A Two-Dimensional Piezothermoelastic Problem in an Orthotropic Plate Exhibiting Crystal Class mm2*

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In the beginning of paper, a solution technique for two-dimensional piezothermoelastic problems in orthotropic solids of crystal class mm2 is proposed. The solution technique is formulated in terms of two piezothermoelastic potential functions, three piezoelastic potential functions and a single piezoelectric potential function. Next, by using the solution technique, a two-dimensional piezothermoelastic problem in an orthotropic thin plate of crystal class mm2 which is subjected to heating on one surface and electric surface charge on both surfaces is analyzed. The numerical calculations are carried out for cadmium selenide exhibiting class mm2 symmetry. The elastic displacements and stresses are compared with those obtained for a similar thermoelastic problem without the piezoeffect. The effects of the electric surface charge on the elastic displacements, stresses, electric potential, electric field intensities and electric displacements are also investigated.

Key Words: Elasticity, Thermal Stresses, Piezothermoelasticity, Orthotropic Plate, Piezoelectricity, Two-Dimensional Problem

1. Introduction

Recently, piezoelastic (piezoelectric) materials have attracted special interest because of their intrinsic electromechanical coupling behavior. In order to use these materials in various engineering applications such as actuators and sensors, further research on piezoelasticity is needed.

Most previous works⁽¹⁾⁻⁽¹⁸⁾ are concerned with theory and analysis of piezoelastic problems, while a few papers^{(19),(20)} are written on the theory of piezothermoelasticity. In these papers, however, we cannot find a general solution technique applicable to piezoelastic or piezothermoelastic problems.

This paper is concerned with two-dimensional

piezothermoelastic problems in orthotropic solids exhibiting class mm2 symmetry. First, we propose a solution technique which is formulated in terms of two piezothermoelastic potential functions, three piezoelastic potential functions and a single piezoelectric potential function. The governing equations for the potential functions are derived from the equations of equilibrium and the equation of electrostatics.

Next, as a practical application, a two-dimensional piezothermoelastic problem in an orthotropic thin plate of crystal class mm2 which is subjected to heating on one surface and electric surface charge on both surfaces is analyzed. The numerical calculations are carried out for a cadmium selenide solid exhibiting the piezoeffect. The elastic displacements and stresses are compared with those obtained for a similar thermoelastic problem without piezoeffect. The effects of the electric surface charge on the elastic displacements, stresses, electric potential, electric field intensities and electric displacements are also investigated.

2. Basic Equations

Let us consider two-dimensional piezothermoelastic problems in orthotropic solids of crystal class

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mm2. The constitutive equations for the elastic field are expressed as follows:

$$\left. \begin{aligned} \sigma_{xx} &= c_{11}\varepsilon_{xx} + c_{13}\varepsilon_{zz} - e_1E_z - \beta_1T \\ \sigma_{yy} &= c_{12}\varepsilon_{xx} + c_{23}\varepsilon_{zz} - e_2E_z - \beta_2T \\ \sigma_{zz} &= c_{13}\varepsilon_{xx} + c_{33}\varepsilon_{zz} - e_3E_z - \beta_3T \\ \sigma_{zx} &= c_{55}\varepsilon_{zx} - e_5E_x \end{aligned} \right\} \quad (1)$$

where σ_{ij} are the stresses, ε_{ij} are the strains, E_i are the electric field intensities, c_{ij} are elastic moduli, e_i are piezoelectric coefficients, β_i are stress-temperature coefficients, and T is the temperature rise from the stress-free state, which is governed by Fourier's heat conduction equation:

$$T_{,xx} + \lambda^2 T_{,zz} = \frac{T_{,t}}{\kappa_x} \quad (2)$$

where

$$T_{,x} = \frac{\partial T}{\partial x}, \quad \lambda^2 = \frac{\lambda_x}{\kappa_x}$$

and where λ_i are coefficients of thermal conductivity, κ_x is thermal diffusivity, and t is time. The strains are expressed in terms of the displacements u_i as follows:

$$\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2} \quad (3)$$

The constitutive equations for the electric field are

$$D_x = e_5\varepsilon_{zx} + \eta_1E_x, \quad D_z = e_1\varepsilon_{xx} + e_3\varepsilon_{zz} + \eta_3E_z + p_3T \quad (4)$$

where D_i are the electric displacements, η_i are dielectric permittivities and p_3 is the pyroelectric constant.

The equations of equilibrium are

$$\sigma_{xx,x} + \sigma_{xz,z} = 0, \quad \sigma_{zx,x} + \sigma_{zz,z} = 0. \quad (5)$$

The equation of electrostatics is

$$D_{x,x} + D_{z,z} = 0. \quad (6)$$

3. Solution Technique

In order to analyze two-dimensional piezothermoelastic problems in orthotropic solids of crystal class mm2, we introduce two displacement potential functions $\mathcal{Q}$ and X and electric potential Φ :

$$u_x = \mathcal{Q}_{,x}, \quad u_z = k\mathcal{Q}_{,z} + X_{,z} \quad (7)$$

$$E_x = -\Phi_{,x}, \quad E_z = -\Phi_{,z} \quad (8)$$

where k is an unknown coefficient.

Substituting Eqs. (7) and (8) into Eqs. (3), (1) and (4), and later into Eqs. (5) and (6), we obtain

$$c_{11}\mathcal{Q}_{,xx} + \{c_{13} + (1+k)c_{55}\}\mathcal{Q}_{,zz} + (c_{13} + c_{55})X_{,zz} = -(e_1 + e_5)\Phi_{,z} + \beta_1T \quad (9)$$

$$[\{c_{13} + (1+k)c_{55}\}\mathcal{Q}_{,xx} + kc_{33}\mathcal{Q}_{,zz} + c_{55}X_{,xx} + c_{33}X_{,zz}]_{,z} = -e_5\Phi_{,xx} - e_3\Phi_{,zz} + \beta_3T_{,z} \quad (10)$$

$$[\{e_1 + (1+k)e_5\}\mathcal{Q}_{,xx} + ke_3\mathcal{Q}_{,zz} + e_5X_{,xx} + e_3X_{,zz}]_{,z} = \eta_1\Phi_{,xx} + \eta_3\Phi_{,zz} - p_3T_{,z} \quad (11)$$

By rearranging Eq. (9), and substituting Eq. (10) into Eq. (11) in order to eliminate the term $\Phi_{,xx}$, the following equations are derived:

$$\mathcal{Q}_{,xx} + \nu\mathcal{Q}_{,zz} + \frac{c_{13} + c_{55}}{c_{11}}X_{,zz} = -\frac{e_1 + e_5}{c_{11}}\Phi_{,z} + \frac{\beta_1}{c_{11}}T \quad (12)$$

$$\begin{aligned} \mathcal{Q}_{,xx} + \nu\mathcal{Q}_{,zz} + \frac{\nu}{k} \left\{ \frac{c_{55} + e_5^2/\eta_1}{c_{33} + e_3e_5/\eta_1} X_{,xx} + X_{,zz} \right\} \\ = \frac{-\nu}{(c_{33} + e_3e_5/\eta_1)k} \left\{ (e_3 - \eta e_5)\Phi_{,z} - \left(\beta_3 - \frac{e_5p_3}{\eta_1} \right) T \right\} \end{aligned} \quad (13)$$

where

$$\begin{aligned} \frac{k c_{13} + (1+k)c_{55}}{c_{11}} \\ = \frac{k(c_{33} + e_3e_5/\eta_1)}{c_{13} + (1+k)c_{55} + e_5\{e_1 + (1+k)e_5\}/\eta_1} = \nu \end{aligned} \quad (14)$$

and

$$\eta = \frac{\eta_3}{\eta_1}.$$

Eliminating k from Eq. (14), we obtain

$$\begin{aligned} c_{11} \left(c_{55} + \frac{e_5^2}{\eta_1} \right) \nu^2 + \left[c_{13}^2 + 2c_{13}c_{55} - c_{11}c_{33} \right. \\ \left. + \frac{e_5\{c_{13}(e_1 + e_5) - c_{11}e_3 + c_{55}e_1\}}{\eta_1} \right] \nu \\ \left. + c_{55} \left(c_{33} + \frac{e_3e_5}{\eta_1} \right) \right] = 0. \end{aligned} \quad (15)$$

Let ν_1 and ν_2 denote the roots of the above equation, so k_1 and k_2 corresponding to ν_1 and ν_2 are given by

$$k_i = \frac{c_{11}\nu_i - c_{55}}{c_{13} + c_{55}}, \quad (i=1, 2). \quad (16)$$

In the case of $\nu = \nu_1$, substituting Eq. (12) into Eq. (13) in order to eliminate the term $\mathcal{Q}_{,xx} + \nu\mathcal{Q}_{,zz}$ yields

$$\begin{aligned} X_{,xx} + \nu_2 X_{,zz} = \left\{ \frac{\nu_2 k_1 (e_1 + e_5)}{c_{55}} - \frac{e_3 - \eta e_5}{c_{55} + e_5^2/\eta_1} \right\} \Phi_{,z} \\ - \left\{ \frac{\nu_2 k_1 \beta_1}{c_{55}} - \frac{\beta_3 - e_5 p_3 / \eta_1}{c_{55} + e_5^2 / \eta_1} \right\} T. \end{aligned} \quad (17)$$

Next, we introduce new displacement potential functions Ψ_1 and Ψ_2 defined as follows:

$$\mathcal{Q} = \Psi_1 + \Psi_2, \quad X = \frac{1}{a} \Psi_2 \quad (18)$$

where a is an unknown coefficient. Substituting Eqs. (18) and (17) into Eq. (12) in order to eliminate the term $\Psi_{2,xx}$, and then expressing a in the following form in order to eliminate the term $\Phi_{,z}$,

$$a = -\frac{e_1 + e_5}{c_{11}\{\nu_2 k_1 (e_1 + e_5) / c_{55} - (e_3 - \eta e_5) / (c_{55} + e_5^2 / \eta_1)\}}, \quad (19)$$

we obtain

$$\Psi_{1,xx} + \nu_1 \Psi_{1,zz} = \xi_1 \Psi_{2,zz} + \delta_1 T \quad (20)$$

where

$$\begin{aligned} \xi_1 = (c_{13} + c_{55}) \left\{ \frac{k_2 \nu_1}{c_{55}} - \frac{e_3 - \eta e_5}{(e_1 + e_5)(c_{55} + e_5^2 / \eta_1)} \right\} \\ \delta_1 = \frac{\beta_1}{c_{11}} - \delta_2. \end{aligned} \quad (21)$$

Substituting the second of Eqs. (18) into Eq. (17) yields

$$\Psi_{2,xx} + \nu_2 \Psi_{2,zz} = \xi_2 \Phi_{,z} + \delta_2 T \quad (22)$$

where

$$\xi_2 = -\frac{e_1 + e_5}{c_{11}}$$

$$\delta_2 = \frac{(e_1 + e_5) \{ \nu_2 k_1 (c_{55} + e_5^2/\eta_1) \beta_1 - c_{55} (\beta_3 - e_5 p_3/\eta_1) \}}{c_{11} \{ \nu_2 k_1 (e_1 + e_5) (c_{55} + e_5^2/\eta_1) - c_{55} (e_3 - \eta e_5) \}}. \quad (23)$$

Substituting Eqs.(18) and (19) into Eq.(11), we obtain

$$\begin{aligned} \Phi_{,xx} + \eta \Phi_{,zz} \\ = \left[a_1 \Psi_{1,xx} + b_1 \Psi_{1,zz} + a_2 \Psi_{2,xx} + b_2 \Psi_{2,zz} + \frac{p_3}{\eta_1} T \right]_{,z} \end{aligned} \quad (24)$$

where

$$\left. \begin{aligned} a_1 &= \frac{e_1 + (1 + k_1)e_5}{\eta_1}, & b_1 &= \frac{k_1 e_3}{\eta_1} \\ a_2 &= \frac{e_1 + (1 + j)e_5}{\eta_1}, & b_2 &= \frac{j e_3}{\eta_1} \\ j &= k_1 - \frac{k_1 \nu_2 c_{11}}{c_{55}} + \frac{c_{11}(e_3 - \eta e_5)}{(e_1 + e_5)(c_{55} + e_5^2/\eta_1)} \end{aligned} \right\}. \quad (25)$$

Eliminating Ψ_2 and Φ from Eqs.(20), (22) and (24), the following uncoupled governing equation for Ψ_1 is derived:

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} + \mu_1 \frac{\partial^2}{\partial z^2} \right) \left(\frac{\partial^2}{\partial x^2} + \mu_2 \frac{\partial^2}{\partial z^2} \right) \left(\frac{\partial^2}{\partial x^2} + \mu_3 \frac{\partial^2}{\partial z^2} \right) \Psi_1 \\ = \delta_1 \left(\frac{\partial^2}{\partial x^2} + \nu_1 \frac{\partial^2}{\partial z^2} \right) \left(\frac{\partial^2}{\partial x^2} + \nu_2 \frac{\partial^2}{\partial z^2} \right) T \end{aligned} \quad (26)$$

where μ_1, μ_2 and μ_3 are the roots of the equation:

$$\begin{aligned} \mu^3 - (\nu_1 + \nu_2 + \eta - a_2 \xi_2) \mu^2 \\ + \{ \nu_1 \nu_2 + \eta (\nu_1 + \nu_2) - (a_2 \nu_1 + b_2) \xi_2 - a_1 \xi_1 \xi_2 \} \mu \\ - \eta \nu_1 \nu_2 + b_2 \nu_1 \xi_2 + b_1 \xi_1 \xi_2 = 0 \end{aligned} \quad (27)$$

and ν_1 and ν_2 are the roots of the equation:

$$\begin{aligned} \nu^2 - \left(\nu_2 + \eta - a_2 \xi_2 + \frac{\delta_2 \xi_1}{\delta_1} \right) \nu \\ + \eta \left(\nu_2 + \frac{\delta_2 \xi_1}{\delta_1} \right) - b_2 \xi_2 + \frac{\xi_1 \xi_2 p_3}{\delta_1 \eta_1} = 0. \end{aligned} \quad (28)$$

We resolve Ψ_1 as follows:

$$\Psi_1 = \phi_1 + \psi_1 + \psi_2 + \psi_3 \quad (29)$$

where ϕ_1 is a piezothermoelastic potential function which is the particular solution of Eq.(26), and ψ_1, ψ_2 and ψ_3 are piezoelectric potential functions which satisfy the nonhomogeneous form of Eq.(26). Their governing equations are expressed by

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} + \mu_1 \frac{\partial^2}{\partial z^2} \right) \left(\frac{\partial^2}{\partial x^2} + \mu_2 \frac{\partial^2}{\partial z^2} \right) \left(\frac{\partial^2}{\partial x^2} + \mu_3 \frac{\partial^2}{\partial z^2} \right) \phi_1 \\ = \delta_1 \left(\frac{\partial^2}{\partial x^2} + \nu_1 \frac{\partial^2}{\partial z^2} \right) \left(\frac{\partial^2}{\partial x^2} + \nu_2 \frac{\partial^2}{\partial z^2} \right) T \end{aligned} \quad (30)$$

$$\psi_{i,xx} + \mu_i \psi_{i,zz} = 0, \quad (i=1 \sim 3). \quad (31)$$

Next, we introduce the second piezothermoelastic potential function ϕ_2 and a piezoelectric potential function χ which are related to Ψ_2 and Φ as follows:

$$\Psi_2 = \phi_2 + \frac{\nu_1 - \mu_1}{\xi_1} \psi_1 + \frac{\nu_1 - \mu_2}{\xi_1} \psi_2 + \frac{\nu_1 - \mu_3}{\xi_1} \psi_3 \quad (32)$$

$$\Phi = \chi + (l_1 \psi_1 + l_2 \psi_2 + l_3 \psi_3)_{,z} \quad (33)$$

where

$$l_i = \frac{(\nu_1 - \mu_i)(\nu_2 - \mu_i)}{\xi_1 \xi_2}, \quad (i=1 \sim 3). \quad (34)$$

Substituting Eqs.(32) and (33) into Eqs.(20) and (22) and considering Eqs.(31), we obtain

$$\phi_{2,zz} = \frac{1}{\xi_1} \{ \phi_{1,xx} + \nu_1 \phi_{1,zz} - \delta_1 T \} \quad (35)$$

$$\chi_{,z} = \frac{1}{\xi_2} \{ \phi_{2,xx} + \nu_2 \phi_{2,zz} - \delta_2 T \}. \quad (36)$$

Substituting Eqs.(18), (19), (29) and (32) into Eqs.(7), the elastic displacements are expressed by

$$\begin{aligned} u_x &= (\phi_1 + \phi_2 + m_1 \psi_1 + m_2 \psi_2 + m_3 \psi_3)_{,x} \\ u_z &= (k_1 \phi_1 + j \phi_2 + n_1 \psi_1 + n_2 \psi_2 + n_3 \psi_3)_{,z} \end{aligned} \quad (37)$$

where

$$m_i = 1 + \frac{\nu_1 - \mu_i}{\xi_1}, \quad n_i = k_1 + \frac{j(\nu_1 - \mu_i)}{\xi_1}, \quad (i=1 \sim 3). \quad (38)$$

In the case of $\nu = \nu_2$, equations similar to those obtained for the case of $\nu = \nu_1$ can be derived, but will be omitted here for brevity.

