Prediction of Fracture Strength of Ceramics with Small Defects Based on *R*-Curve Method*

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Since the critical size of defects in ceramics is usually very small, the conventional method of fracture mechanics prediction often becomes inapplicable. In the present study, the fracture strength of silicon nitride with small defects is evaluated on the basis of the *R*-curve method. The parameters of the *R*-curve of silicon nitride are determined from the experimental relationship between the fracture strength and the defect size. For penny-shaped cracks, the fracture strength is constant for small defects and the critical stress intensity factor is constant for large defects. The equivalent defect size is an excellent parameter to correlate the fracture strength to the size of elliptical cracks with an aspect ratio between 0.3 and 1. For small voids, the fracture strength is nearly the same as that for a penny-shaped crack. On the other hand, for large voids, the fracture strength approaches a constant value which is equal to that of materials without voids divided by the elastic stress concentration factor. When the tip radius of deep long notches is small, the critical stress intensity factor for fracture is independent of the tip radius. A Monte Carlo simulation of bending fracture of silicon nitride was performed by randomly distributing penny-shaped cracks having random sizes in a specimen by a computer. On the basis of twenty-five runs of fracture simulation, the relationship between the bending strength and the equivalent defect length, and cumulative distribution function of the fracture strength were successfully derived.

Key Words: Crack Propagation, Ceramics, Fracture Criterion, Fracture Mechanics, Monte Carlo Simulation

1. Introduction

Since the fracture toughness of ceramics is very small as compared with that of metallic materials, the critical size of defects for fracture becomes small. The conventional fracture mechanics for a long crack cannot be applied to fracture analysis of ceramics. The resistance to crack propagation in ceramics will increase due to the effect of microstructures and the development of crack wake. Therefore, the *R*-curve method will be useful for the estimation of the critical size of defects.

In engineering, the point stress model⁽¹⁾, the mean stress model⁽²⁾ and the fictitious crack model^{(3),(4)} have

been used to analyze the effect of crack size. