Effect of Out-of-Plane Stress on the Forming Limit Strain of Sheet Metals*

Manabu GOTOH**, Taehoon CHUNG** and Noritoshi IWATA***

In press forming of sheet metals, the material sheet is usually subjected to very large plastic strain under in-plane stressing. Moreover, the sheet also very often is subjected to out-of-plane compressive force between tools such as the upper and lower dies, the blank holder and the die, and so forth. However, the forming limit strain of sheet metals has been investigated so far only for the plane stress state without the effect of such out-of-plane force both theoretically and experimentally. In this paper, it is clearly demonstrated theoretically that out-of-plane stress (even as small as one tenth of the yield stress) may notably raise the forming limit strain and thus it can be effectively utilized to avoid earlier fracture of the sheet in press forming. The classical Swift's or Hill's criterion of instability, Stören & Rice's condition together with the J2G plasticity constitutive equation (previously proposed by one of the authors, Gotoh) for localized necking, and the numerical simulation of plane-stress and plane-strain tension of a sheet by the finite-element method (FEM) are applied here to discuss this problem, all leading us to the same conclusion.

Key Words: Plasticity, Sheet Metal, Forming Limit Strain, Out-of-Plane Stress, Plasticity Instability, Finite-Element Method, Localized Necking, Local Bifurcation Condition, Vertex-Hardening Model

1. Introduction

In press forming of sheet metals, the material sheet is usually subjected to very large plastic strain under in-plane stressing by the action of rigid tools. Therefore, it is of paramount importance to predict the forming limit to fracture. This problem has long been investigated by many researchers both experimentally and theoretically. As for the theoretical investigation, the so-called plastic instability condition or localized necking condition has been considered effective for fracture prediction. Many theories on this condition have been proposed so far, among which the classical Swift's or Hill's criterion^{(1),(2)} is very popular even today and still referred to very often, mainly for its simplicity. However, considering only simple or proportional loading, their criterion does not involve the state in which the ratio of minor principal in-plane strain to major one ($=\alpha$) is posi-

tive, especially when the localized necking governs the critical condition of fracture as usually considered. To recover this point, Marciniak & Kuczynski proposed a numerical model which assumes the pre-existing inhomogeneity in sheet metal such as thickness nonuniformity⁽³⁾. However, this model never gives any analytical formula of the critical strain and inevitably requires some numerical calculation.

In this sense, their theory is not as simple as it looks. Stören & Rice modified Hill's localized necking condition into a more sophisticated form⁽⁴⁾ which takes into account the state of positive α . Their theory emphasizes the importance of the mathematical plasticity model, namely the plasticity constitutive equation. One of the authors (Gotoh) previously proposed a new type of constitutive equation which allows vertex evolution at the current stress on the subsequent yield surface. He, together with his co-workers, successfully applied it to various types of forming limit problems of sheet metals⁽⁵⁾⁻⁽¹⁰⁾, adopting Stören & Rice's theory. He also suggests the possibility of fracture prediction by the numerical simulation of deformation by FEM, introducing this constitutive equation^{(11),(12)}, without the use of their theory.

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** Department of Mechanical Engineering, Gifu University, Yanagido 1-1, Gifu 501-11, Japan

*** Toyota Central Res. & Develop. Labs. Inc., Nagakute, Aichi 480-11, Japan

However, all theoretical investigations on the forming limit strain of sheet metals reported so far assume the state of plane stress. They are valid for most of the actual press-forming operations in which the critical portion of the sheet is (almost) free from any out-of-plane force from both sides of its surface. On the other hand, however, press forming of sheet metals is very often operated by using the upper and lower dies or the die and the blank-holder. In these cases, it may happen that the sheet breaks at the portion compressed between two dies under some out-of-plane stressing. In such cases, the assumption of plane stress state can no longer be applied. The theory must be reconstructed.

Actually, one of the authors (Iwata) has recently encountered a phenomenon such that fracture at the bore rim is delayed in a bore-expanding operation on a sheet by the use of a flat-headed punch when a back-up tool is employed.