4. Piezothermoelastic Problem in Orthotropic Plate

Let us consider a two-dimensional piezothermoelastic problem in an orthotropic thin plate exhibiting class mm2 symmetry. The plate is subjected to heating on the top surface and electric surface charge on both surfaces. The plate thickness is denoted by $2b$.

4.1 Temperature field

The boundary conditions for the temperature field are given by

$$\begin{aligned} T_{,z} + h_1 T &= h_1 T_0 f(x) \text{ on } z = b \\ -T_{,z} + h_2 T &= 0 \text{ on } z = -b \end{aligned} \quad (39)$$

where h_i are coefficients of relative surface heat transfer, T_0 is constant temperature and $T_0 f(x)$ represents temperature distribution of heating medium which acts on the top surface of the plate.

The solution for the stationary case of Eq.(2), which satisfies Eqs.(39), is expressed by

$$T = \int_0^\infty \cos(qx) \left\{ A \cosh\left(\frac{qz}{\lambda}\right) + B \sinh\left(\frac{qz}{\lambda}\right) \right\} dq \quad (40)$$

where

$$[A, B] = \frac{2h_1 \bar{f} [\overline{TB_2}, \overline{TA_2}]}{\pi (\overline{TA_1} \overline{TB_2} + \overline{TA_2} \overline{TB_1})}$$

and

$$\begin{aligned} \bar{f} &= \int_0^\infty f(x) \cos(qx) dx \\ \overline{TA_i} &= \frac{q}{\lambda} \sinh\left(\frac{qb}{\lambda}\right) + h_i \cosh\left(\frac{qb}{\lambda}\right) \\ \overline{TB_i} &= \frac{q}{\lambda} \cosh\left(\frac{qb}{\lambda}\right) + h_i \sinh\left(\frac{qb}{\lambda}\right). \end{aligned}$$

4.2 Piezothermoelastic field

The boundary conditions for the elastic and electric fields are taken to be

$$\sigma_{xz} = \sigma_{zz} = 0 \text{ on } z = \pm b \quad (41)$$

$$D_z = D_0 g(x) \text{ on } z = \pm b \quad (42)$$

where D_0 is constant surface charge and $D_0 g(x)$ represents distribution of electric surface charge which acts on both surfaces of the plate.

The piezothermoelastic potential function ϕ_1 is represented in a form similar to the temperature

expression (40).

$$\phi_1 = \int_0^\infty F_1 \cos(qx) \left\{ A \cosh\left(\frac{qz}{\lambda}\right) + B \sinh\left(\frac{qz}{\lambda}\right) \right\} dq \quad (43)$$

Substituting Eqs. (40) and (43) into Eq. (30), F_1 can be determined as follows:

$$F_1 = \frac{\lambda^2(\nu_1 - \lambda^2)(\nu_2 - \lambda^2)\delta_1}{(\mu_1 - \lambda^2)(\mu_2 - \lambda^2)(\mu_3 - \lambda^2)q^2}.$$

Similarly, the second piezothermoelastic potential function and the piezoelectric potential function can be obtained from Eqs. (35) and (36):

$$\phi_2 = \int_0^\infty F_2 \cos(qx) \left\{ A \cosh\left(\frac{qz}{\lambda}\right) + B \sinh\left(\frac{qz}{\lambda}\right) \right\} dq \quad (44)$$

$$\chi = \int_0^\infty F_3 \cos(qx) \left\{ A \sinh\left(\frac{qz}{\lambda}\right) + B \cosh\left(\frac{qz}{\lambda}\right) \right\} dq \quad (45)$$

where

$$F_2 = \frac{(\nu_1 - \lambda^2)q^2 F_1 - \lambda^2 \delta_1}{\xi_1 q^2}, \quad F_3 = \frac{(\nu_2 - \lambda^2)q^2 F_2 - \lambda^2 \delta_2}{\lambda \xi_2 q}.$$

Next, the piezoelectric potential functions ϕ_i ($i=1\sim 3$) which satisfy their governing equations (31) are assumed to be

$$\phi_i = \int_0^\infty \cos(qx) \left\{ G_i \cosh\left(\frac{qz}{\sqrt{\mu_i}}\right) + H_i \sinh\left(\frac{qz}{\sqrt{\mu_i}}\right) \right\} dq, \quad (i=1\sim 3) \quad (46)$$

where G_i and H_i ($i=1\sim 3$) are unknown coefficients.

Substituting Eqs. (43)~(46) into Eqs. (33) and (37) yields the electric potential:

$$\Phi = \int_0^\infty \cos(qx) \left[F_3 \left\{ A \sinh\left(\frac{qz}{\lambda}\right) + B \cosh\left(\frac{qz}{\lambda}\right) \right\} + \sum_{i=1}^3 \frac{l_i q}{\sqrt{\mu_i}} \left\{ G_i \sinh\left(\frac{qz}{\sqrt{\mu_i}}\right) + H_i \cosh\left(\frac{qz}{\sqrt{\mu_i}}\right) \right\} \right] dq \quad (47)$$

and the elastic displacements:

$$u_x = - \int_0^\infty q \sin(qx) \left\{ (F_1 + F_2) \left\{ A \cosh\left(\frac{qz}{\lambda}\right) + B \sinh\left(\frac{qz}{\lambda}\right) \right\} + \sum_{i=1}^3 m_i \left\{ G_i \cosh\left(\frac{qz}{\sqrt{\mu_i}}\right) + H_i \sinh\left(\frac{qz}{\sqrt{\mu_i}}\right) \right\} \right\} dq. \quad (48)$$

$$u_z = \int_0^\infty q \cos(qx) \left\{ \frac{k_1 F_1 + j F_2}{\lambda} \left\{ A \sinh\left(\frac{qz}{\lambda}\right) + B \cosh\left(\frac{qz}{\lambda}\right) \right\} + \sum_{i=1}^3 \frac{n_i}{\sqrt{\mu_i}} \left\{ G_i \sinh\left(\frac{qz}{\sqrt{\mu_i}}\right) + H_i \cosh\left(\frac{qz}{\sqrt{\mu_i}}\right) \right\} \right\} dq. \quad (49)$$

By substituting Eqs. (47)~(49) into Eqs. (3) and (8), and later into Eqs. (1) and (4), expressions for the electric field intensities, stresses and electric displacements are derived:

$$E_x = \int_0^\infty q \sin(qx) \left[F_3 \left\{ A \sinh\left(\frac{qz}{\lambda}\right) + B \cosh\left(\frac{qz}{\lambda}\right) \right\} \right]$$

$$+ \sum_{i=1}^3 \frac{l_i q}{\sqrt{\mu_i}} \left\{ G_i \sinh\left(\frac{qz}{\sqrt{\mu_i}}\right) + H_i \cosh\left(\frac{qz}{\sqrt{\mu_i}}\right) \right\} \right] dq \quad (50)$$

$$E_z = - \int_0^\infty q \cos(qx) \left\{ \frac{F_3}{\lambda} \left\{ A \cosh\left(\frac{qz}{\lambda}\right) + B \sinh\left(\frac{qz}{\lambda}\right) \right\} + \sum_{i=1}^3 \frac{l_i q}{\mu_i} \left\{ G_i \cosh\left(\frac{qz}{\sqrt{\mu_i}}\right) + H_i \sinh\left(\frac{qz}{\sqrt{\mu_i}}\right) \right\} \right\} \right] dq \quad (51)$$

$$\sigma_{xx} = - \int_0^\infty q^2 \cos(qx) \left\{ \left[c_{11}(F_1 + F_2) - \frac{c_{13}(jF_1 + k_1 F_2)}{\lambda^2} - \frac{e_1 F_3}{\lambda q} + \frac{\beta_1}{q^2} \right] \left\{ A \cosh\left(\frac{qz}{\lambda}\right) + B \sinh\left(\frac{qz}{\lambda}\right) \right\} + \sum_{i=1}^3 \left\{ m_i c_{11} - \frac{n_i c_{13}}{\mu_i} - \frac{l_i e_1}{\mu_i} \right\} \left\{ G_i \cosh\left(\frac{qz}{\sqrt{\mu_i}}\right) + H_i \sinh\left(\frac{qz}{\sqrt{\mu_i}}\right) \right\} \right\} \right] dq \quad (52)$$

$$\sigma_{zx} = - \int_0^\infty q^2 \sin(qx) \left\{ \left[\frac{c_{55} \langle (1+j)F_1 + (1+k_1)F_2 \rangle}{\lambda} + \frac{e_5 F_3}{q} \right] \left\{ A \sinh\left(\frac{qz}{\lambda}\right) + B \cosh\left(\frac{qz}{\lambda}\right) \right\} + \sum_{i=1}^3 \left\{ \frac{(m_i + n_i)c_{55}}{\sqrt{\mu_i}} + \frac{l_i e_5}{\sqrt{\mu_i}} \right\} \left\{ G_i \sinh\left(\frac{qz}{\sqrt{\mu_i}}\right) + H_i \cosh\left(\frac{qz}{\sqrt{\mu_i}}\right) \right\} \right\} \right] dq \quad (53)$$

