods to analyze dynamic fracture problems in finite solids, such as laboratory test specimens.

On the other hand, many experimental approaches have been devised to study the process of dynamic fracture. The Charpy test and the dynamic tear test are typically used in impact fracture mechanics, while the tests on the double cantilever beam (DCB) specimen and the compact tension (CT)

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specimen are used in fast fracture mechanics. These laboratory specimens have relatively small dimensions. Moreover, due to the very short time spans of the event, it is difficult to directly measure higher-order physical quantities such as energy distribution, instantaneous dynamic energy release rate and dynamic stress field close to the crack tips in those test specimens, particularly in opaque structural steel specimens of practical interest. To overcome these difficulties and to obtain more detailed information on dynamic fracture processes, hybrid experimental-numerical methods of laboratory tests have been actively developed.

Therefore, the advancement of dynamic fracture mechanics relies heavily on parallel advances in computational methods for dynamic fracture simulation^{(1),(2)}. This paper is an attempt to summarize the state of the art in computational dynamic fracture mechanics, as well as that of some of the author's work in this area.

2. Asymptotic Solutions of Near-Tip Field of Elastodynamically Propagating Crack

2.1 Steady-state elastodynamic crack propagation

The general solutions (eigensolutions) for the stress and displacement fields near the crack tip, under a steady-state condition, were obtained by Nishioka and Atluri⁽³⁾ in a unified fashion for all three of the fracture modes:

$$\sigma_{11n} = \frac{K_n^0 B_I(C)}{(2\pi)^{1/2}} \frac{n(n+1)}{2} \left\{ (1 + 2\beta_1^2 - \beta_2^2) r_1^{n/2-1} \cos\left(\frac{n}{2} - 1\right) \theta_1 - 2h(n) r_2^{n/2-1} \cos\left(\frac{n}{2} - 1\right) \theta_2 \right\} \\ + \frac{K_n^* B_{II}(C)}{(2\pi)^{1/2}} \frac{n(n+1)}{2} \left\{ (1 + 2\beta_1^2 - \beta_2^2) r_1^{n/2-1} \sin\left(\frac{n}{2} - 1\right) \theta_1 - 2h(\bar{n}) r_2^{n/2-1} \sin\left(\frac{n}{2} - 1\right) \theta_2 \right\} \quad (1 \cdot a)$$

$$\sigma_{22n} = \frac{K_n^0 B_I(C)}{(2\pi)^{1/2}} \frac{n(n+1)}{2} \left\{ -(1 + \beta_2^2) r_1^{n/2-1} \cos\left(\frac{n}{2} - 1\right) \theta_1 + 2h(n) r_2^{n/2-1} \cos\left(\frac{n}{2} - 1\right) \theta_2 \right\} \\ + \frac{K_n^* B_{II}(C)}{(2\pi)^{1/2}} \frac{n(n+1)}{2} \left\{ -(1 + \beta_2^2) r_1^{n/2-1} \sin\left(\frac{n}{2} - 1\right) \theta_1 + 2h(\bar{n}) r_2^{n/2-1} \sin\left(\frac{n}{2} - 1\right) \theta_2 \right\} \quad (1 \cdot b)$$

$$\sigma_{12n} = \frac{K_n^0 B_I(C)}{(2\pi)^{1/2}} \frac{n(n+1)}{2} \left\{ -2\beta_1 r_1^{n/2-1} \sin\left(\frac{n}{2} - 1\right) \theta_1 + \frac{(1 + \beta_2^2)}{\beta_2} h(n) r_2^{n/2-1} \sin\left(\frac{n}{2} - 1\right) \theta_2 \right\} \\ + \frac{K_n^* B_{II}(C)}{(2\pi)^{1/2}} \frac{n(n+1)}{2} \left\{ +2\beta_1 r_1^{n/2-1} \cos\left(\frac{n}{2} - 1\right) \theta_1 - \frac{(1 + \beta_2^2)}{\beta_2} h(\bar{n}) r_2^{n/2-1} \cos\left(\frac{n}{2} - 1\right) \theta_2 \right\} \quad (1 \cdot c)$$

$$\sigma_{13n} = \frac{K_n^+ B_{III}(C)}{(2\pi)^{1/2}} \frac{n(n+1)}{2} r_2^{n/2-1} \begin{cases} \sin\left(\frac{n}{2} - 1\right) \theta_2 & : n \text{ odd} \\ \cos\left(\frac{n}{2} - 1\right) \theta_2 & : n \text{ even} \end{cases} \quad (1 \cdot d)$$

$$\sigma_{23n} = \frac{K_n^+ B_{III}(C)}{(2\pi)^{1/2}} \frac{n(n+1)}{2} \beta_2 r_2^{n/2-1} \begin{cases} \cos\left(\frac{n}{2} - 1\right) \theta_2 & : n \text{ odd} \\ -\sin\left(\frac{n}{2} - 1\right) \theta_2 & : n \text{ even} \end{cases} \quad (1 \cdot e)$$

$$\sigma_{33n} = \begin{cases} 0 & : \text{plane stress} \\ \nu(\sigma_{11n} + \sigma_{22n}) & : \text{plane strain} \end{cases} \quad (1 \cdot f)$$

and

$$u_{1n} = \frac{K_n^0 B_I(C)}{2\mu} \left(\frac{2}{\pi}\right)^{1/2} (n+1) \left\{ r_1^{n/2} \cos \frac{n}{2} \theta_1 - h(n) r_2^{n/2} \cos \frac{n}{2} \theta_2 \right\} \\ + \frac{K_n^* B_{II}(C)}{2\mu} \left(\frac{2}{\pi}\right)^{1/2} (n+1) \left\{ r_1^{n/2} \sin \frac{n}{2} \theta_1 - h(\bar{n}) r_2^{n/2} \sin \frac{n}{2} \theta_2 \right\} \quad (2 \cdot a)$$

$$u_{2n} = \frac{K_n^0 B_I(C)}{2\mu} \left(\frac{2}{\pi}\right)^{1/2} (n+1) \left\{ -\beta_1 r_1^{n/2} \sin \frac{n}{2} \theta_1 + \frac{h(n)}{\beta_2} r_2^{n/2} \sin \frac{n}{2} \theta_2 \right\} \\ + \frac{K_n^* B_{II}(C)}{2\mu} \left(\frac{2}{\pi}\right)^{1/2} (n+1) \left\{ \beta_1 r_1^{n/2} \cos \frac{n}{2} \theta_1 - \frac{h(\bar{n})}{\beta_2} r_2^{n/2} \cos \frac{n}{2} \theta_2 \right\} \quad (2 \cdot b)$$

$$u_{3n} = \frac{K_n^+ B_{III}(C)}{2\mu} \left(\frac{2}{\pi}\right)^{1/2} (n+1) r_2^{n/2} \begin{cases} \sin \frac{n}{2} \theta_2 & : n \text{ odd} \\ \cos \frac{n}{2} \theta_2 & : n \text{ even} \end{cases} \quad (2 \cdot c)$$

where C is the crack velocity; β_1 and β_2 are the crack-velocity parameters normalized by the dilatational and shear wave velocities C_d and C_s , and are defined as $\beta_1^2 = 1 - C^2/C_d^2$ and $\beta_2^2 = 1 - C^2/C_s^2$, respectively; n is the integer number ($0 \leq n < \infty$); $\bar{n} = n + 1$ and $h(n) = \{2\beta_1\beta_2/(1 + \beta_2^2) : n \text{ odd, or } (1 + \beta_2^2)/2 : n \text{ even}\}$; (r_1, θ_1) and (r_2, θ_2) are the scaled polar coordinate systems attached to the propagating crack tip. The undetermined parameters K_n^0 , K_n^* and K_n^+ are related to the fracture Modes I, II and III, respectively. The functions of crack velocity $B_M(C)$ ($M = \text{I, II and III}$) are defined by

$$B_I(C) = (1 + \beta_2^2)/D(C), \quad B_{II}(C) = 2\beta_2/D(C), \\ B_{III}(C) = 1/\beta_2 \quad (3 \cdot a, b, c)$$

and

$$D(C) = 4\beta_1\beta_2 - (1 + \beta_2^2)^2. \quad (3 \cdot d)$$

The equation $D(C)=0$ is the well-known Rayleigh equation and has the roots of $C=0$ and C_R (Rayleigh wave speed).

The general solutions given in Eqs. (1) and (2) contain the zero stresses and rigid-body motions ($n=0$), the singular stresses and corresponding displacements ($n=1$), the constant stresses and linear displacements ($n=2$), and the higher-order terms ($n \geq 3$). Thus, the parameters K_I^0 , K_I^* and K_I^+ are equivalent to the dynamic stress-intensity factors K_I , K_{II} and K_{III} , respectively.

The Mode I eigensolution that is related to the constant K_I^0 has been derived by Malluck⁽⁴⁾ following the scheme employed by Rice⁽⁵⁾. However, it is noted that the present expressions for the Mode I eigensolution differ from those of Malluck⁽⁴⁾ in the coefficients including the number n .

Reducing the crack velocity C to zero in Eqs. (1) and (2) and taking the limits as $C \rightarrow 0$ (using L'Hospital's rule) properly, one can obtain the eigenfunctions that are valid for a stationary crack. These can be shown to agree with those obtained independently by Williams⁽⁶⁾.