In this paper, the forming limit strain of sheet metals under out-of-plane compressive force as well as in-plane biaxial stressing is investigated from three points of view. One of them is classical, namely in which Swift's or/and Hill's idea is followed in a modified form. The other one follows Stören & Rice's idea together with the J2G constitutive equation. In both cases, proportional strain with various constant α -values is involved. The third is the direct numerical simulation of deformation in simple tension in the states of both plane stress and plane strain by FEM, in which the elastic-plastic FEM code GOLDA developed previously for the analysis of large deformation by one of the authors (Gotoh)⁽¹³⁾ is utilized with a minor revision of fitting to the present problem.

2. Discussion from the View Point of Swift or Hill

In this section, the sheet material is approximated to be rigid-plastic, neglecting elastic strain as compared with large plastic strain. Furthermore, the material is assumed to obey the so-called Mises' yield condition (i.e., it is the Mises material).

The classical theory on plastic instability both in diffuse and localized senses was derived from the idea that a virtually infinitesimal disturbance in deformation in the current balanced state could grow or be kept stationary and never shrink to nothing. The critical condition is simply formulated as follows^{(1),(14)}:

$$\Delta\bar{\sigma} \leq \delta\bar{\sigma} \quad (\text{instability condition}) \quad (1)$$

where $\Delta\bar{\sigma} = (d\bar{\sigma}/d\bar{\epsilon})\delta\bar{\epsilon}$. This means the increment of the equivalent stress which can be supplied from the material property itself, i.e., the potential hardening of the material, where $\delta\bar{\epsilon}$ is the virtual increment of the equivalent strain due to the virtual disturbance in deformation. On the other hand, $\delta\bar{\sigma}$ is the increment

of the equivalent stress required for realization of the kinematic virtual disturbance $\delta\bar{\epsilon}$. Of course, $(d\bar{\sigma}/d\bar{\epsilon})$ is the instantaneous work-hardening rate of the material and thus given by the slope of the tangential line to the stress-strain curve (showing the work-hardening property) at the current stress state. The instability condition (1) above may be understood in the manner that, if Eq. (1) holds, the material cannot catch up with the given disturbance by its own property and thus may lose the capability to resist growth of the disturbance, which means that unstable deformation could prevail.

In the following, for simplicity, only simple or straight strain-paths with constant strain ratio α are considered, where $\alpha = d\epsilon_2/d\epsilon_1 = \epsilon_2/\epsilon_1$, $\epsilon_1 > 0$, and $|\alpha| \leq 1$.

2.1 Localized necking

It is now commonly believed that any thin sheet may break due to and along a precedent localized neck. This neck grows apparently in the thickness direction within a narrow band whose width is comparable to the sheet thickness (see Fig. 1). Its occurrence is a kind of plastic strain localization and thus a type of localized bifurcation phenomenon.

In Fig. 1, a material small element is being deformed under (almost) uniform stress and strain, being subjected to biaxial stressing in the 1- and 2- directions, where σ_1 and σ_2 are the in-plane principal stresses, $\sigma_1 \geq 0$, and $\sigma_1 \geq \sigma_2$. Additionally, though not shown in the figure, the third principal stress $\sigma_3 (= -p)$ acts in the out-of-plane direction (i.e., in the 3-direction), where p is assumed to be nonnegative and constant throughout loading ($p \geq 0$). A virtual localized neck band is also drawn in the figure whose in-plane normal unit vector $\bar{n}$ is inclined with respect to the 1-axis by the angle ψ and has g_1 and g_2 as its 1- and 2-components. Now, the tensile in-plane force T is assumed to act perpendicularly to the neck band in proportion to unit length, where $T = t\sigma_n$, t = current sheet thickness, and σ_n is the in-plane stress component in the n -direction.

In this balanced state, taking the nature of localized necking into account, an infinitesimal disturbance

Fig. 1 Small element of sheet (localized neck)

δt in thickness is introduced along the virtual localized necking band. Then other various physical quantities also experience corresponding disturbances, which are denoted δ . According to Hill's idea⁽²⁾, the following equation is readily obtained:

$$\delta\sigma_1/\sigma_1 = \delta\sigma_2/\sigma_2 = -\delta t/t = -\delta\varepsilon_3 = \delta\varepsilon_1 + \delta\varepsilon_2. \quad (2)$$

From this equation,

$$\left. \begin{aligned} \delta\bar{\sigma} &= -\bar{\sigma}(1-\beta)\delta\varepsilon_3, \\ \beta &= (3/2)\sigma_3\sigma_3/\bar{\sigma}^2, \sigma_3 = -p \end{aligned} \right\} \quad (3)$$

where, the prime denotes the deviator of stress, and the equivalent stress $\bar{\sigma}$ is given as follows:

$$\bar{\sigma} = [(1/2)\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}]^{1/2} \quad (4)$$