$$D_x = - \int_0^\infty q^2 \sin(qx) \left\{ \left[\frac{e_5 \langle (1+j)F_1 + (1+k_1)F_2 \rangle}{\lambda} - \frac{\eta_1 F_3}{q} \right] \left\{ A \sinh\left(\frac{qz}{\lambda}\right) + B \cosh\left(\frac{qz}{\lambda}\right) \right\} + \sum_{i=1}^3 \left\{ \frac{(m_i + n_i)e_5}{\sqrt{\mu_i}} - \frac{l_i \eta_1}{\sqrt{\mu_i}} \right\} \left\{ G_i \sinh\left(\frac{qz}{\sqrt{\mu_i}}\right) + H_i \cosh\left(\frac{qz}{\sqrt{\mu_i}}\right) \right\} \right\} \right] dq \quad (54)$$

$$D_z = \int_0^\infty q^2 \cos(qx) \left\{ \left[-e_1(F_1 + F_2) + \frac{e_3(jF_1 + k_1 F_2)}{\lambda^2} - \frac{\eta_3 F_3}{\lambda q} + \frac{p_3}{q^2} \right] \left\{ A \cosh\left(\frac{qz}{\lambda}\right) + B \sinh\left(\frac{qz}{\lambda}\right) \right\} + \sum_{i=1}^3 \left\{ -m_i e_1 + \frac{n_i e_3}{\mu_i} - \frac{l_i \eta_3}{\mu_i} \right\} \left\{ G_i \cosh\left(\frac{qz}{\sqrt{\mu_i}}\right) + H_i \sinh\left(\frac{qz}{\sqrt{\mu_i}}\right) \right\} \right\} \right] dq \quad (55)$$

where σ_{yy} and σ_{zz} can be obtained by replacing c_{11} , c_{13} , e_1 and β_1 with c_{12} , c_{23} , e_2 and β_2 and those with c_{13} , c_{33} , e_3 and β_3 in the expression of σ_{xx} , respectively.

From the boundary conditions given by Eqs. (41) and (42), the unknown coefficients G_i and H_i ($i=1\sim 3$) contained in Eqs. (47)~(55) can be determined by solving simple linear equations.

5. Numerical Example

For the numerical calculations, the following dimensionless quantities are introduced:

$$\bar{x} = \frac{x}{b}, \quad \bar{z} = \frac{z}{b}, \quad B_i = b h_i, \quad \bar{T} = \frac{T}{T_0},$$

$$\bar{u}_i = \frac{u_i}{b \alpha_x T_0}, \quad \bar{\sigma}_{ij} = \frac{\sigma_{ij}}{\alpha_x Y_x T_0},$$

$$\bar{\Phi} = \frac{|d_1|\Phi}{b\alpha_x T_0}, \quad \bar{E}_i = \frac{|d_1|E_i}{\alpha_x T_0}, \quad \bar{D}_i = \frac{D_i}{|d_1|\alpha_x Y_x T_0}$$

where α_x is the coefficient of linear thermal expansion, Y_x is Young's modulus and d_1 is the piezoelectric coefficient.

Fig. 1 Comparison of elastic displacements in $\bar{x}$ and $\bar{z}$ directions for cadmium selenide and those for material without piezoeffect

Fig. 2 Comparison of stress in $\bar{x}$ direction for cadmium selenide and that for material without piezoeffect

Fig. 3 Comparison of stress in $\bar{y}$ direction for cadmium selenide and that for material without piezoeffect

Material constants are taken for cadmium selenide as follows⁽²¹⁾:

$$c_{11}=74.1 \text{ GPa}, \quad c_{12}=45.2 \text{ GPa},$$

$$c_{13}=c_{23}=39.3 \text{ GPa}, \quad c_{33}=83.6 \text{ GPa}, \quad c_{55}=13.17 \text{ GPa},$$

$$\beta_1=\beta_2=0.621 \frac{\text{MPa}}{\text{K}}, \quad \beta_3=0.551 \frac{\text{MPa}}{\text{K}},$$

$$e_1=e_2=-0.160 \frac{\text{C}}{\text{m}^2}, \quad e_3=0.347 \frac{\text{C}}{\text{m}^2}, \quad e_5=-0.138 \frac{\text{C}}{\text{m}^2},$$

$$\eta_1=8.25 \times 10^{-11} \frac{\text{C}^2}{\text{Nm}^2}, \quad \eta_3=9.02 \times 10^{-11} \frac{\text{C}^2}{\text{Nm}^2},$$

$$p_3=-2.94 \times 10^{-6} \frac{\text{C}}{\text{m}^2 \text{K}}.$$

The value $\lambda^2=\lambda_z/\lambda_x=1.5$ is assumed, because the coefficients of thermal conductivity for cadmium selenide could not be found in the literature.