The singular stress field and the corresponding displacement field are given by the $n=1$ terms:

$$\begin{aligned} \sigma_{11} = & \frac{K_I B_I(C)}{\sqrt{2\pi}} \left\{ (1 + 2\beta_1^2 - \beta_2^2) r_1^{-1/2} \cos \frac{\theta_1}{2} \right. \\ & \left. - \frac{4\beta_1\beta_2}{(1 + \beta_2^2)} r_2^{-1/2} \cos \frac{\theta_2}{2} \right\} \\ & + \frac{K_{II} B_{II}(C)}{\sqrt{2\pi}} \left\{ -(1 + 2\beta_1^2 - \beta_2^2) r_1^{-1/2} \sin \frac{\theta_1}{2} \right. \\ & \left. + (1 + \beta_2^2) r_2^{-1/2} \sin \frac{\theta_2}{2} \right\} \end{aligned} \quad (4 \cdot a)$$

$$\begin{aligned} \sigma_{22} = & \frac{K_I B_I(C)}{\sqrt{2\pi}} \left\{ -(1 + \beta_2^2) r_1^{-1/2} \cos \frac{\theta_1}{2} \right. \\ & \left. + \frac{4\beta_1\beta_2}{(1 + \beta_2^2)} r_2^{-1/2} \cos \frac{\theta_2}{2} \right\} \\ & + \frac{K_{II} B_{II}(C)}{\sqrt{2\pi}} \left\{ (1 + \beta_2^2) r_1^{-1/2} \sin \frac{\theta_1}{2} \right. \\ & \left. - (1 + \beta_2^2) r_2^{-1/2} \sin \frac{\theta_2}{2} \right\} \end{aligned} \quad (4 \cdot b)$$

$$\begin{aligned} \sigma_{12} = & \frac{K_I B_I(C)}{\sqrt{2\pi}} \left\{ 2\beta_1 r_1^{-1/2} \sin \frac{\theta_1}{2} - 2\beta_1 r_2^{-1/2} \sin \frac{\theta_2}{2} \right\} \\ & + \frac{K_{II} B_{II}(C)}{\sqrt{2\pi}} \left\{ 2\beta_1 r_1^{-1/2} \cos \frac{\theta_1}{2} \right. \\ & \left. - \frac{(1 + \beta_2^2)^2}{2\beta_2} r_2^{-1/2} \cos \frac{\theta_2}{2} \right\} \end{aligned} \quad (4 \cdot c)$$

$$\sigma_{13} = -\frac{K_{III} B_{III}(C)}{\sqrt{2\pi}} r_2^{-1/2} \sin \frac{\theta_2}{2} \quad (4 \cdot d)$$

$$\sigma_{23} = \frac{K_{III} B_{III}(C)}{\sqrt{2\pi}} \beta_2 r_2^{-1/2} \cos \frac{\theta_2}{2} \quad (4 \cdot e)$$

$$\sigma_{33} = \begin{cases} 0 & : \text{plane stress} \\ -\frac{K_{II} B_{II}(C)}{\sqrt{2\pi}} 2\nu(\beta_1^2 - \beta_2^2) r_1^{-1/2} \sin \frac{\theta_1}{2} \\ & : \text{plane strain} \end{cases} \quad (4 \cdot f)$$

and

$$\begin{aligned} u_1 = & \frac{K_I B_I(C)}{\mu} \sqrt{\frac{2}{\pi}} \left\{ r_1^{1/2} \cos \frac{\theta_1}{2} \right. \\ & \left. - \frac{2\beta_1\beta_2}{(1 + \beta_2^2)} r_2^{1/2} \cos \frac{\theta_2}{2} \right\} \\ & + \frac{K_{II} B_{II}(C)}{\mu} \sqrt{\frac{2}{\pi}} \left\{ r_1^{1/2} \sin \frac{\theta_1}{2} \right. \\ & \left. - \frac{(1 + \beta_2^2)}{2} r_2^{1/2} \sin \frac{\theta_2}{2} \right\} \end{aligned} \quad (5 \cdot a)$$

$$\begin{aligned} u_2 = & \frac{K_I B_I(C)}{\mu} \sqrt{\frac{2}{\pi}} \left\{ -\beta_1 r_1^{1/2} \sin \frac{\theta_1}{2} \right. \\ & \left. + \frac{2\beta_1}{(1 + \beta_2^2)} r_2^{1/2} \sin \frac{\theta_2}{2} \right\} \\ & + \frac{K_{II} B_{II}(C)}{\mu} \sqrt{\frac{2}{\pi}} \left\{ \beta_1 r_1^{1/2} \cos \frac{\theta_1}{2} \right. \\ & \left. - \frac{(1 + \beta_2^2)}{2\beta_2} r_2^{1/2} \cos \frac{\theta_2}{2} \right\} \end{aligned} \quad (5 \cdot b)$$

$$u_3 = \frac{K_{III} B_{III}(C)}{\mu} \sqrt{\frac{2}{\pi}} r_2^{1/2} \sin \frac{\theta_2}{2} \quad (5 \cdot c)$$

Although the complete singular stress and the corresponding displacement fields for all three of the fracture modes were obtained in a unified fashion by Nishioka and Atluri⁽³⁾, several works concerned with the derivation of the singular stress field in the individual modes were conducted by Rice⁽⁵⁾, Achenbach and Bazant⁽⁷⁾, Freund⁽⁸⁾, and Burgers⁽⁹⁾.

2.2 Transient elastodynamic crack propagation

If we consider fairly severe transient crack-tip motion, effects of the changes in the crack velocity (C) and coefficient for the asymptotic solutions (K_I^0 , K_I^* and K_I^+) cannot be ignored. The transient feature of Mode I elastodynamic near-tip field during crack propagation has recently been investigated by Freund and Rosakis⁽¹⁰⁾ and Rosakis et al.⁽¹¹⁾ The mixed-mode transient near-tip fields have also been investigated by Deng⁽¹²⁾. Here we rewrite the Mode I transient asymptotic stress field⁽¹¹⁾ using the present notation:

$$\begin{aligned} \frac{\sigma_{11}}{\mu} = & \frac{K_I(t) B_I(C)}{\mu \sqrt{2\pi}} \left\{ (1 + 2\beta_1^2 - \beta_2^2) r_1^{-1/2} \cos \frac{\theta_1}{2} \right. \\ & \left. - \frac{4\beta_1\beta_2}{(1 + \beta_2^2)} r_2^{-1/2} \cos \frac{\theta_2}{2} \right\} \\ & + 2(1 + 2\beta_1^2 - \beta_2^2) A_1(t) + 4\beta_2 B_1(t) \\ & + \left\{ D_1 \{A_0(t)\} \left[\frac{(1 + \beta_1^2)(\beta_1^2 - \beta_2^2) + (1 - \beta_1^2)^2}{2(1 - \beta_1^2)} \cos \frac{\theta_1}{2} \right. \right. \\ & \left. \left. + \frac{1 + 2\beta_1^2 - \beta_2^2}{8} \cos \frac{3\theta_1}{2} \right] \right\} \\ & + \frac{B_1(t)}{2} \left[\frac{(2\beta_1^2 - \beta_2^2 - 1)\beta_1^2}{1 - \beta_1^2} \cos \frac{\theta_1}{2} + \left\{ \frac{5(2\beta_1^2 - \beta_2^2)}{8} \right\} \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{\beta_1^2 - \beta_2^2}{1 - \beta_1^2} - \frac{3}{8} \cos \frac{3\theta_1}{2} \\
& + \frac{1 + 2\beta_1^2 - \beta_2^2}{16} \cos \frac{7\theta_1}{2} \Big] \\
& + \frac{15(1 + 2\beta_1^2 - \beta_2^2)}{4} A_2(t) \cos \frac{\theta_1}{2} \Big\} r_1^{1/2} \\
& + 2\beta_2 \left\{ \frac{1}{8} D_s^1 \{B_0(t)\} \cos \frac{3\theta_2}{2} + \frac{1}{4} B_s(t) \left(\cos \frac{\theta_2}{2} \right. \right. \\
& + \left. \left. \frac{1}{4} \cos \frac{3\theta_2}{2} + \frac{1}{8} \cos \frac{7\theta_2}{2} \right) \right. \\
& + \left. \frac{15}{4} B_2(t) \cos \frac{\theta_2}{2} \right\} r_2^{1/2} + O(r_{1,2}) \quad (6 \cdot a)
\end{aligned}$$