On the other hand, the following equations hold:

$$\left. \begin{aligned} \Delta\bar{\sigma} &= (\bar{\sigma}/Z)\delta\bar{\varepsilon} \\ 1/Z &= (d\bar{\sigma}/d\bar{\varepsilon})/\bar{\sigma} \end{aligned} \right\} \quad (5)$$

From Eqs. (1), (3) and (5), the localized-type instability condition is derived as follows:

$$\frac{1}{Z} \leq \frac{(1+\alpha)(1-\beta)}{(2/\sqrt{3})(1+\alpha+\alpha^2)^{1/2}} \quad (6)$$

where the equivalent strain $\bar{\varepsilon}$ is expressed as follows:

$$\bar{\varepsilon} = (2/\sqrt{3})(\varepsilon_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_2^2)^{1/2} \quad (7)$$

For clarity, only the n -th power law is adopted for the expression of work-hardening property of the sheet metals. That is,

$$\bar{\sigma} = C \cdot \bar{\varepsilon}^n,$$

where C is called the plasticity coefficient and n is the work-hardening exponent simply called the n -value. In this case, $1/Z = n/\bar{\varepsilon}$ and thus, from Eq. (6), the critical condition for the major in-plane strain ε_1 is reduced as follows:

$$\varepsilon_1 \geq \frac{n}{1+\alpha} \cdot \frac{1}{1-\beta}; \quad (1-\beta) > 0 \quad (8)$$

and therefore the forming limit strains are governed by the following equation:

$$(\varepsilon_1)_{cr} + (\varepsilon_2)_{cr} = -(\varepsilon_3)_{cr} = n/(1-\beta) \quad (9)$$

In the equations above, from Eqs. (3) and (4), the factor β is expressed as follows:

$$\beta = \frac{1}{2} \cdot \frac{m'(1+m+2m')}{(1-m+m^2) + m'(1+m+m')}, \quad (10)$$

$$\left. \begin{aligned} m &= \frac{\sigma_2}{\sigma_1} \\ &= \frac{\alpha(2+m') + 1 - m'}{(2+\alpha)}, \\ -1 &\leq \alpha \leq 0, \\ |m| &\leq 1, \\ m' &= \frac{p}{\sigma_1} = -\frac{\sigma_3}{\sigma_1} \end{aligned} \right\} \quad (11)$$

The denominator in Eq. (10) is always positive. Furthermore, the ratio of the minor in-plane stress to the major one (i.e., $m = \sigma_2/\sigma_1$) is confined within unity ($|m| \leq 1$). Therefore, for $p \geq 0$, the factor β is easily found to satisfy the following inequality:

$$0 \leq \beta < 1.$$

Considering this inequality together with Eq. (9), it is found that the limiting strain to fracture increases

upon the application of the out-of-plane compressive stress ($\sigma_3 = -p$; $p > 0$) than that in the common case where the sheet is free from the out-of-plane stress ($p = 0$), if fracture occurs immediately after localized necking begins, as is usually seen in press forming.

To clarify this point, several special cases are considered here.

(i) Plane strain state ($\alpha = 0$): In this case, the following equations are deduced.

$$\left. \begin{aligned} m &= \frac{(1-m')}{2}, \\ \beta &= \frac{m'}{(1+m')}; \quad m' = \frac{\beta}{(1-\beta)} \\ (\varepsilon_1)_{cr} &= \frac{n}{(1-\beta)} = n(1+m') \geq n \end{aligned} \right\} \quad (12)$$

Remembering the conventional statement that the limit strain for $\alpha = 0$ is equal to n itself for an n -th power hardening material, it is interesting to point out from Eq. (12) that this is not true when the out-of-plane loading p is applied, and that the limit strain becomes greater than n with p through the factor β or m' in Eq. (12).

