Stress Singularity Near Apex in Three-Phase Bonded Structure*

(Effect of Elastic Property of Intermediate Material on Order of Stress Singularity)

Tadanobu INOUE**, Hideo KOGUCHI**
and Toshio YADA**

In the present paper, the plane problem for three-phase materials formed from three isotropic homogeneous wedges with arbitrary angles is analyzed. In particular, the eigenequation for the three-phase bonded structure analytically derived in the previous paper is used for investigating the effect of the elastic property of the intermediate material on the order of singularity in the stress field near the apex. For the several cases in which wedge angles φ_1 , φ_2 and φ_3 are equal and those in which $\varphi_1 + \varphi_3$ and φ_2 are constant, the order of singularity in the stress field for different material combinations and that for different wedge angles of regions 1 and 3 are studied in detail.

Key Words: Elasticity, Composite Material, Stress Concentration, Three-Phase Materials, Interface, Eigenvalue, Eigenequation, Stress Singularity

1. Introduction

In recent years, dissimilar materials have come to be used in welded structures, and machine and electronic parts, wherein a three-phase bonded structure is formed from three different materials such as Si chip-die pad-resin in ICs and LSIs. Many investigations⁽¹⁾⁻⁽¹⁸⁾ have so far been carried out on the two-phase bonded structure, but studies of three-phase one have been rare. The results obtained from two-phase structure cannot be applied to stress estimation in the three-phase one structure.

In the previous paper⁽¹⁹⁾, an eigenequation for a three-phase bonded structure (see Fig. 1) composed of three isotropic homogeneous wedges with arbitrary angles subjected to surface tractions, was derived using the Airy stress function through the Mellin transform. It was shown that the order of the stress

singularity at the apex, which is obtained from the eigenequation, can be reduced by suitably arranging the bonded order of materials in the three-phase

Fig. 1 Analytical model for three-phase bonded structure under surface tractions

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** Department of Mechanical Engineering, Nagaoka University of Technology, 1603-1 Kamitomioka, Nagaoka, Niigata 940-21, Japan

bonded structure.

In the present paper, we first consider the cases where the third region with wedge angles φ_3 of $\pi/3$, $\pi/2$ and $2\pi/3$ is bonded to the two-phase structure with wedge angles of $(\varphi_1, \varphi_2) = (\pi/3 - \pi/3)$, $(\pi/2 - \pi/2)$ and $(2\pi/3 - 2\pi/3)$. They show the variations of the order of the stress singularity on the α_{23} - β_{23} plane for several combinations of materials. After that, in those bonded structures, i.e., $\pi/3 - \pi/3 - \pi/3$, $\pi/2 - \pi/2 - \pi/2$ and $2\pi/3 - 2\pi/3 - 2\pi/3$, the effect of the elastic property in the intermediate region on the order of singularity is investigated. The relationship between the wedge angles φ_1 and φ_3 in the two adjacent regions to the intermediate region and the order of stress singularity is estimated in several cases where the wedge angle φ_2 in the intermediate region and the sum of the wedge angles φ_1 and φ_3 are fixed (φ_2 and $\varphi_1 + \varphi_3 = \text{constant}$).

2. Derivation of Eigenequation for Three-Phase Bonded Structure

Since the derivation of the eigenequation for the three-phase bonded structure is described in detail in the previous paper⁽¹⁹⁾, only the necessary results used hereafter are exhibited in this section.

The constant m_δ is denoted by Poisson's ratio ν_δ as follows:

$$m_\delta = \begin{cases} 4(1 - \nu_\delta) & \text{for plane strain} \\ \frac{4}{1 + \nu_\delta} & \text{for plane strain} \end{cases} \quad (\delta = 1, 2, 3). \quad (1)$$

The six elastic constants for the bonded structure with three pairs $((G_1, \nu_1), (G_2, \nu_2) \text{ and } (G_3, \nu_3))$ of shear modulus and Poisson's ratio are represented by four Dundurs composite parameters⁽⁴⁾, α_{12} and β_{12} for materials 1 and 2 and α_{23} and β_{23} for materials 2 and 3:

$$\alpha_{12} = \frac{k_{12}m_2 - m_1}{k_{12}m_2 + m_1}, \quad \beta_{12} = \frac{k_{12}(m_2 - 2) - (m_1 - 2)}{k_{12}m_2 + m_1} \quad (2)$$

$$\alpha_{23} = \frac{k_{23}m_3 - m_2}{k_{23}m_3 + m_2}, \quad \beta_{23} = \frac{k_{23}(m_3 - 2) - (m_2 - 2)}{k_{23}m_3 + m_2},$$

where

$$k_{12} = \frac{G_1}{G_2}, \quad k_{23} = \frac{G_2}{G_3}. \quad (3)$$

The subscript of elastic constants represents each region. The complex variable p is defined by

$$p = -s - 1 = \xi + i\eta, \quad (4)$$

where s is the parameter of the Mellin transform, and ξ and η are the real and imaginary parts of p , respectively.