$$\begin{aligned}
\frac{\sigma_{22}}{\mu} = & \frac{K_I(t) B_1(C)}{\mu \sqrt{2\pi}} \left\{ -(1 + \beta_2^2) r_1^{-1/2} \cos \frac{\theta_1}{2} \right. \\
& + \left. \frac{4\beta_1\beta_2}{(1 + \beta_2^2)} r_2^{-1/2} \cos \frac{\theta_2}{2} \right\} \\
& - 2(1 + \beta_2^2) A_1(t) - 4\beta_2 B_1(t) \\
& + \left\{ D_I^1 \{A_0(t)\} \left[\frac{(1 + \beta_1^2)(\beta_1^2 - \beta_2^2) - (1 - \beta_1^2)^2}{2(1 - \beta_1^2)} \cos \frac{\theta_1}{2} \right. \right. \\
& - \left. \left. \frac{1 + \beta_2^2}{8} \cos \frac{3\theta_1}{2} \right] \right. \\
& + \left. \frac{B_I(t)}{2} \left[\frac{(1 - \beta_2^2)\beta_1^2}{1 - \beta_1^2} \cos \frac{\theta_1}{2} + \left\{ \frac{3 - 5\beta_2^2}{8} \right. \right. \right. \\
& - \left. \left. \frac{\beta_1^2 - \beta_2^2}{1 - \beta_1^2} \right\} \cos \frac{3\theta_1}{2} - \frac{1 + \beta_2^2}{16} \cos \frac{7\theta_1}{2} \right] \right. \\
& - \left. \frac{15(1 + \beta_2^2)}{4} A_2(t) \cos \frac{\theta_1}{2} \right\} r_1^{1/2} \\
& - 2\beta_2 \left\{ \frac{1}{8} D_s^1 \{B_0(t)\} \cos \frac{3\theta_2}{2} + \frac{1}{4} B_s(t) \left(\cos \frac{\theta_2}{2} \right. \right. \\
& + \left. \left. \frac{1}{4} \cos \frac{3\theta_2}{2} + \frac{1}{8} \cos \frac{7\theta_2}{2} \right) \right. \\
& + \left. \frac{15}{4} B_2(t) \cos \frac{\theta_2}{2} \right\} r_2^{1/2} + O(r_{1,2}) \quad (6 \cdot b)
\end{aligned}$$

$$\begin{aligned}
\frac{\sigma_{12}}{\mu} = & \frac{K_I(t) B_1(C)}{\mu \sqrt{2\pi}} 2\beta_1 \left(r_1^{-1/2} \sin \frac{\theta_1}{2} - r_2^{-1/2} \sin \frac{\theta_2}{2} \right) \\
& + 2\beta_1 \left\{ \frac{1}{8} D_I^1 \{A_0(t)\} \sin \frac{3\theta_1}{2} - \frac{1}{4} B_I(t) \left(\sin \frac{\theta_1}{2} \right. \right. \\
& - \left. \left. \frac{1}{4} \sin \frac{3\theta_1}{2} - \frac{1}{8} \sin \frac{7\theta_1}{2} \right) \right. \\
& - \left. \frac{15}{4} A_2(t) \sin \frac{\theta_1}{2} \right\} r_1^{1/2} \\
& - \left\{ \frac{1}{2} D_s^1 \{B_0(t)\} \left[(1 - \beta_2^2) \sin \frac{\theta_2}{2} - \frac{1 + \beta_2^2}{4} \sin \frac{3\theta_2}{2} \right] \right. \\
& - \left. \frac{B_s(t)}{2} \left[\beta_2^2 \sin \frac{\theta_2}{2} + \frac{5\beta_2^2 - 3}{8} \sin \frac{3\theta_2}{2} \right. \right. \\
& + \left. \left. \frac{1 + \beta_2^2}{16} \sin \frac{7\theta_2}{2} \right] \right. \\
& + \left. \frac{15(1 + \beta_2^2)}{4} B_2(t) \sin \frac{\theta_2}{2} \right\} r_2^{1/2} + O(r_{1,2}) \quad (6 \cdot c)
\end{aligned}$$

where $A_m(t)$ and $B_m(t)$ ($m=0, 1, 2$) are the coefficients of the asymptotic expansion, and other coefficients and operators in Eqs. (6·a-c) are defined by

$$\begin{aligned}
B_I(t) &= \frac{2C^2(t) B_1(C)}{\mu \sqrt{2\pi} \beta_1^4 C_d^4} K_I(t) \ddot{a}, \\
B_s(t) &= -\frac{2C^2(t)}{\mu \sqrt{2\pi} \beta_2^4 C_s^4} \frac{2\beta_1 B_1(C)}{(1 + \beta_2^2)} K_I(t) \ddot{a} \quad (7 \cdot a, b)
\end{aligned}$$

and

$$D_I^1 \{A_0(t)\} = -\frac{4C^{1/2}(t)}{\mu \sqrt{2\pi} \beta_1^2 C_d^2} \frac{d}{dt} \{C^{1/2}(t) B_1(C) K_I(t)\} \quad (8)$$

$$D_s^1 \{B_0(t)\} = \frac{4C^{1/2}(t)}{\mu \sqrt{2\pi} \beta_2^2 C_s^2} \frac{d}{dt} \left\{ C^{1/2}(t) \frac{2\beta_1 B_1(C)}{(1 + \beta_2^2)} K_I(t) \right\}. \quad (9)$$

The coefficients $A_m(t)$ and $B_m(t)$ ($m=0, 1, 2, \dots$) can be related to the coefficients K_n^0 ($n \equiv m+1=1, 2, 3, \dots$) in Eqs. (1) and (2):

$$A_{n-1}(t) = \frac{2(n+1)}{(n+2)} \frac{B_1(C)}{\mu \sqrt{2\pi}} K_n^0(t) \quad (10)$$

$$B_{n-1}(t) = -\frac{2(n+1)}{(n+2)} \frac{h(n) B_1(C)}{\mu \sqrt{2\pi} \beta_2} K_n^0(t). \quad (11)$$

Therefore, we can summarize the transient asymptotic stress field for a non-uniformly propagating crack tip, as follows:

$$\begin{aligned}
\sigma_{ij}|_{\text{transient}} = & (\sigma_{ij})_{n=1}|_{\text{steady}} + (\sigma_{ij})_{n=2}|_{\text{steady}} \\
& \quad \text{(singular term)} \quad \text{(constant-stress term)} \\
& + (\sigma_{ij})_{n \geq 3}|_{\text{steady}} + (\sigma_{ij})_{n \geq 3}|_{\text{transient}} \\
& \quad \text{(high-order terms)} \quad \text{(high-order terms)}. \quad (12)
\end{aligned}$$

The transient effects do not appear in the singular and constant-stress terms, but they appear only in the higher-order terms greater than or equal to the third order ($n \geq 3$). Furthermore, since the third-order term has the order of $O(r^{1/2})$, the higher-order terms greater than or equal to the third order ($n \geq 3$) become negligibly small as compared with the singular and constant-stress terms in the vicinity of the crack tip ($r \rightarrow 0$).

3. Dynamic J Integral in Dynamic Fracture Mechanics

3.1 Elastodynamic fracture mechanics

The so-called J integral⁽¹³⁾ has played an important role in static fracture mechanics. From the theoretical and computational points of view, the static J integral has the following salient features: (i) it physically represents energy release rate; (ii) it has the property of the path-independent integral, which gives a unique value for an arbitrary integral path surrounding the crack tip; (iii) it can be related to the stress intensity factors by arbitrarily shrinking the integral path to the crack tip.

In the case of dynamic fracture mechanics, Nishioka and Atluri⁽³⁾ have derived the dynamic J integral (J') which has the three aforementioned features. Using a global Cartesian coordinate system X_k , for a crack propagating at an angle θ_0 measured from X_1 (see Fig. 1), the components of dynamic J integral J'_k that are equivalent to the components of the energy release rate are represented by Nishioka and Atluri⁽³⁾ in a path-independent form:

$$G = J'_1 \cos \theta_0 + J'_2 \sin \theta_0 \quad (13)$$

and

Fig. 1 Definition of integral paths and coordinate systems

$$J_k = \lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon} [(W + K)n_k - t_i u_{i,k}] dS$$

$$= \lim_{\epsilon \rightarrow 0} \left\{ \int_{\Gamma + \Gamma_c} [(W + K)n_k - t_i u_{i,k}] dS \right.$$

$$\left. + \int_{V_r - V_\epsilon} [\rho \ddot{u}_i u_{i,k} - \rho \dot{u}_i \dot{u}_{i,k}] dV \right\} \quad (14)$$

In Eq. (14), W and K are the strain and kinetic energy densities, respectively, and $(\cdot)_{,k} \equiv \partial(\cdot)/\partial X_k$. Integral paths are defined as follows: Γ is a far-field contour that encloses the crack tip and envelops a volume V_r ; Γ_ϵ is a near-field contour arbitrarily close to the propagating crack tip and envelops an infinitesimal volume V_ϵ ; and Γ_c is the crack surface enclosed by Γ . Physically, the near-tip region V_ϵ can be considered as the process zone in which micro-processes associated with fracture occur.