(ii) Nonpositive strain ratio ($-1 \leq \alpha \leq 0$): The critical major strain is expressed as follows:

$$(\varepsilon_1)_{cr} = \frac{\frac{n}{(1+\alpha)}}{1-\beta} \quad (13)$$

Furthermore, the following equations are readily deduced.

$$\left. \begin{aligned} \frac{p}{c} &= A^{(1+n)} \cdot n^n \cdot \frac{\beta}{(1-\beta)^n} \\ A &= \frac{2}{\sqrt{3}} \cdot \frac{A^*}{(1-\alpha)}; \quad A^* = \sqrt{1+\alpha+\alpha^2} \end{aligned} \right\} \quad (14)$$

Therefore, factor β can be evaluated from these equations by giving (p/c) , n and α . First, the following relations are easily found.

i. When $p \rightarrow \infty$, then $\beta \rightarrow 1$, and thus $(\varepsilon_1)_{cr} \rightarrow \infty$.

ii. When $\alpha \rightarrow -1$, then $\beta \rightarrow 0$, and thus $(\varepsilon_1)_{cr} \rightarrow (\varepsilon_1)_{cr}$ for $p = 0$, which means the effect of p decreases for α closer to -1 .

Next, a few numerical examples are given below for $n = 0.5$:

i. $p/c = 0.1$

$$\alpha = 0: (\varepsilon_1)_{cr} \approx n \times 1.12$$

$$\alpha = -0.5: (\varepsilon_1)_{cr} \approx (2n) \times 1.05$$

ii. $p/c = 1.0$

$$\alpha = 0: (\varepsilon_1)_{cr} \approx n \times 3.0$$

$$\alpha = -0.5: (\varepsilon_1)_{cr} \approx (2n) \times 1.64$$

2.2 Diffuse necking

This type of necking corresponds, for example, to the necking phenomenon apparently in the width direction in the tensile test of thin sheet which will appear almost at and after the maximum point in tensile load. Therefore it never induces fracture in the sheet immediately. However, it is also a kind of

plastic instability, though not localized within a narrow region but diffused over some area.

In Fig. 2, a small material element is being deformed under uniform in-plane biaxial stresses together with the constant out-of-plane pressure p . In this balanced state, considering the nature of diffuse necking, virtual disturbances δa and δb in the sides a and b are assumed. Then, following Swift's or Hill's idea and making use of the critical condition (1), the following expression for the critical major strain is finally obtained.

$$(\epsilon_1)_{cr} = n \times \frac{2(1+\alpha+\alpha^2)(1+m')}{(1+\alpha)(2-\alpha+2\alpha^2)-m'\alpha^2(1-\alpha)} \quad (15)$$

Furthermore, the following relations between (p/c) and m' and α are deduced, where $m' = -\sigma_3/\bar{\sigma} = p/\bar{\sigma}$.

$$\left. \begin{aligned} \frac{p}{c} &= B \cdot \frac{m'}{1+m'} \cdot \left[\frac{(1+m')}{C-m'\alpha^2(1-\alpha)} \right]^n \\ B &= (A^*)^{3n-1} \cdot 2^{2n} \cdot n^n (2+\alpha) / (\sqrt{3})^{n+1} \\ C &= (1+\alpha)(2-\alpha+2\alpha^2) \end{aligned} \right\} \quad (16)$$

From Eq. (15), it is again ascertained that the out-of-plane loading makes the critical strain greater than that under no such loading even for this diffuse-type instability.

Using Eqs. (15) and (16), several numerical examples are presented below for the case of $n=0.5$:

i. For $\alpha=0$ (i.e., plane strain state),

$$(\epsilon_1)_{cr} \approx n \times 1.12 \text{ for } p/c=0.1,$$

$$(\epsilon_1)_{cr} \approx n \times 3.0 \text{ for } p/c=1.0.$$

In this case, $(\epsilon_1)_{cr}$ is equal to that for localized necking, as expected, but greater than n -value.

ii. For $\alpha=-0.5$ (uniaxial tension in 1-direction),

$$(\epsilon_1)_{cr} \approx n \times 1.40 \text{ for } p/c=0.1,$$

$$(\epsilon_1)_{cr} \approx n \times 70.4 \text{ for } p/c=0.8,$$

iii. For $\alpha=1.0$ (i.e., equi-biaxial tension),

$$(\epsilon_1)_{cr} \approx n \times 1.105 \text{ for } p/c=0.1,$$

$$(\epsilon_1)_{cr} \approx n \times 2.62 \text{ for } p/c=1.0,$$

From these examples, we may say that the effect of p on the limiting strain becomes weaker for α closer to 1, in contrast with the case of localized necking.

3. Discussion in Light of Authors' Theory : Localized Necking

In this and the next sections the sheet material is considered to be elastic-plastic. As stated in the intro-

Fig. 2 Small element of sheet (diffuse neck)

duction, one of the authors (Gotoh) together with his co-workers has performed much research on the forming limit strains of sheets under various conditions. In such research, a new type of plasticity constitutive equation (J2G, J_2 -Gotoh's corner theory) previously proposed by himself is employed in Stören & Rice's theory on localized necking and gives successful results.

Stören & Rice's theory assumes a localized necking band (as seen in Fig.1) to be a band along which strain gradients (and thus stress gradients as well) are discontinuous. Their critical condition for localized necking is found to be very sensitive to the plasticity constitutive equation employed. For example, the conventional J2F (J_2 -flow theory, the large strain version of classical Prandtl-Reuss' equation) never gives the limiting strain of reasonable magnitude. They used the J2D (J_2 -deformation theory, the large-strain version of the classical Hencky's total strain theory in its rate form). However, in a quantitative sense, their numerical results were not very satisfactory. The J2G recovers it to a great extent. However, all works reported so far assumed plane stress state in deforming sheets, being free from out-of-plane stresses. Therefore, we could not comment on the influence of out-of-plane stress on the forming limit strain. This is the problem to be discussed below.

The J2G constitutive equation is written in tensor form as follows (see, e.g., Ref.(15) for details):

$$\dot{\epsilon} = \frac{1}{2G} \dot{T} + \langle P(\Theta) \rangle \left\{ \frac{1}{2\bar{\sigma}h_0} bT + \frac{1}{2h_0} a\dot{T} \right\} \quad (17)$$

$$P(\Theta) = a + b \cos \Theta;$$

$$\langle A \rangle = A \text{ for } A > 0$$

$$\langle A \rangle = 0 \text{ for } A \leq 0$$

$$\cos \Theta = T : \dot{T} / \{ (T^2) \cdot (\dot{T}^2) \}^{1/2},$$

$$a = h_0/H_0,$$

$$b = 1 - a,$$

$$3h_0 = d\bar{\sigma}/d\bar{\epsilon}^p,$$

$$a = \frac{\cos \Theta_0}{1 + \cos \Theta_0}$$

$$\Theta_0 = \frac{\pi}{2} - \rho \bar{\epsilon}^p; \quad \bar{\epsilon}^p = \int d\bar{\epsilon}^p \quad (A)$$

$$\dot{T} = \dot{T} - \dot{\omega}T + \dot{\omega}, \quad T = \sigma - \sigma_m I$$

$$\dot{\omega} = \frac{1}{2}(\dot{L} - \dot{L}^T), \quad L = \frac{\partial u}{\partial x}$$

$$\dot{\epsilon} = \frac{1}{2}(\dot{L} + \dot{L}^T),$$

$$\dot{\epsilon} = \dot{\epsilon} - (\dot{\epsilon}_v/3)I$$

σ =Cauchy stress, $\dot{\epsilon}$ =Euler strain rate,

$\dot{\epsilon}$ =deviator of $\dot{\epsilon}$, $\sigma_m = \sigma_{kk}/3$, $\dot{\epsilon} = \dot{\epsilon}_{kk}$,

I =unit tensor of second rank,

G =elastic shear modulus

The most characteristic feature of the J2G is that,

as illustrated in Fig. 3, it allows the evolution of a vertex on the subsequent yield surface at the current stress and thus never follows the conventional normality rule for strain rate. This vertexed portion of the yield surface can be understood as a cone in the superspace of deviatoric stress $\mathbf{T}$ whose semi-cone angle is denoted by Θ_0 in the figure and in Eq. (A) above. This vertex is considered to evolve with deformation as described in Eq. (A). That is, in this equation, $\bar{\epsilon}^p$ means the total length of the plastic strain path during deformation. The newly introduced material constant ρ governs the rate of vertex evolution. As easily guessed, the larger ρ is, the faster the vertex evolves. Usually this constant is assumed to be nonnegative and less than unity.

$$0 \leq \rho < 1 \quad (18)$$

If ρ is identically equal to 0, no vertex exists on the yield surface and J2G reduces to J2F. Namely, J2G involves J2F as its special case.