In the numerical calculations, the functions $f(\bar{x})$ and $g(\bar{x})$ in Eqs. (39) and (42) and the values of Biot's number are assumed to be

$$f(\bar{x})=H(1-|\bar{x}|), \quad g(\bar{x})=\exp(-\bar{x}^2)$$

$$B_1=1.0, \quad B_2=0.1$$

where $H(\bar{x})$ is Heaviside's unit step function.

Calculated results on the surfaces of the plate are illustrated in the following figures, where the value of $\bar{D}_0$ defined in Eqs. (42) represents the strength of electric surface charge.

When heat is introduced on the top surface of the plate and no electric surface charge is exerted on both surfaces (i.e., $\bar{D}_0=0$), calculated results for the elastic displacements and stresses are illustrated in Figs. 1~3. The solid lines denote the numerical results for cadmium selenide exhibiting the piezoeffect, while the broken lines denote the corresponding values for a material with the same thermoelastic properties but with the piezoeffect ignored. The values of $(\bar{u}_x)_{\bar{z}=\pm 1}$, $(\bar{\sigma}_{xx})_{\bar{z}=\pm 1}$ and $(\bar{\sigma}_{yy})_{\bar{z}=\pm 1}$ for cadmium selenide are somewhat smaller than those for the material without the piezoeffect, while the value of $[(\bar{u}_x)_{\bar{z}=1} - (\bar{u}_x)_{\bar{z}=-1}]$ for the cadmium selenide is larger than that for the material without the piezoeffect.

In the case that the plate is subjected not only to the heating on the top surface of the plate but also to the electric surface charge on both surfaces, the effects of the electric surface charge on the elastic displacements, stresses, electric potential difference, electric field intensities and electric displacements are illustrated in Figs. 4~11. The solid lines denote the numerical results on the top surface, while the broken lines denote those on the bottom surface. In Figs. 4~7, the values of $(\bar{u}_x)_{\bar{z}=\pm 1}$ and $(\bar{\sigma}_{yy})_{\bar{z}=\pm 1}$ decrease and the values of $[(\bar{u}_x)_{\bar{z}=1} - (\bar{u}_x)_{\bar{z}=-1}]$ and $(\bar{\sigma}_{xx})_{\bar{z}=\pm 1}$ increase with increasing electric surface charge. Since the maximum stress is $(\bar{\sigma}_{yy})_{\bar{z}=1}$, it is possible to decrease the maximum value of induced thermal stresses by application of appropriate electric surface charge. In

Figs. 8~11, the values of $[(\bar{\Phi})_{\bar{z}=1} - (\bar{\Phi})_{\bar{z}=-1}]$, $(\bar{E}_x)_{\bar{z}=\pm 1}$, $(\bar{E}_z)_{\bar{z}=\pm 1}$ and $(\bar{D}_x)_{\bar{z}=\pm 1}$ increase with the electric surface charge. In Figs. 9~11, the heating which acts only on the top surface exerts an influence on the electric field, thus the absolute values of $\bar{E}_x$, $\bar{E}_z$ and $\bar{D}_x$

on the top surface of the plate are different from those on the bottom surface.

The lines for $\bar{D}_0=0$ shown in Figs. 8~11 which denote the electric potential, electric field intensities and electric displacements correspond to the values of

Fig. 4 Effect of electric surface charge on elastic displacement in $\bar{x}$ direction

Fig. 7 Effect of electric surface charge on stress in $\bar{y}$ direction

Fig. 5 Effect of electric surface charge on elastic displacement in $\bar{z}$ direction

Fig. 8 Effect of electric surface charge on electric potential difference between plate surfaces

Fig. 6 Effect of electric surface charge on stress in $\bar{x}$ direction

Fig. 9 Effect of electric surface charge on electric field intensity in $\bar{x}$ direction

Fig. 10 Effect of electric surface charge on electric field intensity in $\bar{z}$ direction

Fig. 11 Effect of electric surface charge on electric displacement in $\bar{x}$ direction

elastic displacements and stresses represented by the solid lines in Figs. 1~3.

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