The crack-axis components of the dynamic J integral can be expressed by the following coordinate transformation:

$$J_k^0 = \alpha_{ki}(\theta_0) J_i^0 \quad (15)$$

Then, using a circular path that shrinks to the propagating crack tip, the dynamic J integral can be related to the instantaneous stress intensity factors for the elastodynamically propagating crack with velocity C , as in Ref. (3):

$$J_1^0 = \frac{1}{2\mu} \{A_I(C)K_I^2 + A_{II}(C)K_{II}^2 + A_{III}(C)K_{III}^2\} \quad (16\cdot a)$$

$$J_2^0 = -\frac{A_{IV}(C)}{\mu} K_I K_{II} \quad (16\cdot b)$$

where $(A_I, \dots, A_{IV})$ are functions of the crack velocity, and are given by

$$A_I(C) = \beta_1(1 - \beta_2^2)/D(C) \quad (17\cdot a)$$

$$A_{II}(C) = \beta_2(1 - \beta_2^2)/D(C) \quad (17\cdot b)$$

$$A_{III}(C) = \frac{1}{\beta_2} \quad (17\cdot c)$$

$$A_{IV}(C) = \frac{(\beta_1 - \beta_2)(1 - \beta_2^2)\bar{D}(C)}{2\{D(C)\}^2} \left[\frac{(2 + \beta_1 + \beta_2)}{\sqrt{(1 + \beta_1)(1 + \beta_2)}} - \frac{4(1 + \beta_2^2)}{\bar{D}(C)} \right] \quad (17\cdot d)$$

where

$$\bar{D}(C) = 4\beta_1\beta_2 + (1 + \beta_2^2)^2. \quad (17\cdot e)$$

For a stationary crack ($C=0$) under impact as well as static conditions, the velocity functions reduce to $A_I(0) = A_{II}(0) = A_{IV}(0) = (\chi + 1)/4$, $A_{III}(0) = 1$, where $\chi = (3 - \nu)/(1 + \nu)$ for plane stress and $\chi = 3 - 4\nu$ for plane strain.

The existence of many path-independent integrals that do not physically represent the energy release rate was demonstrated in Ref. (14). The process zone of brittle (elastic) fracture may be negligibly small compared with the size of the crack itself. Thus, an infinitesimal process zone ($V_\epsilon \rightarrow 0$) can be considered in the evaluation of the dynamic J integral. However, if the dynamic J integral or the other integrals converge to significantly different values for different shapes of infinitesimal path ($\Gamma_\epsilon \rightarrow 0$), those integrals cannot be used as a crack-tip parameter due to non-uniqueness for the same state of crack-tip field.

The invariance of the dynamic J integral with respect to the shape of the infinitesimal process zone was examined by Nishioka⁽¹⁵⁾. It is shown in Ref. (15) that (i) for the J_1^0 integral, its invariance with respect to the shape of the infinitesimal process zone is theoretically verified; (ii) for the J_2^0 integral, numerical demonstration for circular and various rectangular process zones indicates its invariance; and (iii) the other integrals depend on the shape of the infinitesimal process zone. It is also shown in Ref. (15) that the dynamic J integral components are finite even though they have apparent non-integrable singularities in their expressions.

Recently, Van Vroonheven⁽¹⁶⁾ has analytically evaluated the J_2^0 integral for a special case of the rectangular path with the lateral side of 2ϵ and the vertical side of 2δ . After the substitution of the singular stress and displacement fields given by Eqs. (4) and (5), the limit for $\epsilon \rightarrow 0$ is taken first, followed by $\delta \rightarrow 0$. This calculation results in

$$J_2^0 = -\frac{H_{IV}(C)}{\mu} K_I K_{II} \quad (18)$$

where $H_{IV}(C)$ is a function of the crack velocity and given by

$$H_{IV}(C) = \frac{(\beta_1 - \beta_2)(1 - \beta_2^2)\bar{D}(C)}{2\{D(C)\}^2} \times \left[\frac{(\beta_1 + \beta_2)}{\sqrt{\beta_1\beta_2}} - \frac{4(1 + \beta_2^2)}{\bar{D}(C)} \right]. \quad (19)$$

It is seen from Eqs. (17•d) and (19) that the differences of both functions appear only in the first term in the square brackets. Moreover, the differences remain very small up to $C=0.6Cs$ as shown in Ref. (16), where almost all experimentally measured crack velocities are included in this range.

Therefore, we may draw the following conclusions⁽¹⁷⁾.

(1) The dynamic J integral component J_1^0 is “theoretically” invariant with respect to the shape of an infinitesimal process zone.

(2) The dynamic J integral component J_2^0 is “practically” invariant with respect to the shape of an infinitesimal process zone.

(3) The other path-independent integrals⁽¹⁴⁾ that do not physically represent the dynamic energy release rate are not invariant with respect to the shape of an infinitesimal process zone.

The above features of the dynamic J integral including its finiteness are very important for the theory and the numerical analyses in dynamic fracture mechanics.

Usually in numerical analyses far-field integrals are used to evaluate the values of the J integral. In this case it is convenient to consider the following expression taking the limit of $V_\epsilon \rightarrow 0 (\Gamma_\epsilon \rightarrow 0)$:

$$J_k' = \int_{\Gamma + \Gamma_c} [(W + K)n_k - t_i u_{i,k}] dS + \int_{V_r} [\rho \ddot{u}_i u_{i,k} - \rho \dot{u}_i \dot{u}_{i,k}] dV. \quad (20)$$

A comprehensive summary of recent studies on the dynamic J integral in dynamic fracture mechanics is provided in Ref. (17). The features of other types of path-independent integrals^{(18)–(21)} in dynamic fracture mechanics were also summarized in Ref. (17).

3.2 Component separation method for determining mixed-mode dynamic stress-intensity factors

Various procedures for determining mixed-mode dynamic stress-intensity factors from the dynamic J integral values were summarized in Ref. (17). The typical procedures are (i) direct method, (ii) mode decomposition method, (iii) component separation method.

The direct method uses the direct relations between the dynamic J integral components and the stress intensity factors given by Eqs. (16.a and b). For the in-plane mixed-mode problems, the J_k' -integrals are transformed to J_k^0 -values by Eq. (15). Then, from Eqs. (16•a and b), one may evaluate K_I and K_{II} as follows⁽²²⁾:

$$K_I = \pm \left\langle \frac{\mu}{A_{II}(C)} \left\{ (J_1^0) \pm [(J_1^0)^2 - (A_I A_{II}/A_{IV}^2)(J_2^0)^2]^{1/2} \right\} \right\rangle^{1/2} \quad (21\cdot a)$$

$$K_{II} = \pm \left\langle \frac{\mu}{A_{II}(C)} \left\{ (J_1^0) \mp [(J_1^0)^2 - (A_I A_{II}/A_{IV}^2)(J_2^0)^2]^{1/2} \right\} \right\rangle^{1/2} \quad (21\cdot b)$$

The signs of K_I and K_{II} correspond to the signs of the Mode I and Mode II crack-opening displacements δ_I and δ_{II} , respectively ($K_I > 0$ if $\delta_I > 0$, and so on). The term in braces in Equation (21.a) is determined accordingly as:

$$\text{if } |\delta_I/\beta_1| \geq |\delta_{II}/\beta_2|, \text{ take } + [] \quad (22\cdot a)$$

$$\text{if } |\delta_I/\beta_1| < |\delta_{II}/\beta_2|, \text{ take } - [] \quad (22\cdot b)$$

with similar choices for K_{II} .

However, it was discovered in Ref. (22) that, in order to employ the direct method, it is necessary to accurately model the crack-tip region with elements that possess the $r^{-1/2}$ stress singularity known to exist at a sharp crack tip as expressed in Eqs. (4•a–f). Consequently, the use of regular elements without the singularity results in somewhat inaccurate values for the stress intensity factors. This can be attributed to the numerical errors in the evaluation of the J_k' integrals (more specifically J_2^0) along the crack surface path Γ_c .

On the other hand, the mode decomposition method requires decomposition of the stress, displacement and other fields into individual modes^{(22),(23)}, prior to the dynamic J integral calculation. Although this method gives accurate stress intensity factors in most cases, it always requires the field quantities at the mirror image points with respect to the crack axis. Thus, it is difficult to apply this method to an irregular mesh pattern around the crack tip. Moreover, this method cannot be applied for a curved crack.

To overcome the difficulties in the other methods, the component separation method was proposed by Nishioka et al.⁽²⁴⁾

If Eq. (20) is used for numerical calculation of the dynamic J integral, then the numerical integration on Γ_c including the crack tip is required. This may involve a singular integration associated with the singularities in the energy densities W and K . The asymptotic behavior of the energy densities ($W + K$) on Γ_c near the crack tip was examined by Nishioka et al.⁽²⁴⁾ Substituting Eqs. (1) and (2) into W and K , the energy densities can be expressed in the form:

$$(W + K)^+ - (W + K)^- = \frac{A}{\sqrt{r}} + O(r^0) + O(r^{1/2}) + \dots \quad (23)$$

where $()^+$ and $()^-$ indicate the quantities on the upper and lower crack surfaces, respectively; and A is a constant depending on the stress intensity factors and the crack velocity. It is noted that the non-integrable singular term of $O(r^{-1})$ disappears owing to the existence of the same magnitudes of the $O(r^{-1})$

terms on the upper and lower crack surfaces.

Now we consider a characteristic distance δ from the crack tip, such that the singular term ($A/\sqrt{r}$) in the energy densities is predominant for the region $0 \leq r \leq \delta$. The J'_k integrals can then be rewritten as:

$$J'_k = \tilde{J}'_k + \tilde{\tilde{J}}'_k \quad (24)$$

where

$$\begin{aligned} \tilde{J}'_k &= \int_{\Gamma + \Gamma_{c0}} [(W + K)n_k - t_i u_{i,k}] dS \\ &+ \int_{V_r} [\rho \dot{u}_i u_{i,k} - \rho \dot{u}_i \dot{u}_{i,k}] dV \end{aligned} \quad (25 \cdot a)$$

$$\tilde{\tilde{J}}'_k = \int_0^\delta [(W + K)^+ - (W + K)^-] n_k^+ dS \quad (25 \cdot b)$$

The path Γ_{c0} will be taken as $(\Gamma_c - \delta)$, which excludes the crack-tip region. Thus, $\tilde{J}'_k$ can be evaluated very accurately by the ordinary finite-element or boundary-element method. Furthermore, the use of the coordinate transformation (see Eq. (15)) for Eq. (25·b) leads to the following relations:

$$\tilde{J}'_1 = 0 \quad (26 \cdot a)$$

$$\tilde{J}'_2 = - \int_0^\delta [(W + K)^+ - (W + K)^-] dS \quad (26 \cdot b)$$

In the above equations, the normal direction cosines are assumed to be constant in this region.

