Integrated FEM Formulation for Total/Updated-Lagrangian Method in Geometrically Nonlinear Problems*

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This paper presents a new integrated FEM formulation for geometrically nonlinear analyses. There have been two methods to solve the so-called large displacement problem, i.e., the total-Lagrangian method and the updated-Lagrangian method, and two types of FEM programming have been conventionally written. It is shown in the present paper that these two types of programming can be combined by the use of the covariant components of the incremental Green-Lagrange strain tensor and the contravariant components of the second Piola-Kirchhoff stress tensor in the convected coordinate system. The difference is only seen in the transformation of the constitutive tensorial components. The stiffness matrices for the solid and shell elements by this formulation are illustrated in detail, and advantages and disadvantages of the proposed method are also discussed.

Key Words: Finite Element Method, Finite Deformation Theory, Computational Mechanics, Convected Coordinate System, Tensor Analysis

1. Introduction

At present, the total-Lagrangian (T.L.) or the updated-Lagrangian (U.L.) method is generally used for the formulation of the finite element method (FEM) in geometrically nonlinear problems. The difference between these methods is based on the time of reference configuration for the stress and strain tensors in the virtual work principle; that is, the T.L. method refers to time 0 and the U.L. method to time t . All kinds of geometrically nonlinear problems can be formulated by both methods, and the tangent stiffness matrices and the numerical solutions obtained by these methods should be identical.

Practically, the choice of using either the T.L. or the U.L. method depends on the (time of) reference configuration used in the constitutive law. For

instance, the U.L. method is often used for metal, as the constitutive law is composed of the stress and strain rate tensors which refer to the current configuration. On the other hand, for hyper elastic material such as the Mooney-Rivlin model, which refers to the deformation gradient from time 0, the T.L. method is used.

From the viewpoint of general purposes, it is desired that either of the methods be available in the FEM code so that it can be selected in correspondence with the constitutive law. By the conventional method, however, which utilizes the tensorial components of stress and strain in the orthogonal Cartesian coordinate system, the derivation of the tangent stiffness matrix differs greatly between the T.L. method and the U.L. method⁽¹⁾, and two methods of programming are required. Furthermore, it is difficult to see at once the identity of the tangent stiffness matrices derived by both methods in the same analysis.

On the other hand, in the case of the formulation using the tensorial components in the convected coordinate system, the deformation of structure can be simply defined by only the base vectors in the

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convected coordinate system⁽²⁾. However, it is rather complicated to handle, because the base vectors themselves vary spatially and are not always normal to each other generally. Therefore, this method is seldom used for the general-purpose FEM formulation⁽³⁾.

The objective of this paper is to develop the integrated FEM formulation using tensorial components in the convected coordinate system, which can be applied to both the T.L. and the U.L. methods by the appropriate transformation of the constitutive equation.

2. Virtual Work Principle

In this chapter, the virtual work equation based on the T.L. and U.L. methods using the tensorial components in the convected coordinate system is shown. The nomenclature of the symbols used in this paper is shown as follows.

- ${}^0\mathbf{S}, {}^t\mathbf{S}$ 2nd Piola-Kirchhoff stress tensor at time t' , of which reference configuration is time 0 and t
- ${}^0\dot{\mathbf{S}}, {}^t\dot{\mathbf{S}}$ 2nd Piola-Kirchhoff stress rate tensor and Truesdell stress rate tensor at time t
- ${}^t\mathbf{u}$ displacement vector at time t
- $\mathbf{u}$ incremental displacement vector from time 0 to t'
- ${}^0\mathbf{E}, {}^t\mathbf{E}$ Green-Lagrange strain tensor at time t' , of which reference configuration is time 0 and t
- ${}^0\dot{\mathbf{E}}, {}^t\dot{\mathbf{E}}$ Green-Lagrange strain rate tensor and deformation rate tensor at time t
- ${}^0\mathbf{C}, {}^t\mathbf{C}$ fourth-order constitutive tensor assumed between ${}^0\dot{\mathbf{S}}$ and ${}^0\dot{\mathbf{E}}$, and ${}^t\dot{\mathbf{S}}$ and ${}^t\dot{\mathbf{E}}$
- $\mathbf{F}$ deformation gradient tensor between time 0 to t
- $\mathbf{G}_i, {}^t\mathbf{g}_i$ covariant base vector in convected coordinate system at time 0 and t
- $\mathbf{G}_i = \frac{\partial \mathbf{X}}{\partial r^i}, {}^t\mathbf{g}_i = \frac{\partial {}^t\mathbf{x}}{\partial r^i}$
- $\mathbf{G}^i, {}^t\mathbf{g}^i$ contravariant base vector in convected coordinate system at time 0 and t
- $\mathbf{G}^i = \frac{\mathbf{G}_j \times \mathbf{G}_k}{\mathbf{G}_i \cdot (\mathbf{G}_j \times \mathbf{G}_k)}, {}^t\mathbf{g}^i = \frac{{}^t\mathbf{g}_j \times {}^t\mathbf{g}_k}{{}^t\mathbf{g}_i \cdot ({}^t\mathbf{g}_j \times {}^t\mathbf{g}_k)}$
where $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$
- $\mathbf{X}, {}^t\mathbf{x}$ positional vector of material point at time 0 and t
- r^i component of i -direction in convected coordinate system
- $V, {}^t v$ material volume at time 0 and t
- J volume ratio $\left(= \det \mathbf{F} = \frac{d {}^t v}{d V} \right)$

2.1 Incremental decomposition of virtual work equation

The virtual work equations at time $t' (= t + \Delta t)$ in

the T.L. and the U.L. methods can be written as follows.

$$\int_V {}^0\mathbf{S} : \delta {}^0\mathbf{E} dV = {}^t\delta R \quad (1)$$

$$\int_V {}^t\mathbf{S} : \delta {}^t\mathbf{E} d {}^t v = {}^t\delta R \quad (2)$$

Here δ shows the variation due to the virtual displacement and ${}^t\delta R$, the virtual work by the external force.

The Green-Lagrange strain tensor is decomposed by the contravariant base vectors $\mathbf{G}^i$ and ${}^t\mathbf{g}^i$ in the convected coordinate system as

$${}^0\mathbf{E} = \frac{1}{2} ({}^0\mathbf{g}_i \cdot {}^0\mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j) \mathbf{G}^i \otimes \mathbf{G}^j, \quad (3)$$

$${}^t\mathbf{E} = \frac{1}{2} ({}^t\mathbf{g}_i \cdot {}^t\mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j) \mathbf{G}^i \otimes \mathbf{G}^j, \quad (4)$$

and

$${}^t\mathbf{E} = \frac{1}{2} ({}^t\mathbf{g}_i \cdot {}^t\mathbf{g}_j - {}^t\mathbf{g}_i \cdot {}^t\mathbf{g}_j) {}^t\mathbf{g}^i \otimes {}^t\mathbf{g}^j. \quad (5)$$

From Eqs. (3), (4) and (5), the incremental Green-Lagrange strains ${}^0\mathbf{E}$ and ${}^t\mathbf{E}$ are obtained as Eqs. (6) and (7).

$$\begin{aligned} {}^0\mathbf{E} &= {}^0E_{ij} \mathbf{G}^i \otimes \mathbf{G}^j = {}^0\mathbf{E} - {}^t\mathbf{E} \\ &= \frac{1}{2} ({}^0\mathbf{g}_i \cdot {}^0\mathbf{g}_j - {}^t\mathbf{g}_i \cdot {}^t\mathbf{g}_j) \mathbf{G}^i \otimes \mathbf{G}^j \end{aligned} \quad (6)$$

$$\begin{aligned} {}^t\mathbf{E} &= {}^tE_{ij} {}^t\mathbf{g}^i \otimes {}^t\mathbf{g}^j = {}^t\mathbf{E} - {}^0\mathbf{E} \\ &= \frac{1}{2} ({}^t\mathbf{g}_i \cdot {}^t\mathbf{g}_j - {}^0\mathbf{g}_i \cdot {}^0\mathbf{g}_j) {}^t\mathbf{g}^i \otimes {}^t\mathbf{g}^j \end{aligned} \quad (7)$$

Equations (6) and (7) indicate that ${}^0\mathbf{E}$ and ${}^t\mathbf{E}$ have the same components if they are decomposed by the proper base vectors, although they are different kinds of tensors. On the other hand, ${}^t\mathbf{g}_i$ at time t' can be written using ${}^t\mathbf{g}_i$ and the incremental displacement vector $\mathbf{u}$ from time t to t' , as

$${}^t\mathbf{g}_i = {}^t\mathbf{g}_i + \frac{\partial \mathbf{u}}{\partial r^i}. \quad (8)$$

Substituting Eq. (8) into the far right-hand side of Eqs. (6) and (7), the following equation is obtained.

$${}^0E_{ij} = {}^tE_{ij} = e_{ij} + \eta_{ij} \quad (9)$$

where

$$e_{ij} = \frac{1}{2} \left({}^t\mathbf{g}_i \cdot \frac{\partial \mathbf{u}}{\partial r^j} + {}^t\mathbf{g}_j \cdot \frac{\partial \mathbf{u}}{\partial r^i} \right), \quad \eta_{ij} = \frac{1}{2} \frac{\partial \mathbf{u}}{\partial r^i} \cdot \frac{\partial \mathbf{u}}{\partial r^j}. \quad (10)$$

Taking the variation of Eq. (9), we have

$$\delta {}^0E_{ij} = \delta {}^tE_{ij} = \delta e_{ij} + \delta \eta_{ij}. \quad (11)$$

where

$$\begin{aligned} \delta e_{ij} &= \frac{1}{2} \left({}^t\mathbf{g}_i \cdot \frac{\partial \delta \mathbf{u}}{\partial r^j} + {}^t\mathbf{g}_j \cdot \frac{\partial \delta \mathbf{u}}{\partial r^i} \right), \\ \delta \eta_{ij} &= \frac{1}{2} \left(\frac{\partial \delta \mathbf{u}}{\partial r^i} \cdot \frac{\partial \mathbf{u}}{\partial r^j} + \frac{\partial \mathbf{u}}{\partial r^i} \cdot \frac{\partial \delta \mathbf{u}}{\partial r^j} \right). \end{aligned} \quad (12)$$

Assuming that the state at time t is known, the variations of ${}^0\mathbf{E}$ and ${}^t\mathbf{E}$ can be regarded as those of ${}^0\mathbf{E}$ and ${}^t\mathbf{E}$, respectively.

$$\begin{aligned} \delta {}^0\mathbf{E} &= \delta {}^0E_{ij} \mathbf{G}^i \otimes \mathbf{G}^j = \delta {}^0\mathbf{E} \\ &= (\delta e_{ij} + \delta \eta_{ij}) \mathbf{G}^i \otimes \mathbf{G}^j \end{aligned} \quad (13)$$

$$\begin{aligned} \delta {}^t\mathbf{E} &= \delta {}^tE_{ij} {}^t\mathbf{g}^i \otimes {}^t\mathbf{g}^j = \delta {}^t\mathbf{E} \\ &= (\delta e_{ij} + \delta \eta_{ij}) {}^t\mathbf{g}^i \otimes {}^t\mathbf{g}^j \end{aligned} \quad (14)$$

The 2nd Piola - Kirchhoff stress tensor is decomposed by the covariant base vector $\mathbf{G}_i$ and ${}^t\mathbf{g}_i$ in the convected coordinate system, as Eqs.(15) and (16).

$${}^t\mathbf{S} = {}^t\mathbf{S} + {}_0\mathbf{S} = ({}^tS^{ij} + {}_0S^{ij})\mathbf{G}_i \otimes \mathbf{G}_j \quad (15)$$

$${}^t\mathbf{S} = {}^t\mathbf{T} + {}_t\mathbf{S} = ({}^tT^{ij} + {}_tS^{ij}){}^t\mathbf{g}_i \otimes {}^t\mathbf{g}_j \quad (16)$$

where ${}^t\mathbf{T}$ ($= {}^t\mathbf{S}$) is the Cauchy stress tensor at time t . Substituting Eqs.(13)~(16) into Eqs.(1) and (2), the following incremental decomposition forms of the virtual work can be obtained.

$$\int_V {}_0S^{ij}(\delta e_{ij} + \delta \eta_{ij})dV + \int_V {}^tS^{ij}\delta \eta_{ij}dV \\ = {}^t\delta R - \int_V {}^tS^{ij}\delta e_{ij}dV \quad (17)$$

$$\int_{tv} {}^tS^{ij}(\delta e_{ij} + \delta \eta_{ij})d^t v + \int_{tv} {}^tT^{ij}\delta \eta_{ij}d^t v \\ = {}^t\delta R - \int_{tv} {}^tT^{ij}\delta e_{ij}d^t v \quad (18)$$

2.2 Derivation of tangent stiffness matrix

Dividing the left-hand sides of Eqs.(17) and (18) by Δt and setting $\Delta t \rightarrow 0$, we can linearize them and have the tangent stiffness for solving the nonlinear equation. Taking account of ${}_0S^{ij}/\Delta t \rightarrow {}^t\dot{S}^{ij}$, ${}_tS^{ij}/\Delta t \rightarrow {}^t\dot{S}^{ij}$ and $\mathbf{u}/\Delta t \rightarrow {}^t\dot{\mathbf{u}}$, we have Eqs. (19) and (20) from the left-hand side of Eqs.(17) and (18).

the left-hand side of Eq.(17)

$$\rightarrow \int_V {}^t\dot{S}^{ij}\delta e_{ij}dV + \int_V {}^t\dot{S}^{ij}(\delta \eta_{ij})\cdot dV \quad (19)$$

the left-hand side of Eq.(18)

$$\rightarrow \int_{tv} {}^t\dot{S}^{ij}\delta e_{ij}d^t v + \int_{tv} {}^tT^{ij}(\delta \eta_{ij})\cdot d^t V \quad (20)$$

where

$$(\delta \eta_{ij})\cdot = \frac{1}{2} \left(\frac{\partial \delta \mathbf{u}}{\partial r^i} \cdot \frac{\partial {}^t\dot{\mathbf{u}}}{\partial r^j} + \frac{\partial {}^t\dot{\mathbf{u}}}{\partial r^i} \cdot \frac{\partial \delta \mathbf{u}}{\partial r^j} \right). \quad (21)$$

In this paper, the following constitutive equations between the stress rate tensor and the strain rate tensor are assumed.

$${}^t\dot{\mathbf{S}} = {}_0\mathbf{C} : {}^t\dot{\mathbf{E}} \quad (22)$$

$${}^t\dot{\mathbf{S}} = {}_t\mathbf{C} : {}^t\dot{\mathbf{E}} \quad (23)$$

$${}^t\dot{\mathbf{E}} = \dot{e}_{ij}\mathbf{G}^i \otimes \mathbf{G}^j \quad (24)$$

$${}^t\dot{\mathbf{E}} = \dot{e}_{ij}{}^t\mathbf{g}^i \otimes {}^t\mathbf{g}^j \quad (25)$$

where

$$\dot{e}_{ij} = \frac{1}{2} \left(\frac{\partial {}^t\dot{\mathbf{u}}}{\partial r^i} \cdot {}^t\mathbf{g}_j + {}^t\mathbf{g}_i \cdot \frac{\partial {}^t\dot{\mathbf{u}}}{\partial r^j} \right). \quad (26)$$

By decomposing ${}_0\mathbf{C}$ and ${}_t\mathbf{C}$ as shown in Eqs. (27) and (28), we have Eqs.(29) and (30).

$${}_0\mathbf{C} = {}_0C^{ijkl}\mathbf{G}_i \otimes \mathbf{G}_j \otimes \mathbf{G}_k \otimes \mathbf{G}_l \quad (27)$$

$${}_t\mathbf{C} = {}_tC^{ijkl}{}^t\mathbf{g}_i \otimes {}^t\mathbf{g}_j \otimes {}^t\mathbf{g}_k \otimes {}^t\mathbf{g}_l \quad (28)$$

$${}^t\dot{S}^{ij} = {}_0C^{ijkl}\dot{e}_{kl} \quad (29)$$

$${}^t\dot{S}^{ij} = {}_tC^{ijkl}\dot{e}_{kl} \quad (30)$$

Therefore, Eqs.(19) and (20) can be rewritten as Eqs. (31) and (32).

$$\text{Eq. (19)} = \int_V {}_0C^{ijkl}\dot{e}_{kl}\delta e_{ij}dV \\ + \int_V {}^tS^{ij}(\delta \eta_{ij})\cdot dV \quad (31)$$

$$\text{Eq. (20)} = \int_{tv} {}_tC^{ijkl}\dot{e}_{kl}\delta e_{ij}d^t v \\ + \int_{tv} {}^tT^{ij}(\delta \eta_{ij})\cdot d^t v \quad (32)$$

On the other hand, generally, there exist the following relations.

$${}^t\dot{\mathbf{S}} = \frac{1}{J}\mathbf{F} \cdot {}^t\dot{\mathbf{S}} \cdot \mathbf{F}^T \quad (33)$$

$${}^t\mathbf{T} = \frac{1}{J}\mathbf{F} \cdot {}^t\mathbf{S} \cdot \mathbf{F}^T \quad (34)$$

$$\mathbf{F} = \delta_{ij}{}^t\mathbf{g}_i \otimes \mathbf{G}^j \quad (35)$$

Then we have the following equation.

$${}^t\dot{S}^{ij}{}^t\mathbf{g}_i \otimes {}^t\mathbf{g}_j = \frac{1}{J}\delta_{kl}({}^t\mathbf{g}_k \otimes \mathbf{G}^l) \\ \cdot {}^t\dot{S}^{mn}(\mathbf{G}_m \otimes \mathbf{G}_n)\delta_{pq}(\mathbf{G}^p \otimes {}^t\mathbf{g}_q) \\ = \frac{1}{J}{}^t\dot{S}^{kq}({}^t\mathbf{g}_k \otimes {}^t\mathbf{g}_q) \\ \therefore {}^t\dot{S}^{ij} = \frac{1}{J}{}^t\dot{S}^{ij} \quad (36)$$

In the same manner, we have

$${}^tT^{ij} = \frac{1}{J}{}^tS^{ij}. \quad (37)$$

Furthermore, we have Eq.(38) from Eqs.(29), (30) and (31).

$${}_0C^{ijkl} = J {}_tC^{ijkl} = \frac{d^t v}{dV} {}_tC^{ijkl} \quad (38)$$

Finally, the identity of the tangent stiffness matrices obtained by Eqs.(31) and (32) can easily be seen from Eqs.(37) and (38).

3. Identity of the Total-Lagrangian and the Updated-Lagrangian Method in Programming

3.1 B matrix ($\mathbf{B}_L$, $\mathbf{B}_{NL}$)

According to the conventional notation, the $\mathbf{B}$ matrix shows the transformation matrix from the displacement vector to the strain in the element or to the spatial derivative of it in the finite element discretization. $\mathbf{B}_L$ and $\mathbf{B}_{NL}$ correspond to e_{ij} and η_{ij} , respectively, which are strictly defined in the following section. $\mathbf{B}_L$ is used for the calculation of the internal force vector (the equivalent nodal force vector), as well as the tangent stiffness matrix.

As shown in the previous chapter, if the tensorial components in the convected coordinate system are used in the formulation, there is no distinction of the strain components in the virtual work equation between the T.L. and the U.L. methods. Therefore, the same $\mathbf{B}_L$ and $\mathbf{B}_{NL}$ matrices can be used in either the T.L. or the U.L. method.

By the proposed method, the initial displacement matrix ($\mathbf{B}_{Li}$) is not seen explicitly, which is usually necessary in the conventional formulation which uses the tensorial components in the orthogonal coordinate system. This can be interpreted as follows. Substituting ${}^t\mathbf{g}_i = \mathbf{G}_i + \frac{\partial {}^t\mathbf{u}}{\partial r^i}$ into Eqs.(26) and (12), the rate and

the variation of e_{ij} can be divided into the two terms as

$$\begin{aligned} \dot{e}_{ij} = & \frac{1}{2} \left(\mathbf{G}_i \cdot \frac{\partial^t \dot{\mathbf{u}}}{\partial r^j} + \mathbf{G}_j \cdot \frac{\partial^t \dot{\mathbf{u}}}{\partial r^i} \right) \\ & + \frac{1}{2} \left(\frac{\partial^t \mathbf{u}}{\partial r^i} \cdot \frac{\partial^t \dot{\mathbf{u}}}{\partial r^j} + \frac{\partial^t \mathbf{u}}{\partial r^j} \cdot \frac{\partial^t \dot{\mathbf{u}}}{\partial r^i} \right) \\ \delta e_{ij} = & \frac{1}{2} \left(\mathbf{G}_i \cdot \frac{\partial \delta \mathbf{u}}{\partial r^j} + \mathbf{G}_j \cdot \frac{\partial \delta \mathbf{u}}{\partial r^i} \right) \\ & + \frac{1}{2} \left(\frac{\partial^t \mathbf{u}}{\partial r^i} \cdot \frac{\partial \delta \mathbf{u}}{\partial r^j} + \frac{\partial^t \mathbf{u}}{\partial r^j} \cdot \frac{\partial \delta \mathbf{u}}{\partial r^i} \right). \end{aligned} \quad (39)$$

By FE discretization, the first term yields the $\mathbf{B}_{L0}$ matrix in the small displacement analysis and the second term, $\mathbf{B}_{L1}$ the matrix contributed by the initial displacement vector ${}^t\mathbf{u}$ in the conventional T.L. method.

3.2 Tangent stiffness matrix ($\mathbf{K}$)

As shown in the previous section, the tangent stiffness matrix can be derived from Eqs.(31) and (32). The regional integration is conducted in the convected coordinate system and the usual numerical integration scheme can also be adopted if the natural coordinate system in the isoparametric element is selected as the convected coordinate system. Using the relations of $dV = [\mathbf{G}_1 \mathbf{G}_2 \mathbf{G}_3] dr^1 dr^2 dr^3$ and $d^t v = [{}^t\mathbf{g}_1 {}^t\mathbf{g}_2 {}^t\mathbf{g}_3] dr^1 dr^2 dr^3$, where $[\]$ shows the scalar triple product, the following equations are obtained.

$$\begin{aligned} \text{Eq. (31)} = & \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [\mathbf{G}_1 \mathbf{G}_2 \mathbf{G}_3] {}_0C^{ijkl} \\ & \times \dot{e}_{\kappa i} \delta e_{ij} dr^1 dr^2 dr^3 + \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [\mathbf{G}_1 \mathbf{G}_2 \mathbf{G}_3] \\ & \times {}^tS^{ij} (\delta \eta_{ij}) dr^1 dr^2 dr^3 \end{aligned} \quad (40)$$

$$\begin{aligned} \therefore \mathbf{K} = & \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [\mathbf{G}_1 \mathbf{G}_2 \mathbf{G}_3] (\mathbf{B}_L^T {}_0\bar{\mathbf{D}} \mathbf{B}_L \\ & + \mathbf{B}_{NL}^T \bar{\mathbf{S}} \mathbf{B}_{NL}) dr^1 dr^2 dr^3 \end{aligned} \quad (41)$$

$$\begin{aligned} \text{Eq. (32)} = & \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [{}^t\mathbf{g}_1 {}^t\mathbf{g}_2 {}^t\mathbf{g}_3] {}_tC^{ijkl} \\ & \times \dot{e}_{\kappa i} \delta e_{ij} dr^1 dr^2 dr^3 + \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [{}^t\mathbf{g}_1 {}^t\mathbf{g}_2 {}^t\mathbf{g}_3] \\ & \times {}^tT^{ij} (\delta \eta_{ij}) dr^1 dr^2 dr^3 \end{aligned} \quad (42)$$

$$\begin{aligned} \therefore \mathbf{K} = & \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [{}^t\mathbf{g}_1 {}^t\mathbf{g}_2 {}^t\mathbf{g}_3] (\mathbf{B}_L^T {}_t\bar{\mathbf{D}} \mathbf{B}_L \\ & + \mathbf{B}_{NL}^T \bar{\mathbf{T}} \mathbf{B}_{NL}) dr^1 dr^2 dr^3 \end{aligned} \quad (43)$$

The far right-hand sides of Eqs.(42) and (43) include the $\mathbf{B}$ matrix defined in the previous section. ${}_0\bar{\mathbf{D}}$ and ${}_t\bar{\mathbf{D}}$ show the constitutive matrices and $\bar{\mathbf{S}}$ and $\bar{\mathbf{T}}$ stress matrices. In order to distinguish matrix from tensor, an upper bar is added to the symbol in the equation. From the above, if the component in the convected (natural) coordinate system is used, the tangent stiffness matrices can be calculated in the same way by either the T.L. method or the U.L. method, and the $\mathbf{B}$ matrix can be used in common. In other words, it is possible to regard one program as the one determined by the T.L. method or the U.L. method according to the base vectors selected. The way of selecting the base vectors depends conventionally on the

effectiveness with which we can determine ${}_0C^{ijkl}$ or ${}_tC^{ijkl}$ from the material constitutive law.

4. Discretization and Programming

In this chapter, the programming procedure for the 3-dimensional solid element and the shell element are shown. The conventional procedure is described precisely in the references⁽¹⁾ and omitted here.

4.1 3-dimensional solid element (continuum element)

The procedure for the derivation of the tangent stiffness matrix and the equivalent nodal force vector at time t is shown in this section.

(a) Calculate ${}^t\mathbf{g}_i$, the covariant base vector at time t .

$${}^t\mathbf{g}_i = \frac{\partial^t \mathbf{x}}{\partial r^i} = \frac{\partial N^k}{\partial r^i} {}^t\mathbf{x}^k = \mathbf{G}_i + \frac{\partial^t \mathbf{u}}{\partial r^i} = \mathbf{G}_i + \frac{\partial N^k}{\partial r^i} {}^t\mathbf{u}^k \quad (44)$$

where

$${}^t\mathbf{x} = N^k {}^t\mathbf{x}^k, \quad {}^t\mathbf{u} = N^k {}^t\mathbf{u}^k. \quad (45)$$

N is a shape function. The right superscript k shows the value at the node k and follows the summation convention with regard to all the nodes connected to the element.

(b) Calculate the scalar triple product $[{}^t\mathbf{g}_1 {}^t\mathbf{g}_2 {}^t\mathbf{g}_3]$ and the contravariant base vector ${}^t\mathbf{g}^i$ using ${}^t\mathbf{g}_i$. Also calculate $[\mathbf{G}_1 \mathbf{G}_2 \mathbf{G}_3]$, if necessary.

(c) Construct $\mathbf{B}_L$ matrix using ${}^t\mathbf{g}_i$, where

$$\bar{\mathbf{e}} = \mathbf{B}_L \bar{\mathbf{u}}. \quad (46)$$

$\bar{\mathbf{e}}$ is a vector representation of $\dot{e}_{ij}$, such that $\bar{\mathbf{e}}^T = \{\dot{e}_{11} \dot{e}_{22} \dot{e}_{33} 2\dot{e}_{12} 2\dot{e}_{23} 2\dot{e}_{31}\}$, and $\bar{\mathbf{u}}$ is the nodal displacement rate vector in the element, that is, $\bar{\mathbf{u}}^T = \{\dot{u}_1^1 \dot{u}_2^1 \dot{u}_3^1 \cdots \dot{u}_1^k \dot{u}_2^k \dot{u}_3^k \cdots \dot{u}_1^n \dot{u}_2^n \dot{u}_3^n\}$. The size of $\mathbf{B}_L$ is $6 \times 3n$ (n is the total number of nodes in one element) and can be written as Eq.(47).

$$\mathbf{B}_L = [\mathbf{B}_L^1 \mathbf{B}_L^2 \cdots \mathbf{B}_L^k \cdots \mathbf{B}_L^n] \quad (47)$$

Letting $B_L^k(L, M)$ be the component at the L -th row and the M -th column of the partial matrix $\mathbf{B}_L^k$ and using Eq.(48), which is derived from Eqs.(26) and (45)₂, we have Eq.(49).

$$\dot{e}_{ij} = \frac{1}{2} \left(\frac{\partial N^k}{\partial r^i} \dot{\mathbf{u}}^k \cdot {}^t\mathbf{g}_j + {}^t\mathbf{g}_i \cdot \frac{\partial N^k}{\partial r^j} \dot{\mathbf{u}}^k \right) \quad (48)$$

$$B_L^k(L, M) = ({}^t\mathbf{g}_L)_M \frac{\partial N^k}{\partial r^L} \quad (L, M = 1, 2, 3)$$

$$B_L^k(L, M) = ({}^t\mathbf{g}_{L2})_M \frac{\partial N^k}{\partial r^{L1}} + ({}^t\mathbf{g}_{L1})_M \frac{\partial N^k}{\partial r^{L2}} \quad (L = 4, 5, 6, M = 1, 2, 3) \quad (49)$$

where the values of $(L1, L2)$ are $(2, 1), (3, 2)$ and $(1, 3)$ corresponding to the values of $L, 4, 5$ and 6 , respectively and $(\mathbf{g}_L)_M$ shows the M -th component of the vector $\mathbf{g}_L$ in the orthogonal Cartesian coordinate system which is spatially fixed.

(d) Construct $\mathbf{B}_{NL}$ matrix in the same manner as

(c).

Equation (21) is discretized as follows.

$$(\delta\eta_{ij})^* = \frac{1}{2} \left(\frac{\partial N^k}{\partial r^i} \delta \mathbf{u}^k \cdot \frac{\partial N^m}{\partial r^j} \delta \mathbf{u}^m + \frac{\partial N^k}{\partial r^i} \delta \mathbf{u}^k \cdot \frac{\partial N^m}{\partial r^j} \delta \mathbf{u}^m \right) \quad (50)$$

 $\mathbf{B}_{NL}$ is defined so that Eq.(51) can be satisfied.

$$\bar{\mathbf{d}} = \mathbf{B}_{NL} \bar{\mathbf{u}} \quad (51)$$

where

$$\bar{\mathbf{d}} = \left\{ \frac{\partial \dot{u}_1}{\partial r^1} \frac{\partial \dot{u}_2}{\partial r^1} \frac{\partial \dot{u}_3}{\partial r^1} \frac{\partial \dot{u}_1}{\partial r^2} \frac{\partial \dot{u}_2}{\partial r^2} \frac{\partial \dot{u}_3}{\partial r^2} \frac{\partial \dot{u}_1}{\partial r^3} \frac{\partial \dot{u}_2}{\partial r^3} \frac{\partial \dot{u}_3}{\partial r^3} \right\}.$$

The size of $\mathbf{B}_{NL}$ is $9 \times 3n$ and $\mathbf{B}_{NL}$ can be written as Eq. (52).

$$\mathbf{B}_{NL} = [\mathbf{B}_{NL}^1 \mathbf{B}_{NL}^2 \cdots \mathbf{B}_{NL}^k \cdots \mathbf{B}_{NL}^n] \quad (52)$$

By dividing $\mathbf{B}_{NL}^k$ into the 3 partial matrices whose size is 3×3 , as

$$\mathbf{B}_{NL}^k = [\mathbf{B}_{NL,1}^k \mathbf{B}_{NL,2}^k \mathbf{B}_{NL,3}^k]^T, \quad (53)$$

and letting $\mathbf{B}_{NL,j}^k(L, M)$ be the component at the L -th row and the M -th column of the partial matrix $\mathbf{B}_{NL,j}^k$, we have Eq. (54).

$$\mathbf{B}_{NL,j}^k(L, M) = \frac{\partial N^k}{\partial r^j} \delta_{LM} \quad (J, L, M = 1, 2, 3) \quad (54)$$

(e) Calculate the contravariant components of the constitutive tensor.

Let $\mathbf{e}_i$ be the unit base vector in the orthogonal Cartesian coordinate system and add $\sim$ to the component of the constitutive tensor which is decomposed by $\mathbf{e}_i$.

$$\begin{aligned} {}_0\mathbf{C} &= {}_0\tilde{\mathbf{C}}_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \\ &= {}_0\mathbf{C}^{mnop} \mathbf{G}_m \otimes \mathbf{G}_n \otimes \mathbf{G}_o \otimes \mathbf{G}_p \end{aligned} \quad (55)$$

From Eq.(55), we have

$${}_0\mathbf{C}^{mnop} = {}_0\tilde{\mathbf{C}}_{ijkl} (\mathbf{e}_i \cdot \mathbf{G}^m) (\mathbf{e}_j \cdot \mathbf{G}^n) (\mathbf{e}_k \cdot \mathbf{G}^o) (\mathbf{e}_l \cdot \mathbf{G}^p). \quad (56)$$

In the same way, we also have

$${}_t\mathbf{C}^{mnop} = {}_t\tilde{\mathbf{C}}_{ijkl} (\mathbf{e}_i \cdot {}^t\mathbf{g}^m) (\mathbf{e}_j \cdot {}^t\mathbf{g}^n) (\mathbf{e}_k \cdot {}^t\mathbf{g}^o) (\mathbf{e}_l \cdot {}^t\mathbf{g}^p). \quad (57)$$

The material constant ${}_0\tilde{\mathbf{C}}_{ijkl}$ or ${}_t\tilde{\mathbf{C}}_{ijkl}$ obtained can be transformed into the contravariant component by Eq. (56) or (57).

(f) Construct the tangent stiffness matrix.

The constitutive matrix $\bar{\mathbf{D}}$ is defined as Eq.(58) by the contravariant components of the constitutive tensor in Eqs.(56) or (57).

$$\begin{aligned} \bar{\mathbf{D}} &= [\mathbf{G}_1 \mathbf{G}_2 \mathbf{G}_3] {}_0\bar{\mathbf{D}} = [\mathbf{G}_1 \mathbf{G}_2 \mathbf{G}_3] \\ &\times \begin{pmatrix} {}_0C^{1111} & {}_0C^{1122} & {}_0C^{1133} & {}_0C^{1112} & {}_0C^{1123} & {}_0C^{1131} \\ {}_0C^{2211} & {}_0C^{2222} & {}_0C^{2233} & {}_0C^{2212} & {}_0C^{2223} & {}_0C^{2231} \\ {}_0C^{3311} & {}_0C^{3322} & {}_0C^{3333} & {}_0C^{3312} & {}_0C^{3323} & {}_0C^{3331} \\ {}_0C^{1211} & {}_0C^{1222} & {}_0C^{1233} & {}_0C^{1212} & {}_0C^{1223} & {}_0C^{1231} \\ {}_0C^{2311} & {}_0C^{2322} & {}_0C^{2333} & {}_0C^{2312} & {}_0C^{2323} & {}_0C^{2331} \\ {}_0C^{3111} & {}_0C^{3122} & {}_0C^{3133} & {}_0C^{3112} & {}_0C^{3123} & {}_0C^{3131} \end{pmatrix} \\ &= [{}^t\mathbf{g}_1 {}^t\mathbf{g}_2 {}^t\mathbf{g}_3] {}_t\bar{\mathbf{D}} = [{}^t\mathbf{g}_1 {}^t\mathbf{g}_2 {}^t\mathbf{g}_3] \end{aligned} \quad (58)$$

$$\times \begin{pmatrix} {}^tC^{1111} & {}^tC^{1122} & {}^tC^{1133} & {}^tC^{1112} & {}^tC^{1123} & {}^tC^{1131} \\ {}^tC^{2211} & {}^tC^{2222} & {}^tC^{2233} & {}^tC^{2212} & {}^tC^{2223} & {}^tC^{2231} \\ {}^tC^{3311} & {}^tC^{3322} & {}^tC^{3333} & {}^tC^{3312} & {}^tC^{3323} & {}^tC^{3331} \\ {}^tC^{1211} & {}^tC^{1222} & {}^tC^{1233} & {}^tC^{1212} & {}^tC^{1223} & {}^tC^{1231} \\ {}^tC^{2311} & {}^tC^{2322} & {}^tC^{2333} & {}^tC^{2312} & {}^tC^{2323} & {}^tC^{2331} \\ {}^tC^{3111} & {}^tC^{3122} & {}^tC^{3133} & {}^tC^{3112} & {}^tC^{3123} & {}^tC^{3131} \end{pmatrix} \quad (58)$$

Then, the stress matrix $\bar{\mathbf{S}}$ is defined by the contravariant components of the stress tensor as follows, where $\bar{\mathbf{S}}_{ij}$ is the partial matrix whose size is 3×3 and $\mathbf{I}$ is a unit matrix.

$$\bar{\mathbf{S}} = [\mathbf{G}_1 \mathbf{G}_2 \mathbf{G}_3] \bar{\mathbf{S}} = [{}^t\mathbf{g}_1 {}^t\mathbf{g}_2 {}^t\mathbf{g}_3] \bar{\mathbf{T}} \quad (59)$$

where

$$\bar{\mathbf{S}} = \begin{pmatrix} \bar{\mathbf{S}}_{11} & \bar{\mathbf{S}}_{12} & \bar{\mathbf{S}}_{13} \\ \bar{\mathbf{S}}_{21} & \bar{\mathbf{S}}_{22} & \bar{\mathbf{S}}_{23} \\ \bar{\mathbf{S}}_{31} & \bar{\mathbf{S}}_{32} & \bar{\mathbf{S}}_{33} \end{pmatrix}, \quad (60)$$

$$\bar{\mathbf{S}}_{ij} = [\mathbf{G}_1 \mathbf{G}_2 \mathbf{G}_3] {}^t\mathbf{S}^{ij} \mathbf{I} = [{}^t\mathbf{g}_1 {}^t\mathbf{g}_2 {}^t\mathbf{g}_3] {}^t\mathbf{T}^{ij} \mathbf{I}. \quad (61)$$

Finally, referring to Eqs.(42) and (43), the tangent stiffness matrix can be derived by the following equation.

$$\begin{aligned} \mathbf{K} &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \{ \mathbf{B}_L^T \bar{\mathbf{D}} \mathbf{B}_L + \mathbf{B}_{NL}^T \bar{\mathbf{S}} \mathbf{B}_{NL} \} \\ &\times dr^1 dr^2 dr^3 \end{aligned} \quad (62)$$

(g) Calculate the equivalent nodal force vector.

Letting $\bar{\mathbf{S}}$ be the stress vector as

$$\begin{aligned} \bar{\mathbf{S}} &= [\mathbf{G}_1 \mathbf{G}_2 \mathbf{G}_3] \\ &\times \{ {}^t\mathbf{S}^{11} {}^t\mathbf{S}^{22} {}^t\mathbf{S}^{33} {}^t\mathbf{S}^{12} {}^t\mathbf{S}^{23} {}^t\mathbf{S}^{31} \}^T \\ &= [{}^t\mathbf{g}_1 {}^t\mathbf{g}_2 {}^t\mathbf{g}_3] \\ &\times \{ {}^t\mathbf{T}^{11} {}^t\mathbf{T}^{22} {}^t\mathbf{T}^{33} {}^t\mathbf{T}^{12} {}^t\mathbf{T}^{23} {}^t\mathbf{T}^{31} \}^T, \end{aligned} \quad (63)$$

we can calculate the equivalent nodal force vector according to the following equation.

$$\mathbf{F}_{\text{int}} = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \mathbf{B}_L^T \bar{\mathbf{S}} dr^1 dr^2 dr^3 \quad (64)$$

4.2 Isoparametric shell element (structural element)

The procedure for discretization and programming of the 'Ahmad type'^{(4),(5)} degenerated shell element is exemplified in this section. The major differences from the sold (continuum) element in the previous section are 1) the element has rotational degrees of freedom as well as translation degrees of freedom and 2) the plane stress state is assumed in the local orthogonal coordinate system on the element and the not all of the stress components are considered. In the following part, only the procedure differing from that in the previous section is shown.

(a)' Calculate ${}^t\mathbf{g}_i$, the covariant base vector at time t .As shown in Fig. 1, the positional vector ${}^t\mathbf{x}$ at any point in the element at time t and the incremental displacement vector $\mathbf{u}$ during Δt can be written as Eqs.(65) and (66).

$${}^t\mathbf{x} = N^k {}^t\mathbf{x}^k + \frac{\gamma^3}{2} a N^k {}^t\mathbf{V}_s^k \quad (65)$$

Fig. 1 Isoparametric shell element⁽³⁾

$$\mathbf{u} = N^k \mathbf{u}^k + \frac{r^3}{2} a N^k (-{}^t\mathbf{V}_2^k \dot{\alpha}^k + {}^t\mathbf{V}_1^k \dot{\beta}^k) \quad (66)$$

From Eq. (65), we have

$${}^t\mathbf{g}_i = \frac{\partial N^k}{\partial r^i} {}^t\mathbf{x}^k + \frac{r^3}{2} a \frac{\partial N^k}{\partial r^i} {}^t\mathbf{V}_3^k, \quad (i=1, 2) \quad (67)$$

$${}^t\mathbf{g}_i = \frac{1}{2} a N^k {}^t\mathbf{V}_3^k, \quad (i=3) \quad (68)$$

where N^k shows a shape function and the summation convention is used with regard to k . In Eq. (66), the rotational increment is assumed to be small during Δt .

(c)' Construct $\mathbf{B}_L$ matrix using ${}^t\mathbf{g}_i$.

As the plane stress state is assumed in the shell element, the value of the stress component in the direction of shell thickness becomes 0. Therefore, only the five strain components except for that in the direction of shell thickness are taken into account for the derivation of the tangent stiffness matrix. From Eq. (66), we have

$$\frac{\partial \dot{\mathbf{u}}}{\partial r^i} = \frac{\partial N^k}{\partial r^i} \dot{\mathbf{u}}^k + \frac{r^3}{2} a \frac{\partial N^k}{\partial r^i} (-{}^t\mathbf{V}_2^k \dot{\alpha}^k + {}^t\mathbf{V}_1^k \dot{\beta}^k), \quad (i=1, 2) \quad (69)$$

$$\frac{\partial \dot{\mathbf{u}}}{\partial r^i} = \frac{1}{2} a N^k (-{}^t\mathbf{V}_2^k \dot{\alpha}^k + {}^t\mathbf{V}_1^k \dot{\beta}^k), \quad (i=3) \quad (70)$$

Substituting the above equations, $\bar{\mathbf{e}}^T = \{\dot{\epsilon}_{11} \quad \dot{\epsilon}_{22} \quad 2\dot{\epsilon}_{12} \quad 2\dot{\epsilon}_{23} \quad 2\dot{\epsilon}_{31}\}$ and $\bar{\mathbf{u}}^T = \{\dot{u}_1^1 \quad \dot{u}_2^1 \quad \dot{u}_3^1 \quad \dot{\alpha}^1 \quad \dot{\beta}^1 \dots \dot{u}_1^n \quad \dot{u}_2^n \quad \dot{u}_3^n \quad \dot{\alpha}^n \quad \dot{\beta}^n\}$, into Eq. (26), the $\mathbf{B}_L$ matrix in Eq. (46), whose size is $5 \times 5n$, can be divided into n blocks, and each component of the partial matrix is given as follows.

for $M=1, 2, 3$

$$B_L^k(L, M) = ({}^t\mathbf{g}_L)_M \frac{\partial N^k}{\partial r^L} \quad (L=1, 2)$$

$$B_L^k(3, M) = ({}^t\mathbf{g}_2)_M \frac{\partial N^k}{\partial r^1} + ({}^t\mathbf{g}_1)_M \frac{\partial N^k}{\partial r^2}$$

$$B_L^k(4, M) = ({}^t\mathbf{g}_3)_M \frac{\partial N^k}{\partial r^2}$$

$$B_L^k(5, M) = ({}^t\mathbf{g}_3)_M \frac{\partial N^k}{\partial r^1}$$

for $M=4, 5$

$$B_L^k(L, M) = \frac{a}{2} r^3 \frac{\partial N^k}{\partial r^L} ({}^t\mathbf{g}_L \cdot {}^t\mathbf{V}_M^k) \quad (L=1, 2)$$

$$B_L^k(3, M) = \frac{a}{2} r^3 \frac{\partial N^k}{\partial r^1} ({}^t\mathbf{g}_2 \cdot {}^t\mathbf{V}_M^k)$$

$$+ \frac{a}{2} r^3 \frac{\partial N^k}{\partial r^2} ({}^t\mathbf{g}_1 \cdot {}^t\mathbf{V}_M^k)$$

$$B_L^k(4, M) = \frac{a}{2} N^k ({}^t\mathbf{g}_2 \cdot {}^t\mathbf{V}_M^k)$$

$$B_L^k(5, M) = \frac{a}{2} N^k ({}^t\mathbf{g}_1 \cdot {}^t\mathbf{V}_M^k) \quad (71)$$

In Eq. (71), ${}^t\mathbf{V}_4^k = -{}^t\mathbf{V}_2^k$ and ${}^t\mathbf{V}_5^k = {}^t\mathbf{V}_1^k$ are used for brevity of expression, and the transverse shear strains are assumed to be constant through the thickness.

(d)' Construct $\mathbf{B}_{NL}$ matrix.

The components of the $\mathbf{B}_{NL}$ matrix in Eq. (51) can be written using $\bar{\mathbf{d}}^T = \left\{ \frac{\partial \dot{u}_1}{\partial r^1} \frac{\partial \dot{u}_2}{\partial r^1} \frac{\partial \dot{u}_3}{\partial r^1} \frac{\partial \dot{u}_1}{\partial r^2} \frac{\partial \dot{u}_2}{\partial r^2} \frac{\partial \dot{u}_3}{\partial r^2} \right\}$ and $\bar{\mathbf{u}}^T = \{\dot{u}_1^1 \quad \dot{u}_2^1 \quad \dot{u}_3^1 \quad \dot{\alpha}^1 \quad \dot{\beta}^1 \dots \dot{u}_1^n \quad \dot{u}_2^n \quad \dot{u}_3^n \quad \dot{\alpha}^n \quad \dot{\beta}^n\}$ as Eq. (72).

The size of $\mathbf{B}_{NL}$ is $9 \times 5n$ and $\mathbf{B}_{NL}$ can be divided into 3 partial matrices whose size is 3×5 . Referring to Eqs. (69) and (70), we have

for $M=1, 2, 3$

$$B_{NL,J}^k(L, M) = \frac{\partial N^k}{\partial r^J} \delta_{LM} \quad (J=1, 2)$$

$$B_{NL,J}^k(L, M) = 0 \quad (J=3)$$

for $M=4, 5$

$$B_{NL,J}^k(L, M) = \frac{a}{2} r^3 \frac{\partial N^k}{\partial r^J} ({}^t\mathbf{V}_M^k)_L \quad (J=1, 2)$$

$$B_{NL,J}^k(L, M) = \frac{a}{2} N^k ({}^t\mathbf{V}_M^k)_L \quad (J=3) \quad (72)$$

(e)' Calculate the contravariant components of the constitutive tensor.

In the case of the shell element, the differences from the continuum element, as stated the first part of this section, are i) the plane stress state is assumed in the r^1 and r^2 direction on the shell element and ii) not all of the stress and strain components are used. With respect to i), $\mathbf{e}_i$ in the local orthogonal coordinate system on the shell element can be used in place of $\bar{\mathbf{e}}_i$ in Eqs. (55)~(57). With respect to ii), as the value of the stress component in the thickness direction is assumed to be 0, the related tensorial components have to be omitted from the vector and matrix representation in programming.

(f)' Construct the tangent stiffness matrix.

The constitutive matrix $\bar{\mathbf{D}}$ in the shell element can be written as

$$\bar{\mathbf{D}} = [\mathbf{G}_1 \mathbf{G}_2 \mathbf{G}_3] \\ \times \begin{pmatrix} {}_0C^{1111} & {}_0C^{1122} & {}_0C^{1112} & 0 & 0 \\ {}_0C^{2211} & {}_0C^{2222} & {}_0C^{2212} & 0 & 0 \\ {}_0C^{1211} & {}_0C^{3122} & {}_0C^{1212} & 0 & 0 \\ 0 & 0 & 0 & {}_0C^{2323} & {}_0C^{2331} \\ 0 & 0 & 0 & {}_0C^{3123} & {}_0C^{3131} \end{pmatrix} \\ = [{}^t\mathbf{g}_1 {}^t\mathbf{g}_2 {}^t\mathbf{g}_3]$$

$$\times \begin{pmatrix} {}^tC^{1111} & {}^tC^{1122} & {}^tC^{1112} & 0 & 0 \\ {}^tC^{2211} & {}^tC^{2222} & {}^tC^{2212} & 0 & 0 \\ {}^tC^{1211} & {}^tC^{3122} & {}^tC^{1212} & 0 & 0 \\ 0 & 0 & 0 & {}^tC^{2323} & {}^tC^{2331} \\ 0 & 0 & 0 & {}^tC^{3123} & {}^tC^{3131} \end{pmatrix} \quad (73)$$

Then, the stress matrix $\bar{\Sigma}$ is also obtained using the notation in Eq. (59).

$$\bar{\Sigma} = \begin{pmatrix} \bar{\Sigma}_{11} & \bar{\Sigma}_{12} & \bar{\Sigma}_{13} \\ \bar{\Sigma}_{21} & \bar{\Sigma}_{22} & \bar{\Sigma}_{23} \\ \bar{\Sigma}_{31} & \bar{\Sigma}_{32} & 0 \end{pmatrix} \quad (74)$$

(g)' Calculate the equivalent nodal force vector.

The stress vector $\bar{S}$ is defined as Eq. (75) in the case of the shell element.

$$\begin{aligned} \bar{S} &= [G_1 G_2 G_3] \{ {}^tS^{11} {}^tS^{22} {}^tS^{12} {}^tS^{23} {}^tS^{31} \}^T \\ &= [{}^t g_1 {}^t g_2 {}^t g_3] \{ {}^t T^{11} {}^t T^{22} {}^t T^{12} {}^t T^{23} {}^t T^{31} \}^T \end{aligned} \quad (75)$$

4.3 Comparison with the conventional method

(1) By the present method, the formulation for geometrically nonlinear problems can be unified with no distinction between the T.L. method and the U.L. methods.

(2) It is necessary to construct the initial displacement matrix (B_{L1}) explicitly in the strain-displacement (B) matrix with the conventional T.L. method, but is not with the present method.

(3) In the case of the isoparametric element, under the conventional method, the spatial derivatives with respect to the natural coordinates have to be calculated first and then they are transformed into the derivatives with respect to the orthogonal Cartesian coordinates using the chain rule. Under the present method, however, the spatial derivatives with respect to the natural coordinates can be used directly in the formulation when the natural coordinate system is taken as the convected coordinate system.

(4) In the case of a structural element, such as the shell or the beam element, the constitutive law is usually defined by the components in the local orthogonal coordinate system in the element. Therefore, by the conventional method, the coordinate transformation, from the local to the global system or from the global to the local system, has to be conducted with respect to the stress and strain components. On

the other hand, by the present method, the stress and strain components in the convected coordinate system can be used directly. However, the transformation of the components of the constitutive tensor, as shown in Eq. (56) or (57), generally has to be conducted at every loading step.

(5) As the stress and strain components calculated by the present methods are the contravariant components and the covariant components in the convected coordinate system, respectively, their values depend on the size and the shape of the finite element mesh, and it is rather difficult to understand their values intuitively.

5. Conclusions

(1) It has been shown that the T.L. and the U.L. methods can be combined together by the use of the tensorial components in the convected (natural) coordinate system.

(2) The tangent stiffness matrices for the 3-dimensional solid element and the shell element have been exemplified.

(3) By comparison with the conventional method, the advantages and disadvantages of the proposed method have been discussed.

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