The eigenequation for the three-phase bonded structure is denoted as

$$\begin{aligned} & \mathbf{S}(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p) \\ &= \mathbf{A}(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p) \\ &+ \mathbf{B}(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p) \\ &+ \mathbf{C}(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p) \end{aligned}$$

$$+ \mathbf{D}(\varphi_1, \varphi_2, \varphi_3; p), \quad (5)$$

where

$$\begin{aligned} & \mathbf{A}(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p) \\ &= A_1(\varphi_1, \varphi_2, \varphi_3; p)\beta_{12}^2\beta_{23}^2 \\ &+ A_2(\varphi_1, \varphi_2, \varphi_3; p)\beta_{12}\beta_{23} \\ &+ A_3(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{12}\alpha_{23}\beta_{12}\beta_{23} \\ &+ A_4(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{12}^2\alpha_{23}^2 \\ &+ A_5(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{12}\alpha_{23} \\ & \mathbf{B}(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p) \\ &= \sum_{i=1}^2 \{ B_i(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{2,n}\beta_{2,m}^2\beta_{2,n} \\ &+ B_2(\varphi_1, \varphi_2, \varphi_3; p)\beta_{2,m}^2 \\ &+ B_3(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{2,m}^2\beta_{2,n}^2 \\ &+ B_4(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{2,m}\beta_{2,m} \\ &+ B_5(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{2,m}\beta_{2,n} \\ &+ B_6(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{2,m}^2\alpha_{2,n}\beta_{2,n} \\ &+ B_7(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{2,m}^2 \} \\ & \mathbf{C}(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p) \\ &= 2 \sum_{i=1}^2 (-1)^{i-1} \{ C_i(\varphi_1, \varphi_2, \varphi_3; p)\beta_{2,m}\beta_{2,n}^2 \\ &+ C_2(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{2,m}\beta_{2,n}^2 \\ &+ C_3(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{2,n}\beta_{2,m}\beta_{2,n} \\ &+ C_4(\varphi_1, \varphi_2, \varphi_3; p)\beta_{2,m} \\ &+ C_5(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{2,n}^2\beta_{2,m} \\ &+ C_6(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{2,m}\alpha_{2,n}\beta_{2,n} \\ &+ C_7(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{2,m}\alpha_{2,n}^2 \\ &+ C_8(\varphi_1, \varphi_2, \varphi_3; p)\alpha_{2,m} \} \\ & \mathbf{D}(\varphi_1, \varphi_2, \varphi_3; p) = K[p, (\varphi_1 + \varphi_2 + \varphi_3)] \quad (6) \\ & A_1(\varphi_1, \varphi_2, \varphi_3; p) = 16K(p, \varphi_1)K(p, \varphi_2)K(p, \varphi_3) \\ & A_2(\varphi_1, \varphi_2, \varphi_3; p) = 8\{K(p, \varphi_1)K(p, \varphi_3) \\ &+ 2p^2Q(p, \varphi_1, \varphi_2, \varphi_3)\} \\ & A_3(\varphi_1, \varphi_2, \varphi_3; p) = -A_2 + 32p^4R_1(p, \varphi_1, \varphi_3)\sin^2\varphi_2 \\ & A_4(\varphi_1, \varphi_2, \varphi_3; p) = K[p, (\varphi_1 - \varphi_2 + \varphi_3)] \\ & - 4p^2[U(p, \varphi_1, \varphi_2, \varphi_3) \\ &+ (p^2 - 1)\{\sin\varphi_1\sin 2\varphi_2\sin\varphi_3\sin(\varphi_1 + \varphi_3) \\ &- \sin^2\varphi_2(\sin^2\varphi_1 + \sin^2\varphi_3)\}] \\ & A_5(\varphi_1, \varphi_2, \varphi_3; p) \\ &= 4\{K(p, \varphi_1) - K(p, \varphi_2) + K(p, \varphi_3)\} \\ &- 8\{K(p, \varphi_1)\{\sin^2(p\varphi_2) + p^2\sin^2\varphi_3\} \\ &- K(p, \varphi_2)\{\sin^2(p\varphi_1) + \sin^2(p\varphi_3)\} \\ &+ K(p, \varphi_3)\{\sin^2(p\varphi_1) + \sin^2(p\varphi_2)\}\} \\ &+ 16p^2Q(p, \varphi_1, \varphi_2, \varphi_3) \\ & B_1(\varphi_1, \varphi_2, \varphi_3; p) = 16p^2K(p, \varphi_m)R_1(p, \varphi_2, \varphi_n) \\ & B_2(\varphi_1, \varphi_2, \varphi_3; p) = 4K(p, \varphi_m)K[p, (\varphi_2 + \varphi_n)] \\ & B_3(\varphi_1, \varphi_2, \varphi_3; p) = 4K(p, \varphi_n)\left\{U\left(p, \varphi_m, \varphi_2, \frac{\pi}{2}\right) \right. \\ & \left. + K[p, (\varphi_m - \varphi_2)]\right\} \\ & B_4(\varphi_1, \varphi_2, \varphi_3; p) = 4p^2R_1[p, \varphi_m, (\varphi_2 + \varphi_n)] \\ & B_5(\varphi_1, \varphi_2, \varphi_3; p) = 8p^2\{K(p, \varphi_m)\sin^2\varphi_n \\ &+ R_2(p, \varphi_2, \varphi_n) - 2Q(p, \varphi_m, \varphi_2, \varphi_n)\} \\ & B_6(\varphi_1, \varphi_2, \varphi_3; p) = 4p^2\{U(p, \varphi_m, \varphi_2, \varphi_n) \\ &- 2K(p, \varphi_n)\sin^2\varphi_m + R_1[p, (\varphi_m - \varphi_2), \varphi_n]\} \\ &- 4U\left(p, \varphi_m, \varphi_2, \frac{\pi}{2}\right)K(p, \varphi_n) \end{aligned}$$

$$\begin{aligned}
B_7(\varphi_1, \varphi_2, \varphi_3; p) &= U\left[p, \varphi_m, (\varphi_2 + \varphi_n), \frac{\pi}{2}\right] \\
&\quad + K[p, (\varphi_m - \varphi_2 - \varphi_n)] \\
C_1(\varphi_1, \varphi_2, \varphi_3; p) &= 8p^2 K(p, \varphi_n) T(p, \varphi_m, \varphi_2) \\
C_2(\varphi_1, \varphi_2, \varphi_3; p) &= 4K(p, \varphi_n) \{K(p, \varphi_m) \\
&\quad - K(p, \varphi_2)\} - C_1 \\
C_3(\varphi_1, \varphi_2, \varphi_3; p) &= -4K(p, \varphi_n) \{K(p, \varphi_m) \\
&\quad + 2p^2 \sin^2(p\varphi_m) \sin^2 \varphi_2\} - 8p^4 R_2(p, \varphi_m, \varphi_2) \sin^2 \varphi_n \\
C_4(\varphi_1, \varphi_2, \varphi_3; p) &= -2p^2 R_2[p, \varphi_m, (\varphi_2 + \varphi_n)] \\
C_5(\varphi_1, \varphi_2, \varphi_3; p) &= -2p^2 \{2K(p, \varphi_m) \sin^2 \varphi_n \\
&\quad + R_2[p, \varphi_m, (\varphi_2 - \varphi_n)]\} - 2U(p, p\varphi_m, \varphi_2, \varphi_n) \\
C_6(\varphi_1, \varphi_2, \varphi_3; p) &= 4p^2 \sin^2 \varphi_n \{K(p, \varphi_m) \\
&\quad + 2p^2 R_2(p, \varphi_m, \varphi_2)\} \\
&\quad + 4p^2 \{2 \sin^2 \varphi_2 \sin^2(p\varphi_m) K(p, \varphi_n) \\
&\quad - R_1(p, \varphi_2, \varphi_n)\} \\
C_7(\varphi_1, \varphi_2, \varphi_3; p) &= K(p, \varphi_m) \cos 2[p(\varphi_2 - \varphi_n)] \\
&\quad - \cos 2(p\varphi_m) \left\{ U\left(p, \varphi_2, \varphi_n, \frac{\pi}{2}\right) + K[p, (\varphi_2 - \varphi_n)] \right\} \\
C_8(\varphi_1, \varphi_2, \varphi_3; p) &= K(p, \varphi_m) - K[p, (\varphi_2 + \varphi_n)] - C_4.
\end{aligned} \tag{7}$$

The auxiliary functions $K(p, a)$, $Q(p, a, b, c)$, $R_1(p, a, b)$, $R_2(p, a, b)$, $T(p, a, b)$, $U(p, a, b, c)$ and $m, n, \alpha_{2,1}, \beta_{2,1}, \alpha_{2,3}, \beta_{2,3}$ are defined by

$$\begin{aligned}
K(p, a) &= \sin^2(pa) - p^2 \sin^2(a) \\
Q(p, a, b, c) &= \sin^2(pa) \sin^2(b) \sin^2(pc) \\
&\quad - p^2 \sin^2(a) \sin^2(pb) \sin^2(c) \\
R_1(p, a, b) &= \sin^2(a) K(p, b) + \sin^2(b) K(p, a) \\
R_2(p, a, b) &= \sin^2(a) K(p, b) - \sin^2(b) K(p, a) \\
T(p, a, b) &= \sin^2(pa) \sin^2(b) - \sin^2(a) \sin^2(pb) \\
U(p, a, b, c) &= 4p^2(p^2 - 1) \sin^2(a) \sin^2(b) \sin^2(c)
\end{aligned} \tag{8}$$

$$\begin{aligned}
m &= 2i - 1, \quad n = -2i + 5 \\
\alpha_{12} &= \alpha_{2,1}, \quad \beta_{12} = \beta_{2,1}, \quad \alpha_{23} = \alpha_{2,3}, \quad \beta_{23} = \beta_{2,3}.
\end{aligned} \tag{9}$$

It can be proven that Eq. (5) generally fulfills the following relation.

$$\begin{aligned}
S(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p) \\
= S(\varphi_3, \varphi_2, \varphi_1, -\alpha_{23}, -\beta_{23}, -\alpha_{12}, -\beta_{12}; p)
\end{aligned} \tag{10}$$

The order of stress singularity is given by the real part of the root of $S=0$ existing in $0 < \text{Re}(p) < 1$ (stress σ and root p are related by $\sigma \propto r^{p-1}$). Here, root p satisfying the following relations is determined.

$$\begin{aligned}
S(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p_1) &= 0 \\
\frac{d}{dp} S(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p_1) \Big|_{p_1 \rightarrow 1} &= 0
\end{aligned} \tag{11}$$

The second equation is the condition required of the stress field to possess logarithmic singularity⁽⁶⁾. The root will be represented as p_1 . Furthermore, when the eigenequation has multiple roots in $0 < \text{Re}(p) < 1$, the root having the smallest real part will be chosen as p_1 .

3. Results of Numerical Analysis

3.1 Loci for roots on α_{23} - β_{23} plan with $\varphi_1 = \varphi_2 = \varphi_3 = \text{constant}$

The variations of root p_1 for several material

combinations in the bonded wedge geometry of $\pi/3$ - $\pi/3$ - $\pi/3$, $\pi/2$ - $\pi/2$ - $\pi/2$ and $2\pi/3$ - $2\pi/3$ - $2\pi/3$ are investigated. Elastic constants for each material are within the following ranges.

$$0 < G_1, G_2, G_3 < \infty, \quad 0 \leq \nu_1, \nu_2, \nu_3 \leq 0.5 \tag{12}$$

The values of α_{12} , β_{12} , α_{23} and β_{23} of Dundurs parameters defined in Eq. (2) fall within a parallelogram⁽⁵⁾ with the values of α_{12} , α_{23} between -1 and 1 . Since the domain corresponding to plane stress are included within that to plane strain, the results of plane stress are not shown here.

First, a point on the α_{12} - β_{12} plane determined from the elastic properties of materials 1 and 2 is fixed. The loci for root p_1 are then drawn on the α_{23} - β_{23} plane. That is, root p_1 in the case where material 3 is bonded to the two-phase bonded structure of materials 1 and 2 is examined and the loci for root p_1 are drawn on the α_{23} - β_{23} plane. Here, root determination is carried out on materials 1 and 2 which have $(\alpha_{12}, \beta_{12}) = (0.875, 0.25)$ and $(-0.875, -0.25)$, respectively; these will be referred to as *cases 1* and *2*. Also, the root obtained for the two-phase bonded structure of materials 1 and 2 is drawn as the loci for p'_1 on the α_{12} - β_{12} plane.

Figures 2, 4 and 6 show, respectively, the loci for p'_1 plotted for two-phase bonded structures of $\pi/3$ - $\pi/3$, $\pi/2$ - $\pi/2$ and $2\pi/3$ - $2\pi/3$ on the α_{12} - β_{12} plane; points indicated in the plane as *cases 1* and *2* correspond to the combinations of materials 1 and 2 above. Figure 3 shows the loci for p_1 for the three-phase bonded structure ($\pi/3$ - $\pi/3$ - $\pi/3$), in which material 3 with $\varphi_3 = \pi/3$ is bonded to the $\pi/3$ - $\pi/3$ two-phase one. Figure 5 shows those for ($\pi/2$ - $\pi/2$ - $\pi/2$), where material 3 with $\varphi_3 = \pi/2$ is bonded to the $\pi/2$ - $\pi/2$ two-phase one. Figure 7 shows those for ($2\pi/3$ - $2\pi/3$ - $2\pi/3$), where material 3 with $\varphi_3 = 2\pi/3$ is bonded to the $2\pi/3$ - $2\pi/3$ two-phase one. In Figs. 2~7, $p_1 \rightarrow 1.00$ ($p'_1 \rightarrow 1.00$) noted on the α_{23} - β_{23} (α_{12} - β_{12}) plane represents the loci for roots exhibiting logarithmic singularity. Furthermore, the problem for $2\pi/3$ - $2\pi/3$ - $2\pi/3$ corresponds to that containing a straight crack which intersects the bonded plane at an arbitrary angle.

3.2 Roots for bonded wedge with $\varphi_2, \varphi_1 + \varphi_3 = \text{fixed}$

So far results of analysis in the cases where the three wedge angles φ_1, φ_2 and φ_3 are equal were shown. It is here demonstrated how the order of stress singularity varies with the wedge angle φ_1 under the condition of $(\alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23})$. The bonded wedge has the geometry where $\varphi_2, \varphi_1 + \varphi_3 = \text{fixed}$. In particular, roots for the bonded wedge with the conditions of $(\alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}) = \text{I}) (0.875, 0.25, -0.818, -0.234)$ and $\text{II}) (-0.875, -0.25, 0.818, 0.234)$ are investigated. *Conditions I* is equivalent to the bonded

Fig. 2 Loci for roots $p=p'$ for two-phase bonded structure of $\varphi_1=\varphi_2=\pi/3$

(a)

(b)

Fig. 3 Loci for roots $p=p_1$ of $S(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p)=0$ for $\varphi_1=\varphi_2=\varphi_3=\pi/3$ of (a) case 1, (b) case 2

Fig. 4 Loci for roots $p=p'$ for two-phase bonded structure of $\varphi_1=\varphi_2=\pi/2$

structure with the relation $G_1 < G_3 \ll G_2$, and Conditions II, conversely, to that with the relation $G_1 > G_3 \gg G_2$. That is, the former corresponds to the case where the intermediate material is very soft in comparison to the two adjacent materials, and the latter corresponds to that where it is very rigid.

Roots in three configurations of the bonded structure A) $\varphi_2=60^\circ$ and $\varphi_1+\varphi_3=120^\circ$, B) $\varphi_2=90^\circ$ and $\varphi_1+\varphi_3=180^\circ$ and C) $\varphi_2=120^\circ$ and $\varphi_1+\varphi_3=240^\circ$, are examined in detail for each case. The variations of roots p for the wedge angle φ_1 on each wedge geometry are shown in Figs. 8~10. The horizontal axis

(a)

(b)

Fig. 5 Loci for roots $p=p_1$ of $S(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p)=0$ for $\varphi_1=\varphi_2=\varphi_3=\pi/2$ of (a) case 1, (b) case 2

Fig. 6 Loci for roots $p=p'$ for two-phase bonded structure of $\varphi_1=\varphi_2=2\pi/3$

Fig. 7 Loci for roots $p = p_1$ of $S(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p) = 0$ for $\varphi_1 = \varphi_2 = \varphi_3 = 2\pi/3$ of (a) case 1, (b) case 2

Fig. 8 Variations of roots p of $S(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p) = 0$ existing in $0 < \text{Re}(p) < 1$ with varying φ_1 when bonded wedges consist of A) $\varphi_2 = \pi/3$, $\varphi_1 + \varphi_3 = 2\pi/3$ and elastic constants have the relations I) $(\alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}) = (0.875, 0.25, -0.818, -0.234) - G_1 > G_3 \gg G_2$ and II) $(\alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}) = (-0.875, -0.25, 0.818, 0.234) - G_1 < G_3 \ll G_2$

Fig. 9 Variations of roots p of $S(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p) = 0$ existing in $0 < \text{Re}(p) < 1$ with varying φ_1 when bonded wedges consist of B) $\varphi_2 = \pi/2$, $\varphi_1 + \varphi_3 = \pi$ and elastic constants have the relations of I) $(\alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}) = (0.875, 0.25, -0.818, -0.234) - G_1 > G_3 \gg G_2$ and II) $(\alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}) = (-0.875, -0.25, 0.818, 0.234) - G_1 < G_3 \ll G_2$

Fig. 10 Variations of roots p of $S(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p) = 0$ existing in $0 < \text{Re}(p) < 1$ with varying φ_1 when bonded wedges consist of C) $\varphi_2 = 2\pi/3$, $\varphi_1 + \varphi_3 = 4\pi/3$ and elastic constants have the relations I) $(\alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}) = (0.875, 0.25, -0.818, -0.234) - G_1 > G_3 \gg G_2$ and II) $(\alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}) = (-0.875, -0.25, 0.818, 0.234) - G_1 < G_3 \ll G_2$

represents wedge angle φ_1 and the vertical axis root p . In these figures, the left end ($\varphi_1=0^\circ$) corresponds to the two-phase bonded structure with wedge angles φ_2 and φ_3 , and the right end ($\varphi_3=0^\circ$) that with wedge angles φ_1 and φ_2 . Real and imaginary parts of the complex root $p (=Re(p) + iIm(p))$ are plotted as solid and broken lines, respectively. Furthermore, all the roots p existing in $0 < Re(p) < 1$ are shown in these figures.

4. Discussion

4.1 Loci for roots p_1 on α_{23} - β_{23} plane

In case 1, the existing domains of α_{23} and β_{23} do not form a parallelogram for all combinations of materials 2 and 3, because the parameter $(\alpha_{23}, \beta_{23})$ depends on $(\alpha_{12}, \beta_{12})$. When $(\alpha_{12}, \beta_{12}) = (0.875, 0.25)$, Poisson's ratio ν_2 of material 2 derived from Eq. (2) is within $0.286 \leq \nu_2 \leq 0.318$ ($0.4 \leq \nu_2 \leq 0.433$ for plane stress). Hence, the values of α_{23} and β_{23} fall within a trapezoid, as shown in Figs. 3(a), 5(a) and 7(a).

It can be seen from the relation of Dundurs parameters that the conditions of $(\alpha_{23}, \beta_{23}) = (-0.875, -0.25)$ in case 1 and $(\alpha_{23}, \beta_{23}) = (0.875, 0.25)$ in case 2 mean that the properties of material 1 are the same as those of material 3. A condition of $(\alpha_{23}, \beta_{23}) = (0, 0)$ in each case indicates that properties of materials 2 and 3 are identical. Moreover, it is found from Eq. (10) that the condition of $(\alpha_{23}, \beta_{23}) = (0.875, 0.25)$ in case 1 and that of $(\alpha_{23}, \beta_{23}) = (-0.875, -0.25)$ in case 2 yield the same roots p .

Case 1 corresponds to a two-phase bonded structure with the relations $0 \leq \nu_1 \leq 0.5$, $0.286 \leq \nu_2 \leq 0.318$ and $G_1 \gg G_2$, and Case 2 corresponds to that with $0.286 \leq \nu_1 \leq 0.318$, $0 \leq \nu_2 \leq 0.5$ and $G_1 \ll G_2$. Roots at $\alpha_{23}=1$ ($k_{23} \rightarrow \infty$) yield ones in the fixed-free single wedge⁽¹⁾ with the angle φ_3 . However, roots at $\alpha_{23}=1$ in $\pi/2-\pi/2-\pi/2$ and $2\pi/3-2\pi/3-2\pi/3$ also yield ones in two-phase bonded structure of materials 1 and 2 (see Figs. 5 and 7), because the following relation always holds.

$$S\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \pm 0.875, \pm 0.25, 1, \beta_{23}; 0.7753\right) = 0$$

$$S\left(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \pm 0.875, \pm 0.25, 1, \beta_{23}; 0.6193\right) = 0 \quad (13)$$

For three pairs of the bonded wedge geometry, the order $O(r^{p-1})$ of stress singularity in case 1 is large in comparison with that in case 2. It can especially be seen that root p_1 is larger as $\alpha_{23} \rightarrow 1$ and smaller as $\alpha_{23} \rightarrow -1$ on the α_{23} - β_{23} plane in case 1. That is, this indicates that the order $O(r^{p-1})$ of stress singularity tends to increase when $G_1, G_3 \gg G_2$, and to decrease when $G_1, G_3 \ll G_2$. All roots existing in $0 < Re(p) < 1$ when $G_1 = G_3 \gg G_2$ and $G_1 = G_3 \ll G_2$ for three pairs of the bonded wedge geometry are shown in

Table 1 Roots p of $S(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p) = 0$ existing in $0 < Re(p) < 1$ when properties of materials 1 and 3 are identical

Wedge Angle	Dundurs Parameter	
	$\alpha_{12} = 0.875, \beta_{12} = 0.25$ $\alpha_{23} = -0.875, \beta_{23} = -0.25$ ($G_1 = G_3 \gg G_2$)	$\alpha_{12} = -0.875, \beta_{12} = -0.25$ $\alpha_{23} = 0.875, \beta_{23} = 0.25$ ($G_1 = G_3 \ll G_2$)
$\pi/3-\pi/3-\pi/3$	0.6201, 0.6882, 0.9418	0.9590
$\pi/2-\pi/2-\pi/2$	0.2210, 0.3332	0.6945, 0.8396
$2\pi/3-2\pi/3-2\pi/3$	0.1895, 0.1939, 0.9946	0.6116, 0.6210

Table 1. When the intermediate material is very soft in comparison to the two adjacent materials, it is found that the order $O(r^{p-1})$ of stress singularity is very large.

4.2 Loci for roots for structure with $\varphi_2, \varphi_1 + \varphi_3 = \text{fixed}$

A) $\varphi_2 = \pi/3$ and $\varphi_1 + \varphi_3 = 2\pi/3$ (see Fig. 8) : When the intermediate material is very soft in comparison to the two adjacent materials, i.e., I) $G_1 > G_3 \gg G_2$, root p attains a maximum at $\varphi_1 = 0$ and $2\pi/3$ (two-phase bonded structure) and a minimum at $\varphi_1 \cong 22^\circ$ and 95° . Hence, the order $O(r^{p-1})$ of stress singularity yields a minimum at $\varphi_1 = 0$ and $2\pi/3$ and maximum at $\varphi_1 \cong 22^\circ$ and 95° . Conversely, when the intermediate material is very rigid in comparison to the two adjacent materials, i.e., II) $G_1 < G_3 \ll G_2$, the locus for root p varies gradually and the order $O(r^{p-1})$ is small in comparison with I) $G_1 > G_3 \gg G_2$.

B) $\varphi_2 = \pi/2$ and $\varphi_1 + \varphi_3 = \pi$ (see Fig. 9) : When I) $G_1 > G_3 \gg G_2$, two real roots ($p_1 < p_2$) always appear. The order $O(r^{p-1})$ of both roots yields a minimum at $\varphi_1 = 0, \pi$ (two-phase bonded structure). The order $O(r^{p_1-1})$ of the smallest root p_1 which yields the most dominant factor for the stress field is very large within $\varphi_1 \cong 35 \sim 120^\circ$. The order of root p_2 yields a maximum at $\varphi_1 \cong 85^\circ$. Therefore, the bonded geometry with $\varphi_1 \cong 85^\circ$, $\varphi_2 = 90^\circ$ and $\varphi_3 \cong 95^\circ$ is the worst combination under the condition of $(\alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}) = (0.875, 0.25, -0.818, -0.234)$. When II) $G_1 < G_3 \ll G_2$, two real roots ($p_1 < p_2$) appear within $\varphi_1 \cong 0 \sim 154^\circ$. They combine at $\varphi_1 \cong 155^\circ$ and become a complex root $p (=Re(p) + iIm(p))$. The order $O(r^{p-1})$ yields a maximum at $\varphi_1 = 0$ on the two roots and maximum at $\varphi_1 = \pi$ on the complex root. Also, the order $O(r^{p_1-1})$ of root p_1 yields a minimum at $\varphi_1 \cong 85^\circ$. It is found that the order of stress singularity in II) $G_1 < G_3 \ll G_2$ is small in comparison with that in I) $G_1 > G_3 \gg G_2$; however, it conversely becomes large near $\varphi_1 = 0$ and π .