If non singular regular elements are used around the crack tip under a mixed-mode condition, the value of $\tilde{J}'_2$ (and J'_2) cannot be obtained accurately owing to the finiteness of the energy densities in the regular elements, instead of the singularity (see Eq. (23)) at the crack tip ($r=0$). However, the J'_1 component is not affected by the modeling of the crack tip, and can be evaluated accurately, even by the global components of the dynamic J integral, J'_k ($k=1, 2$):

$$J'_1 (= G) = J'_1 \cos \theta_0 + J'_2 \sin \theta_0 \quad (27)$$

This is because the errors caused by the missing singularity in the J'_1 and J'_2 calculations cancel each other in the right-hand side of Eq. (27).

Then J'_2 can be evaluated if the following ratio is accurately determined:

$$\gamma = J'_2 / J'_1 \quad (28)$$

Substituting Eqs. (16·a and b) into Eq. (28), the component ratio can be expressed as

$$\gamma = \frac{-2A_{1V}}{(A_1/\alpha + \alpha A_{11})} \quad (29)$$

where

$$\alpha = K_{II}/K_I = \delta_{II}\beta_1/(\delta_I\beta_2). \quad (30)$$

Using the above relations (Eqs. (28)-(30)) in the formulae of the direct method given by Eqs. (21.a and b), we obtain

$$\begin{aligned} K_I &= \delta_I \left\{ \frac{2\mu J_1^0 \beta_2}{A_I(\delta_I^2 \beta_2 + \delta_{II}^2 \beta_1)} \right\}^{1/2} \\ &= \delta_I \left\{ \frac{2\mu \beta_2 (J'_1 \cos \theta_0 + J'_2 \sin \theta_0)}{A_I(\delta_I^2 \beta_2 + \delta_{II}^2 \beta_1)} \right\}^{1/2} \end{aligned} \quad (31 \cdot a)$$

$$\begin{aligned} K_{II} &= \delta_{II} \left\{ \frac{2\mu J_1^0 \beta_1}{A_{II}(\delta_I^2 \beta_2 + \delta_{II}^2 \beta_1)} \right\}^{1/2} \\ &= \delta_{II} \left\{ \frac{2\mu \beta_1 (J'_1 \cos \theta_0 + J'_2 \sin \theta_0)}{A_{II}(\delta_I^2 \beta_2 + \delta_{II}^2 \beta_1)} \right\}^{1/2}. \end{aligned} \quad (31 \cdot b)$$

It is noted that the signs of the stress intensity factors are automatically determined by the signs of the crack-opening displacements.

If we consider a static crack problem, we have $A_I(0) = A_{II}(0) = (\chi + 1)/4$ and $\beta_1 = \beta_2 = 1$. From Eqs. (31.a and b), the relations between the stress intensity factors and the static J integral components can be derived as

$$K_I = \delta_I \left\{ \left(\frac{8\mu}{\chi + 1} \right) \frac{(J_1 \cos \theta_0 + J_2 \sin \theta_0)}{\delta_I^2 + \delta_{II}^2} \right\}^{1/2} \quad (32 \cdot a)$$

$$K_{II} = \delta_{II} \left\{ \left(\frac{8\mu}{\chi + 1} \right) \frac{(J_1 \cos \theta_0 + J_2 \sin \theta_0)}{\delta_I^2 + \delta_{II}^2} \right\}^{1/2}. \quad (32 \cdot b)$$

It is further noted that recently Eischen⁽²⁵⁾ has also modified the direct method⁽²²⁾ to overcome the difficulty in the calculation of the static J_2^0 integral. However, Eischen's method is difficult to extend to the dynamic J integral.

3.3 Nonlinear dynamic fracture mechanics

We now consider dynamic fracture mechanics in elastic-plastic and elastic-viscoplastic materials. Under an incremental flow theory of plasticity, it is natural to consider an incremental measure of the strength of the crack-tip field. On the basis of this idea, Atluri et al⁽²⁶⁾ have derived a general form of path-independent integral T^* that is valid for any material-constitutive relation under quasi-static as well as dynamic conditions. The T^* integral can be expressed in an incremental form or in a total form. Here, only the total form of the T^* integral is shown:

$$T_k^* = \int_{\Gamma_c} [(W + K)n_k - t_i u_{i,k}] dS \quad (33 \cdot a)$$

$$\begin{aligned} &= \int_{\Gamma + \Gamma_c} [(W + K)n_k - t_i u_{i,k}] dS + \int_{V_r - V_c} [\sigma_{ij} \epsilon_{ij,k} \\ &- W_{,k} + \rho \dot{u}_i u_{i,k} - \rho \dot{u}_i \dot{u}_{i,k}] dV \end{aligned} \quad (33 \cdot b)$$

where W is the stress working density, which can be expressed by

$$\begin{aligned} W &= \int_0^t \dot{W} dt = \Sigma \Delta W; \\ \text{and } \Delta W &= \Sigma (\sigma_{ij} + \frac{1}{2} \Delta \sigma_{ij}) \Delta \epsilon_{ij}. \end{aligned} \quad (34 \cdot a, b)$$

Note that in an elastic-plastic material under arbitrary loading history, W is the total accumulated increments of stress working density, and is not the strain energy density in general. Since σ_{ij} is not a single-valued function of ϵ_{ij} , in general, we have $W_{,k} \neq \sigma_{ij} \epsilon_{ij,k}$. For elastic materials, since W corresponds to the strain energy density, the T_k^* integral reduces to the dynamic J integral (J'_k) for elastodynamic cases. Moreover, the T_k^* integral reduces to the J

integral derived by Budiansky and Rice⁽²⁷⁾ for elastostatic cases. Since the T^* integral is a natural extension of the so-called Rice's static J integral and the dynamic J integral⁽³⁾, it can be considered as the generalized J integral for any material.

The path independence of the T^* integral can be easily shown without using a constitutive relation. Thus, the T^* integral is valid for any constitutive model. The near-field path Γ_ϵ in Eqs. (33•a and b) will be taken along the boundary of a fracture process zone. Usually the size of fracture process zone may be finite for growing cracks in somewhat ductile (elastic-plastic) materials. In these cases, the T^* integral can be interpreted as the rate of the energy flow into the process zone due to a unit crack extension.

3.4 Equivalent domain integral expression of the dynamic J integral

For numerical analyses of three-dimensional crack problems, the original version of the dynamic J integral (see Eq. (20)) can be used by directly carrying out the surface integration on $\Gamma + \Gamma_c$ ($\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2$) and the volume integration in V_r for each crack-front segment (see Fig. 2). However, it is cumbersome to carry out both surface and volume integra-

Fig. 2 Crack-front segment

Fig. 3 Curved crack front

tion, especially for curved crack front as depicted in Fig. 3.

To overcome this difficulty, a convenient expression for the dynamic J integral was derived in Ref. (28). For each crack-front segment as depicted in Fig. 2, we introduce a continuous function s over the domain V_r . Then, using the s function in the integrand in Eq. (20), the following path-independent integral can be derived:

$$J_k f = \int_{\Gamma + \Gamma_c} [(W + K)n_k - t_i u_{i,k}] s \, dS + \int_{V_r} [\sigma_{ij} u_{i,k} s_{,j} - (W + K)s_{,k} + (\rho \ddot{u}_i u_{i,k} - \rho \dot{u}_i \dot{u}_{i,k}) s] dV \quad (35)$$

where f is a weighted coefficient⁽²⁹⁾ that depends on the distribution of the s function along the crack front. If we use $s=1$ for the entire domain, Eq. (35) reduces to the original version of the dynamic J integral, i.e., Eq. (20). Then if we choose $s=0$ on $\Gamma + \Gamma_c$, the dynamic J integral can be expressed only by the volume integration:

$$J'_k = \frac{1}{f} \int_{V_r} [\sigma_{ij} u_{i,k} s_{,j} - (W + K)s_{,k} + (\rho \ddot{u}_i u_{i,k} - \rho \dot{u}_i \dot{u}_{i,k}) s] dV \quad (36)$$

In general, the s function can be interpolated using the same shape function used for the finite-element method. For example, the s function for 20-noded isoparametric elements can be expressed by

$$s = \sum_{q=1}^{20} N_q(\xi, \eta, \zeta) s_q \quad (37)$$

where N_q and s_q are the shape functions and the nodal values of the s function, respectively. Substitution of Eq. (37) into Eq. (36) leads to

$$J'_k = \frac{1}{f} \sum_{n=1}^{N_r} \sum_{q=1}^{20} R_{knq} s_q \quad (38\cdot a)$$

and

$$R_{knq} = \int_{V_n} [\sigma_{ij} u_{i,k} N_{q,j} - (W + K)N_{q,k} + (\rho \ddot{u}_i u_{i,k} - \rho \dot{u}_i \dot{u}_{i,k}) N_q] dV \quad (38\cdot b)$$

in which N_r is the total number of elements in V_r , and V_n is the volume of the N -th element. Typical s functions tested for static problems⁽²⁹⁾ and for dynamic problems^{(28),(30)} are shown in Fig. 4. Type A

Fig. 4 Typical s functions

consists of two elements while type C consists of one element along each crack-front segment. The use of type A in Eq. (38) gives the value of the dynamic J integral at the corner node between the two elements, while type C gives the dynamic J value at the midside node of the element.

To evaluate the crack-axis components of the dynamic J integral J_i^0 with respect to a local coordinate system x_i^0 as depicted in Fig. 3, the global components of the dynamic J integral obtained by Eq. (38) can be converted using the following coordinate transformation:

$$J_i^0 = \alpha_{ik} J_k' \quad (39)$$

where α_{ik} is the coordinate transformation tensor and varies along the curved crack front.

4. Computational Modeling of Dynamic Crack Propagation

Due to the difficulties in analytical and experimental methods as described in the introduction of this article, computational methods provide most solutions for dynamic fracture mechanics problems. Major computational tools for dynamic fracture simulation are (i) the finite difference method (FDM), (ii) the finite-element method (FEM), and (iii) the boundary-element method (BEM). In fast fracture mechanics problems, one must treat a moving-singularity boundary-value problem. Since the moving singularity induces various errors in numerical methods, it is critical to develop accurate solution procedures for the advancement of fast fracture mechanics. In these computational methods, the solid continuum is represented by a finite number of elements or grids. To simulate the crack propagation in solids, two different concepts of computational modeling can be considered, i.e., (i) the stationary (fixed) element procedure, and (ii) the moving (distorting) element procedure, in each computational method. In this section, various computational models for dynamic crack propagation are critically reviewed.

4.1 Finite difference method

The finite difference method is the oldest one among the three methods. In early applications of the

numerical methods for dynamic fracture simulation, this method was considered to be more suitable⁽¹⁾, owing to its simplicity and the ease with which free crack surfaces can be modeled. However, the inherent difficulties in modeling of complicated geometries make FDM unattractive. For this reason, there seems to be less work on the development and application of FDM at the present.

4.1.1 Stationary element (grid) procedure

In this procedure, the crack tip basically moves from one node to the next in the finite difference grid, with a sudden node relaxation scheme^{(31)–(33)} or a smoothing scheme⁽³⁴⁾. The sudden node relaxation usually induces spurious oscillations on the propagating crack surfaces.

To avoid the sudden release of the crack-tip node, the following smoothing scheme was developed by Chen and Wilkins⁽³⁴⁾. To describe partial fracture in a zone (see Fig. 5), a smoothing parameter F is used such that $F=0$ denotes intact material and $F=1$ denotes complete fracture. The smoothing parameter varies with

$$F = \frac{(t - t_c)}{(d/C)} \quad (40)$$

where t_c is the time when fracture begins, t is some time after t_c , d is the grid spacing, and C is the crack velocity. The parameter F is used in the acceleration equations to apply free-surface boundary conditions to the zone. This procedure allows a crack to propagate smoothly through the grid independent of the zone size.

4.1.2 Moving element (grid) procedure This procedure may be further classified into two categories as (1) a moving entire-grid procedure and (2) a moving local-grid procedure.

(1) Moving entire-grid procedure

In the moving entire-grid procedure^{(35),(36)}, the whole mesh for the finite difference grids moves with the propagating crack tip. The governing equations are expressed in terms of the moving coordinate system attached to the crack tip. Although the governing equations become more complicated due to the convection terms associated with the movement, this

Fig. 5 Stationary grid procedure with smoothing process (FDM)

procedure can avoid the node relaxation processes. However, the main drawback of this method is that it cannot be applied to non-self-similar crack propagation problems and to finite-body problems. Using the moving entire-grid procedure, a FDM was used in the development of a computational method for analyzing dynamic crack initiation and growth under stress wave loading conditions⁽³⁷⁾.

(2) Moving local-grid procedure

The moving local-grid procedure⁽³⁸⁾ combines the advantages of both the moving entire-grid and the stationary procedures. Thus, this procedure also can avoid the node relaxation processes. In this procedure, the moving domain grid D_m that moves with the propagating crack tip (see Fig. 6) is used in addition to the stationary domain grid D_s for the entire domain. The governing equations in D_m are expressed in terms of the moving coordinate system (ξ, η) , while those in D_s are expressed by the fixed coordinate system (x, y) , as shown in Fig. 6. First, the governing equations of D_s are solved, and the nodal displacements along the boundary of D_m (i.e., B_m) are calculated by an interpolation procedure for the displacement field of D_s . Then using the nodal displacements on B_m and the crack-face boundary conditions, the governing equations of D_m are solved. This procedure can improve the local accuracy at the vicinity of the propagating crack tip⁽³⁸⁾.

4.2 Finite-element method

The finite-element method is the most popular one among the three typical numerical methods because of its versatility. Two detailed reviews on finite-element modeling of dynamic crack propagation appearing in the literature up to 1986 was given by Nishioka and Atluri^{(2),(39)}. In this paper, relevant developments in the intervening years are reviewed together with earlier developments.

4.2.1 Stationary element (mesh) procedure

In early applications^{(40),(41)} of the finite-element method to dynamic crack propagation, the crack-tip motion was modeled by discontinuous jumps. This was done in the finite-element method by changing the

location of the crack tip from one node to the next at prescribed time intervals. To obtain accurate finite-element solutions for wave propagation problems, one must use a small time increment Δt , which is usually the time required for dilatational stress waves to travel the distance between the two closest finite-element nodes. Since, in general, the velocity of crack propagation is significantly lower than the wave velocity, the crack tip during this time increment Δt will move to a position somewhere in between the two successive nodal points. Also the sudden increase of crack length and the release of constraints on the displacement induce spurious high-frequency oscillations in the finite-element solutions. To overcome these difficulties, several attempts have been made at releasing the nodes gradually over a period of time. These are discussed below.

(1) Nodal relaxation techniques

Keegstra and co-workers^{(42),(43)} have proposed a model shown in Fig. 7(a), which allows for the presence of non-zero "holding-back forces" at more than one node behind the advancing crack tip. Using a viscous boundary damping technique, an effectively similar model to the nodal relaxation mechanism of Keegstra et al. was also developed by Valliappan and Multi⁽⁴⁴⁾. This method is applicable to mixed-mode dynamic crack propagation.

Yagawa et al.⁽⁴⁵⁾ use a Lagrange multiplier approach to enforce the displacement boundary condition on the "uncracked" part of the surface of the crack-tip element [marked as segment (CD) in Fig. 7(b)] in situations when, in time Δt , the crack tip actually propagates to a location between two adjacent nodes. Note that the traction T_2 (see Fig. 7(b)) is used as a Lagrange multiplier on segment (CD).

Other "gradual" nodal relaxation mechanisms were presented in the literature⁽⁴⁶⁾⁻⁽⁴⁸⁾, with different rates of release of nodal forces. Suppose that the actual crack tip is located at point C in between the finite-element nodes B and D as shown in Fig. 7(c). Note that "b" and "d" are the lengths of segments (BC) and (BD), respectively. The "holding-back" force F at node B is gradually reduced to zero as the crack tip reaches node D. Various schemes for accomplishing this process are summarized below:

$$\text{Malluck and King}^{(46)} : F/F_0 = (1 - b/d)^{1/2} \quad (41.a)$$

$$\text{Rydholm et al.}^{(47)} : F/F_0 = (1 - b/d)^{3/2} \quad (41.b)$$

$$\text{Kobayashi et al.}^{(48)} : F/F_0 = (1 - b/d) \quad (41.c)$$

where F_0 is the original reaction force when the crack tip is located at node B. It is important to recognize that the above procedure of moving the crack tip from B to D involves several time steps. Equations (41.a and b) are derived by assuming a constant stress-intensity factor and a constant energy release rate,

Fig. 6 Moving local-grid procedure (FDM)

respectively, during the time that the crack tip traverses the distance between nodes B and D. On the other hand, Eq. (41•c) is based on pure intuition.

In the use of eight-noded isoparametric elements, it was demonstrated in Ref. (49) that both the corner and midside nodes on the crack elements should be released simultaneously. In Ref. (49) the linear relaxation technique expressed by Eq. (41•c) was used for both the corner and midside nodes. From the present author's experience, the use of linear relaxation in eight-noded elements yields a smoother result while use of other nonlinear relaxation techniques gives oscillatory results during the crack propagation between nodes B and D. Furthermore, the physical background for the nonlinear relaxation techniques cannot be extended to the problems of crack propagation in nonlinear materials. Therefore, among the relaxation procedures given in Eqs. (41•a-c), the linear relaxation technique which is the most simple to implement is recommended. Studies using different gradual nodal release mechanisms expressed in Eqs. (41•a-c) were also discussed in Refs. (50) and (51).

Dynamic crack propagation in micropolar elastic materials was modeled using the nodal relaxation schemes⁽⁵²⁾. For the micropolar elasticity case, a nodal moment as well as a nodal force must be released.