[Note that the classical Tresca's yield surface also possesses vertices on it. However, its vertices exist initially and currently at only fixed stress states, not at the arbitrary current stress state, with fixed vertex angles. The plastic constitutive equation due to Tresca's yield criterion never takes the form of Eq. (17) above. Therefore, J2G differs essentially from Tresca's theory. The vertex model, such as J2G, was introduced from the analogy to polycrystalline plasticity, whose yield surface is usually vertexed at the current stress state.]

The main role of such a vertex on the yield surface is to give some flexibility to the direction of strain rate. If the normality rule is applied like a conventional plastic potential theory such as J2F, always with a smoothed yield surface, then the direction of plastic strain rate is definite to be parallel to the normal vector of the yield surface at the current stress state. It depends only on the direction of the current stress and not on that of the subsequent stress rate. On the other hand, regarding J2G, the strain rate is dependent not only on the current stress but also on the subsequent stress rate. In this sense, the response of strain rate and thus plastic deformation increment

Fig. 3 Vertexed yield surface used in J2G

to the subsequent load increment is stiffer in the conventional plastic potential theory than in the J2G constitutive equation. This fact reflects the material behavior, especially when strain localization takes place. A vertex-hardening model such as J2G is more advantageous for the expression of localized plastic deformation than J2F. This is why the J2G can reasonably predict the limiting strain of sheet metals.

The limiting strain to localized necking predicted by the J2G together with Stören & Rice's theory is dependent on the ρ -value and the instantaneous hardening rate h_0 (=the slope of the stress-strain curve). Its dependency on the n -value is included in that on h_0 , remembering that the n -th power law is an approximation of the stress-strain relationship and employed here only for convenience of description.

Considering again only simple or straight strain paths with the constant in-plane strain ratio α , and following the similar procedure of algebraic manipulation for reduction of the critical condition to that in the case of no out-of-plane stress^{(5),(6)}, the following fourth-order algebraic equation governing the existence of a localized neck is derived for the case where out-of-plane stress σ_3 is involved. In doing this, the virtual discontinuity in strain rate gradients along a localized neck is presumed and, introducing the J2G constitutive equation in the form with σ_3 , the balance of force across the band is taken into consideration. Then the primary governing equation is obtained. In the following equations, g_1 and g_2 are the coordinate components of the unit normal vector to the presumed localized necking band (see Fig. 1). Note that m' denotes (σ_3/σ_1) , unlike that in the previous section.

$$Ag_1^4 + Cg_1g_2 + Eg_2^4 = 0, \quad (19)$$

$$\left. \begin{aligned} A &= (a_{11} - \sigma_1)(a_{33} - \bar{\tau}), \\ E &= (a_{22} - \sigma_2)(a_{33} + \bar{\tau}), \\ C &= (a_{11} - \sigma_1)(a_{22} + \sigma_2)(a_{33}^2 - \bar{\tau}^2) \\ &\quad - (a_{12} + a_{33} - \bar{\sigma})(a_{21} + a_{33} - \bar{\sigma}), \end{aligned} \right\} \quad (20)$$

$$\bar{\sigma} = (1/2)(\sigma_1 + \sigma_2),$$

$$\bar{\tau} = (1/2)(\sigma_1 - \sigma_2),$$

$$a_{11} = (1/3D^*)[(1/H^*) + (b/4h_0M^2) \times \{(2m-1-m')^2 + am'(4m+1-5m')\}],$$

$$a_{22} = (1/3D^*)[(1/H^*) + (b/4h_0M^2) \times \{(2-m-m')^2 + am'(m+4-5m')\}],$$

$$a_{12} = (1/6D^*)[(1/H^*) - (b/2h_0M^2) \times \{(2m-1-m'-am') \times (2-m-m')\}],$$

$$a_{21} = (1/6D^*)[(1/H^*) - (b/2h_0M^2) \times \{(2-m+m'-am') \times (2m-1-m')\}],$$

$$a_{33} = 1/[(1/H^*) + (ab/2h_0M^2)m'(1+m-2m')],$$

$$m' = \sigma_3/\sigma_1,$$

$$1/H^* = (1/G) + (1/H_0),$$

$$M^2 = 1 - m + m^2 + m'(m' - m - 1),$$

$$D^* = (1/12H^*)\{(1/G) + (1/h_0)\}$$

$$\begin{aligned}
& + (h_1^*/M^2)m'(1+m-2m') \times (1/2)\{(1/G) \\
& + (a/H_0)\} + (2h_2^*/9) \\
& + (2h_1^*/3)\{4 + (m'/M^2)(1+m-2m')\}, \\
h_1^* & = (b/4H_0), \\
h_2^* & = (3b^2/4h_0).
\end{aligned}$$