C) $\varphi_2 = 2\pi/3$ and $\varphi_1 + \varphi_3 = 4\pi/3$ (see Fig. 10) :

This bonded configuration corresponds to that containing a semi-infinite crack intersecting the interface of regions 1 and 3 at an arbitrary angle. When I) $G_1 > G_3 \gg G_2$, three real roots ($p_1 < p_2 < p_3$) appear. Two of them, p_1 and p_2 , combine at $\varphi_1 = 1.5^\circ$ and $\varphi_1 = 237.8^\circ$, and a complex root $p (=Re(p) + iIm(p))$ appears within $\varphi_1 < 1.5^\circ$ and $\varphi_1 > 237.8^\circ$. The order $O(r^{p-1})$ yields a minimum at $\varphi_1 = 0, 4\pi/3$ (two-phase bonded structure), similarly as in other bonded configurations A) and B). The third root p_3 disappears within $\varphi_1 \cong \pi/3 \sim \pi$; however, two roots p_1 and p_2 become minimum within the same angle φ_1 , i.e., the order of two roots p_1, p_2 is a maximum within $\varphi_1 \cong \pi/3 \sim \pi$. When II) $G_1 < G_3 \ll G_2$, the root is a complex root $p (=Re(p) + iIm(p))$ at $\varphi_1 = 0^\circ$, which then separates into two real roots ($p_1 < p_2$) at $\varphi_1 \cong 85^\circ$. After that, those roots again combine at $\varphi_1 \cong 147^\circ$ and become a complex root p . The order of stress singularity yields a maximum $\varphi_1 = 0, 4\pi/3$. The third root p_3 appears within $\varphi_1 \cong 57 \sim 115^\circ, 124 \sim 183^\circ$ and becomes minimum at $\varphi_1 \cong 88^\circ, 147^\circ$. It is found that the order of singularity in II) $G_1 < G_3 \ll G_2$ is small in comparison with that in I) $G_1 > G_3 \gg G_2$. However, it conversely becomes large, similar to that in B), near $\varphi_1 = 0, 4\pi/3$.

In the cases where the conditions of elastic constants in three materials are I) and II), the loci for root are almost symmetrical with respect to $(\varphi_1 + \varphi_3)/2$, because the properties of materials 1 and 3 are nearly the same (the loci for root are symmetrical with respect to $(\varphi_1 + \varphi_3)/2$ when properties of materials 1 and 3 are identical). When material 2 is very soft in comparison to materials 1 and 3, the order $O(r^{p-1})$ of stress singularity yields a minimum at $\varphi_1 = 0$ or $\varphi_3 = 0$, and becomes larger as $\varphi_1 \rightarrow (\varphi_1 + \varphi_3)/2$. On the contrary, when material 2 is very rigid in comparison to materials 1 and 3, the order yields a maximum at $\varphi_1 = 0$ or $\varphi_3 = 0$ and becomes smaller as $\varphi_1 \rightarrow (\varphi_1 + \varphi_3)/2$. Comparing the results for each case of II) with those for each case of I) in Figs. 8~10, the order for the former is found to be smaller near $(\varphi_1 + \varphi_3)/2$ than that for the latter, and is conversely, larger near $\varphi_1 = 0$ or $\varphi_3 = 0$. Consequently, it is necessary for reducing the order of stress singularity to make either φ_1 or φ_3 small when the intermediate material is very soft in comparison to the two adjacent materials, and to equalize φ_1 and φ_3 when it is very rigid.

5. Conclusions

The order of stress singularity in a three-phase bonded structure with equal wedge angles ($\pi/3 - \pi/3 - \pi/3, \pi/2 - \pi/2 - \pi/2$ and $2\pi/3 - 2\pi/3 - 2\pi/3$) found using the eigenequation analytically derived in the previous

paper, was shown on the $\alpha_{23} - \beta_{23}$ plane. The combination of materials suitable for reducing the stress concentration at the apex was presented. The relation between the wedge angle φ_1 of material 1 and the order of stress singularity was also examined in the case of $\varphi_2, \varphi_1 + \varphi_3 = \text{fixed}$ under the condition $(\alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23})$. Reduction of the stress concentration at the apex by varying the wedge geometry was presented. When $\varphi_1 = \varphi_2 = \varphi_3$, the order $O(r^{p-1})$ of stress singularity in $G_1, G_3 > G_2$ becomes large in comparison with that in $G_1, G_3 < G_2$. Then, the order can be reduced by making either φ_1 or φ_3 small where $\varphi_1 + \varphi_2 + \varphi_3 = \text{constant}$.

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