In the problem of prediction of crack propagation history, it may happen that the crack may arrest somewhere between the nodes of a fixed mesh. Thus, all of the fixed mesh procedures may encounter some difficulty in analyzing the post-crack-arrest behavior of the solid.

(2) Special joint element techniques

In order to prevent spurious oscillations due to sudden nodal release, several types of joint elements or breakable elements have been developed.

Hoff et al.⁽⁵³⁾ have employed spring and gap elements (see Fig. 8(a)) in a FEM scheme for dynamic

crack propagation in viscoplastic solids. The gap elements are formulated so that they can transmit only compressive loads, i.e., no tensile or shear loads. Suppose that the actual crack tip is located between nodes $n+2$ and $n+1$ as shown in Fig. 8(a). The gap elements connected to the nodes behind $n+3$ are open while those connected to the nodes ahead of n are closed. Spring node $m+2$ is displaced upwards in proportion to how far the effective crack tip has advanced through element i : e.g., if the crack tip has advanced 25% of the way between nodes $n+2$ and n , then node $m+2$ is increased by 25% of its maximum value, and the maximum displacement of $m+2$ is almost twice what the final nodal displacement of node $n+2$ will be. Thus, about halfway through the incremental gap-opening procedure, node $n+2$ will separate from the gap (i.e., the gap opens) and will find its own equilibrium position. The variations of the dynamic J integral as expressed by Eq. (20) were evaluated for rapid fracture in viscoplastic materials.

Special joint elements were developed by Ramakrishnan et al.⁽⁵⁴⁾ to simulate behavior within the fracture-process zone for dynamic crack propagation problems in viscoplastic materials. The joint elements are shown in Fig. 8(b) together with their constitutive behavior. The gradual increase in displacement during crack advance is modeled by a strain-softening response as shown in descending portion BC of the curve. In actual computational

Fig. 7 Gradual-nodal-relaxation techniques⁽²⁾ (FEM)

Fig. 8 Special joint element techniques (FEM)

model, since the convergence rate becomes unacceptable as the yield stress approaches zero ($\epsilon \rightarrow C$), a yield-function cutoff procedure was adapted⁽⁵⁴⁾.

A FEM algorithm based on breakable element was developed to simulate dynamic crack propagation⁽⁵⁵⁾. A row of additional and fictitious elements is attached along the expected crack propagation path, as shown in Fig. 8(c). The boundary conditions are as shown in the same figure. The density of the breakable elements (ρ) is set to a very high value (ρ_0), which is reduced to zero as the crack crosses the element:

$$\rho = \rho_0(1 - b/d) \quad (42)$$

This technique is analogous to a large Young's modulus used for quasi-static crack propagation by Hsu and Zhai⁽⁵⁶⁾. It was concluded in Ref. (55) that the numerical results of this technique for dynamic crack propagation are comparable with those of the nodal relaxation techniques.

A type of breakable element similar to that of Hsu and Zhai⁽⁵⁶⁾ was developed by Kostylev and Margolin⁽⁵⁷⁾ for non-self-similar dynamic crack propagation. The breakable elements are shown by the hatched area in Fig. 8(d). The bonding forces of the crack edges, proportional to the stiffness of the elements of the crack cavity, are characterized by the elasticity modulus of the crack E_T . The elastic

moduli of the breakable element (E_T) decrease to almost zero (E^*) in a manner similar to that using Eq. (42):

$$E_T = E(1 - b/d) + E^* \quad (43)$$

where E is the elastic modulus of the material considered. The distances b and d are defined in the same manner as shown in Fig. 7(c). The use of Eq. (43) leads to a smooth reduction of the bonding forces to almost zero during the crack propagation along the element, and to the absence of non-characteristic high-frequency oscillations.

All of the special joint elements presented here for modeling dynamic crack propagation result in effectively the gradual nodal-relaxation mechanism. However, these models are based somewhat on pure intuition, and do not exactly satisfy the boundary conditions near the propagating crack tip.

4.2.2 Moving element (mesh) procedure

Next we present a synopsis of the moving element procedures in which the mesh moves with the crack tip. Similar to the FDM, the moving element procedures for the FEM may be further classified into two categories: (1) the moving entire-mesh procedure and (2) the moving local-mesh procedure.

(1) Moving entire-mesh procedure

Bazant et al.⁽⁵⁸⁾ and Östlund and Gudmundson⁽⁵⁹⁾ developed a nonsingular finite-element method for

moving the entire mesh with the crack tip. However, this procedure has two obvious limitations : (i) it is restricted to infinite bodies whose surface and/or bimaterial interfaces are parallel to the direction of crack propagation; and (ii) more importantly, it cannot be applied to bodies having finite dimensions in the direction of crack propagation.

Despite these limitations, this procedure can be a useful tool for the study of asymptotic behavior around a propagating crack tip under steady-state conditions, especially in nonlinear materials. Using this procedure for 4-noded isoparametric elements, Freund and Douglas⁽⁶⁰⁾, and Lam and Freund⁽⁶¹⁾ investigated steady-state dynamic crack growth of a semi-infinite crack in elastoplastic materials. Freund et al.⁽⁶²⁾ studied the near-tip behavior in viscoplastic materials.

Recently, Östlund⁽⁶³⁾ incorporated a variable-order singular element at the element positions adjacent to the crack tip in the moving entire-mesh procedure. However it was shown that this type of element does not improve the numerical behavior for dynamic steady-state small-scale yielding crack propagation in an elastic-viscoplastic material.

(2) Moving local-mesh procedure

Early attempts toward a singular element for propagating cracks were made by several researchers^{(64)–(66)}. All these moving singular elements move in a discontinuous fashion even for a continuously propagating crack. The detailed features of these discontinuously moving element procedures were critically reviewed in Ref. (2).

In developing a special element, certain simple requirements must be met in order to ensure the convergence of the solution. The displacement field in the element must (i) include the appropriate rigid body motions, (ii) be able to represent a constant strain (stress) field, and (iii) be continuous at the interfaces between elements. In addition, to avoid kinematic modes in the special element, i.e., zero energy modes other than rigid modes, certain requirements on the number of parameters in the element basis functions must be satisfied. Suppose a special element has N unknown parameters, i.e., N eigenfunctions including appropriate rigid body motions and M unconstrained nodal degrees of freedom. It can then be shown^{(67)–(69)} that the inequality $N \geq M$ must be satisfied. The developments in Refs. (64) to (66) violate some or all of these criteria.

A moving singular element which satisfies all the above criteria for convergence and which employs eigenfunctions^{(3),(4)} (see also Eq. (2)) for a dynamically propagating crack, was extensively developed by Nishioka and Atluri^{(67),(68)}. Thus, in the singular

element (model A_0 in Fig. 9), it is assumed that

$$u_i^s(x_1^0, x_2^0, t) = U_{ij}(x_1^0, x_2^0, C) \alpha_j(t) \quad (i=1, 2; j=0, 1, 2, \dots, N-1) \quad (44)$$

where the functions U_{ij} are given by Eq. (2) and the coefficients $\alpha_j(t)$ correspond to K_I^0 in Eq. (2). Thus, α_1 is equivalent to the dynamic stress intensity factor K_I . The displacement functions given by Eq. (44) satisfy the requirements (i) and (ii) mentioned above for the convergence of the solution. The compatibility of displacements, velocities and accelerations at the boundary between the singular element and regular elements (the type-B elements in Fig. 9) can be satisfied through a least squares approach^{(67),(68)}.

In the "moving mesh" procedure^{(67),(68),(70)}, the singular element translates in each time step for which crack growth occurs as illustrated in Fig. 9. Thus the crack tip always remains at the center of the singular element throughout the analysis. The regular elements (type-B elements in Fig. 9) surrounding the moving singular element are continuously distorted. To simulate a large amount of crack propagation, the mesh pattern around the moving element is periodically readjusted as also shown in Fig. 9.

An important feature that distinguishes the moving mesh procedures from the nodal relaxation techniques and the special joint techniques is that the size of the increment of crack growth is not affected by nodal spacing but can be made as small as desired. Therefore, the displacement boundary conditions ahead of the actual crack tip can be satisfied exactly in the moving mesh procedures. Another attractive feature of the moving singular element procedure as developed by Nishioka and Atluri^{(67),(68)} is that the dynamic stress-intensity factor at each time step can be calculated directly as the coefficient of one of the eigenfunctions, e.g., α_1 for a Mode I case.

Later a similar moving singular element procedure was reported⁽⁷¹⁾ based upon a hybrid-displacement finite-element formulation. This quadrilateral singular element has 5 nodes and is used in conjunction with regular 4-noded elements. However the results of its use are not as accurate as those of Refs. (67) and (68). Recently Aminpour and Holsapple⁽⁷²⁾ extended the above formulation to a discontinuously moving singular element for propagating interface cracks. In the singular element, the near-field solutions for propagating cracks at the interface of dissimilar anisotropic elastic materials were successfully implemented.

To simplify the moving singular-element procedure, studies were undertaken in Refs. (69), (73) and (74) to address the effects of alternatively modeling region A in Fig. 9 with either the "quarter-point"

Fig. 9 Moving element procedures (FEM)

singular elements (model A₁) or the regular (non-singular) isoparametric elements (model A₂). It was found in Ref. (74) that the combined use of the moving isoparametric elements (model A₂) and the path-independent dynamic J integral given by Eq. (20) leads to excellent results, and the moving quarter-point singular elements (model A₁) gives considerably lower results for the case of $C=0.6C_s$. Model A₁ can therefore be applied to slower crack propagation ($C=0\sim 0.3C_s$).

In models A₀–A₂, a relatively elaborate "book-keeping" system of nodal numbering to readjust the mesh pattern periodically is required, whereas this is not the case when a simplified moving-element procedure with model A₃ is used⁽⁷⁵⁾. However, the simplified moving-element procedure requires a more refined mesh to accurately model the crack-tip region. Thesken et al.⁽⁷⁶⁾ also used model A₃ for elastic-visco-

plastic dynamic fracture analysis, in which eighteen rectangular elements were incorporated in region A.

(3) Mixed Eulerian-Lagrangian kinematic description (ELD) procedure

A version of the mixed Eulerian-Lagrangian kinematic description (ELD)⁽⁷⁷⁾ for the analysis of elastodynamic crack propagation was developed by Haber and Koh^{(78)–(80)}. Figure 10(a) shows the initial mesh prior to dynamic crack propagation. A mesh of linear-strain triangular isoparametric elements is used to model the upper half of a square plate with an edge crack. As the crack propagates, the mesh changes continuously to model the crack propagation. The final mesh at $a=0.5W$ (W : the width of the plate) is shown in Fig. 10(b). Continuous Eulerian mesh motion models the crack propagation as shown in Fig. 10. For each element, the Eulerian mapping is defined locally by the isoparametric mapping and

Fig. 10 Eulerian-Lagrangian finite-element mesh for a propagating crack (FEM)

nodal coordinates. Auxiliary regional mappings are used to relate the changes of the nodal coordinates to the crack-tip motion. Thus, interpolation techniques are not needed for continuous mesh movement, while they are still needed for the global remeshing if the mesh topology is altered.

When the quarter-point singular elements are used around the running crack tip with the ELD procedure, the appropriate singularity of order $r^{-1/2}$ in the material velocity field at the crack tip is automatically generated due to the convective term in the material time derivatives of the displacement field. The convective terms produce an apparent unsymmetric damping matrix and additional unsymmetric matrices in the stiffness matrix.

Therefore, a major disadvantage of this procedure is that the final generalized coefficient matrix for the ELD finite-element equation results in a full-sized unsymmetric matrix. This is unlike the moving singular element procedure^{(67),(68)} explained in the previous subsection, in which the unsymmetry is confined mainly to the rows and columns corresponding to nodes around the moving singular element. The slightly unsymmetric finite-element equation of the moving singular element procedure can be solved efficiently using the symmetric part of the generalized stiffness matrix in an iterative manner⁽⁶⁷⁾. Solving the full-sized unsymmetric finite-element equations, as appeared in the ELD FEM, greatly increases the required computational time.

(4) Remeshing and interactive graphics procedure

Swenson and Ingraffea⁽⁸¹⁾ developed a remeshing and interactive graphics procedure to model mixed-

Fig. 11 Remeshing and interactive graphics procedure (FEM)

mode dynamic crack propagation. Figure 11 illustrates the remeshing process. As the crack tip moves, only the near-tip mesh pattern is distorted as shown in Fig. 11(a) until the mesh becomes excessively distorted. The remeshing process consists of the following steps. (1) Using a radius based on local element size, a center for remeshing is defined by projecting ahead of the crack tip (Fig. 11(b)). All elements within a distance of twice the radius are deleted. (2) A rosette of quarter-point element is added around the crack tip (Fig. 11(c)). (3) The annulus between the quarter-point elements and the boundary of undeleted elements is filled with elements (Fig. 11(d)) using the modified Suhara-Fukuda algorithm⁽⁸²⁾. (4) Data are transferred from the old mesh to the new mesh, as is done for the remeshing procedure illustrated in Fig. 9.

Interactive computer graphics is essential for reliable application of this procedure. Because elements are continuously being added and deleted, the topology of the problem is always changing. Example analyses of straight and curving crack propagation reveal somewhat spurious oscillatory responses in the dynamic stress intensity factors⁽⁸¹⁾.

4.3 Boundary-element method

The boundary-element method is the youngest one among the three typical numerical methods. In general, the boundary-element method for elastodynamic problems can be classified into two approaches. The direct method employs physical quantities such as the displacements and the tractions in the BEM formulation while the indirect method utilizes artificial quantities such as sources or source densities. Furthermore, elastodynamic problems can

be solved by either a time-domain formulation or frequency-domain (or Laplace domain) formulation. Most of the boundary-element methods applied to dynamic crack propagation are based upon the direct time-domain formulation. Compared with FEM and FDM, the main advantage of BEM is the reduction of the dimensionality of the problem by one. However, its disadvantage is the involvement of non-symmetric and fully populated matrices in the final BEM equations to be solved.

4.3.1 Stationary element procedure Early applications of the BEM or the boundary integral equation method for dynamic crack propagation were briefly reviewed by Beskos⁽⁸³⁾. Most of these studies are limited to infinite domains because they concern the simulation of earthquake sources. Das and co-workers^{(84)–(86)} developed 2-D and 3-D BEM in conjunction with a node release mechanism to solve crack propagation problems in geophysics. Early numerical studies for dynamic crack propagation were also done by Burridge⁽⁸⁷⁾ and Bergkvist⁽⁸⁸⁾ using particular formulations related to the BEM.

Mettu and Kim⁽⁸⁹⁾ developed a linear superposition technique to derive the solution for the case of moving cracks from that of a stationary crack under dynamic loading. They used the time domain formulation together with piecewise-constant approximation of displacements and tractions within each boundary element as well as during a time increment. Let the element just ahead of the initial crack tip be the K th element. The solution of the propagating crack problem at time step $(I+1)$ was obtained by the superposition of the solution of the following two problems as shown in Fig. 12: (a) a stationary crack problem with the initial crack face subjected to a given load, and (b) a stationary crack problem with the crack

Fig. 12 Superposition technique for a dynamically propagating crack (BEM)

length increased by one element-length and subjected to a specified traction on the K th element only. This specified traction varies with time: it is zero until the time step (I) and, during the time step $(I+1)$, is equal and opposite to the traction obtained in case (a). Thus the superposed solution of cases (a) and (b) has the net traction which vanishes on the K th element during the time step $(I+1)$, thus simulating crack propagation. The same procedure can be repeated up to the desired number of time steps.

As an alternative form of the aforementioned single-step superposition method, they developed also the multistep superposition method, performing the above procedure for sub-intervals of the original time step. They found, however, that the single- and multistep methods yielded practically identical results using the time step sizes of 0.5 and 1 μ s.

4.3.2 Moving element procedure For the analysis of dynamic crack propagation in infinite and arbitrary finite bodies, the direct time-domain BEM in conjunction with a moving singular element and a remeshing technique was recently developed by Gallego and Domínguez⁽⁹⁰⁾. The moving boundary-element procedure used in Ref. (90) is essentially same as the moving finite-element procedure for model A₁ explained in Fig. 9. Two quarter-point singular elements, one before and one after the propagating crack tip, are used as shown in Fig. 13. As the crack tip advances from left to right, the two singular elements translate together with the crack tip, and the size of other regular elements to the left is increased and that to the right is decreased (see Fig. 13(b)). When the ratio of the sizes of the regular elements at both sides becomes greater than 5/3 (Fig. 13(c)), the number of elements to the left is increased by one and that to the right is decreased by one. The element discretization of both sides is then redefined as shown in Fig. 13(d). However, as pointed out by Haber⁽⁹¹⁾, there were

Fig. 13 Movement of the elements and remeshing process (BEM)

Fig. 14 Crack extension model with remeshing process at each time step (BEM)

minor errors in the original figure in Ref. (90).

Aoki et al.⁽⁹²⁾ developed the direct time-domain boundary-element method together with an alternative remeshing technique for the analysis of dynamic crack propagation in a semi-infinite body. They used 400 elements along the crack-axis direction (see Fig. 14). The displacements and tractions were assumed to be constant in each element. The entire boundary along the crack axis was remeshed at every time step as shown in Fig. 14. This procedure is analogous to the moving entire-mesh procedure for the finite-element method as explained in Section 4.2.2(a) except that the convection terms associated with the movement are not formulated in this procedure.

5. Concluding Remarks

In this article, fundamental aspects of computational dynamic fracture mechanics were reviewed in conjunction with the asymptotic behaviors of propagating cracks and the dynamic J integral. In particular, the computational modelings of dynamic crack propagation were critically reviewed. Due to page limitations, applications of numerical methods such as the FDM, FEM and BEM in dynamic fracture mechanics problems were not included here. These will be reviewed together with pertinent numerical results, in a separate article⁽⁹³⁾ and in a forthcoming book⁽⁹⁴⁾. There still are many unresolved problems in dynamic fracture phenomena even under self-similar Mode I elastic situations, as reviewed in Ref. (95). Most of non-self-similar dynamic fracture phenomena such as fast crack curving and fast crack branching require clarification of their mechanism. Numerical simulation will play a key role in finding physical quantities that govern those dynamic fracture processes. Future developments in computational dynamic fracture mechanics are needed to establish non-self-similar elastodynamic fracture mechanics and non-linear dynamic fracture mechanics.

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