The sufficient condition for occurrence of localized necking is that Eq. (19) above possesses at least one real root regarding the combination of g_1 and g_2 , because, in this state, a localized neck can exist along the direction in which the unit normal $\bar{n}$ lies, whose coordinate components are g_1 and g_2 . Of course, Eq. (19) generally possesses no real root at an earlier stage of deformation. After some stable plastic deformation, Eq. (19) will capture at least one real root, at which localized necking will begin to evolve, which means that fracture will be induced immediately. More precisely, when Eq. (19) yields real root, then more than one mode of velocity field for the boundary condition depicted in Fig.1 can exist (one is uniform deformation and the other may be localized necking), which means that a kind of bifurcation of solution to the boundary-value problem takes place. In this sense, localized necking is a kind of bifurcation phenomenon, particularly in a localized mode.

Two cases may be distinguished with respect to Eq. (19).

(i) If $A=0$ is met first, then a localized neck will appear along the 2-axis (i.e., with $\bar{n}$ parallel to 1-axis).

(ii) If $C=-2\sqrt{AE}$ is met first, while $A>0$ and $E>0$, then a localized neck will appear along the direction inclined both to 1- and 2-axes with the angle ϕ (see Fig. 1) defined by

$$\tan\phi = g_2/g_1 = (A/E)^{1/4}.$$

Note that $E>0$ is always satisfied by virtue of the condition $|a|\leq 1$ or $|m|\leq 1$.

Using the equations in the statements of (i) and (ii) above, the so-called FLDs (the forming limit strain diagrams) for various simple (proportional) strainings can be readily drawn for various n -values and various values of the out-of-plane stress $\sigma_3 (= -p)$. The new material constant ρ is also varied.

Several examples of FLDs thus calculated are illustrated below, in which G and c are set at 80 GPa and 20 MPa, respectively, and kept constant.

Figure 4 shows the FLD nets in the absence of out-of-plane stress ($p=0$). Figure 4(a) is for the constant ρ of 0.8, varying the n -value from 0.0 to 0.5 with interval of 0.05. Figure 4(b) is for the constant n -value of 0.05, varying the ρ -value from 0.05 to 1.0 with interval of 0.05. The same variations of n and ρ are adopted in all the following FLD nets.

Figure 5 shows the FLD nets for the case of $p/c =$

0.1. Figure 5(a) is for the constant ρ of 0.8, and (b) is for the constant n -value of 0.05. Comparing Fig. 5 with Fig. 4, it is ascertained that the FLD nets are

(a) $\rho=0.8$

(b) $n=0.05$

Fig. 4 FLD nets for $p=0$

(a) $\rho=0.8$

(b) $n=0.05$

Fig. 5 FLD nets for $p/c=0.1$

raised by the presence of nonzero p , though p is not very large in Fig. 5. Moreover, in Fig. 5, the limiting strains on the ordinate (i.e., for plane strain state of $\alpha = 0$) are greater than the corresponding n -values, as mentioned on the previous section.

Figure 6 shows the FLD nets for the case of $p/c = 0.5$. Figure 6(a) is for the constant ρ of 0.8, and (b) is for the constant n -value of 0.05. Comparing Fig. 6 with Fig. 4 and Fig. 5, it is ascertained that the effect of p gets evidently more remarkable for larger values of p . Especially, comparing Fig. 6(a) with Fig. 4(a), it is again confirmed that the effect of p decreases with α getting closer to -1 , as commented in the previous section using the classical theory.

4. Discussions by Use of Finite-Element Analysis

To study the effect of the out-of-plane stress $\sigma_3 (= -p)$ on the forming limit strain, simple tension and plane strain tension of a sheet under action of p are simulated numerically by the use of the elastic-plastic finite-element method (E.-P. FEM).

The program code used here is GOLDA, which was developed previously by one of the authors (Gotoh)⁽¹³⁾ and can analyze any large elastic-plastic deformation. It was already used successfully for several such analyses^{(11),(13),(16)-(22)} after checking of its

accuracy by a benchmark test. This code employs the so-called quasi-static explicit method (i.e., an extension of the r-min method by Yamada et al.⁽²³⁾) using the linear crossed-triangles elements in which the maximum increment of the equivalent plastic strain in every element is controlled within 0.002 throughout the deformation history.

The material constants employed here are as follows:

Young's modulus $E = 200$ GPa, Poisson's ratio $\nu = 0.29$, initial yield stress $\sigma_0 = 400$ MPa, $C = 546$ MPa, $n = 0.05$.

The plasticity constitutive equation employed here is J2F for simple tension, and J2G for plane strain tension with ρ of 0.8. In simple tension, the localized necking phenomenon can be simulated even by J2F. However, the strain localization, i.e., the shear-band-type bifurcation in the plane strain state, cannot be simulated by J2F, which is why J2G is employed for plane strain tension. For symmetry, only a quarter of the testpiece is analyzed.

In Fig. 7(a), the patterns of deformed shape in simple tension at the extension ratio of 10% are illustrated for $p = 0$ and $p/\sigma_0 = 0.4$. In these figures, the dark portion with small circles is in the plastic state. Therefore, a localized neck band is clearly formed in the case of $p = 0$, whereas it is rather obscured and spread to some extent in the case of $p/\sigma_0 = 0.4$, which means that localization of strain is delayed by the presence of p .

In Fig. 7(b), tensile load variations with deformation are illustrated for various p/σ_0 . The arrows point to the state at which localized necking begins and thus the load drops rapidly. These points can be considered the limiting states for the corresponding p/σ_0 . Therefore, from this figure, it is evident that the limiting extension is larger for a larger value of p/σ_0 , which is the same result as that obtained in the two previous sections. Moreover, Fig. 7(b) suggests that tensile load is more remarkably lowered by the application of greater out-of-plane compressive stress p , and that the peak point of tensile load is also delayed by p . The latter agrees with the result obtained in section 2.2, because diffuse necking is considered as a bifurcation corresponding to the maximum load point.

The plane strain tension is analyzed using an element subdivision similar to that in Fig. 7(a), constraining deformation in the out-of-plane direction. In Fig. 8, tensile load variations with deformation are illustrated for various p/σ_0 in the case of plane strain tension. The initiation points of shear-band-type bifurcation in this figure confirm again that the presence of greater p further delays the limiting state. These arrowed points are the points at which tensile load decreases in a stepwise manner, and have been

Fig. 6 FLD nets for $p/c = 0.5$

(a) Comparison of the deformation state at the extension ratio 0.1

(b) Comparison with the extension ratio-tensile load (P : tensile load, A_0 : initial cross-sectional area, σ_0 : initial yield stress, U : edge displacement, L : initial half length of specimen)

Fig. 7 Effect of the out-of-plane stress in the uniaxial tension of sheet metals

Fig. 8 Effect of the out-of-plane stress in the plane strain tension of sheet metals (cf. Fig. 7(b))

verified to correspond to the onset of shear band seen in the deformed patterns in the numerical simulation (see, e.g., Ref.(18)). The tensile load is assured to be lowered more by larger p .

The maximum point of tensile load is also

assured to be delayed more by larger p . All these results are coincident with those in the case of simple tension.

5. Conclusions

The effect of out-of-plane stress p on the forming limit strain of sheets under in-plane biaxial loading is studied from three points of view, i.e., the classical Swift's or Hill's theory, the authors' idea together with Stören & Rice's theory, and a numerical simulation of simple and plane-strain tension of sheets by the elastic-plastic FEM. The following conclusions are derived.

(1) All results obtained here ascertain that greater compressive stress p in the out-of-plane direction further increases the forming limit strain, in which the localized necking is employed as the fracture condition.

(2) This effect is found to decrease with α getting closer to -1, where α is the ratio of in-plane strain components.

(3) The conventional statement that the limiting strain for plane strain state ($\alpha=0$) is equal to the n -value does not hold in the presence of p , because it becomes greater for larger p .

(4) Diffuse necking is also confirmed to be delayed by such out-of-plane stressing.

(5) In simple and plane-strain tension, tensile load decreases with increasing p .

(6) The peak point of tensile load is also delayed more by larger p .

Finally, it is emphasized that the effect of out-of-plane loading clarified here gives us a clue to avoid earlier fracture of sheets in actual press-forming operations.

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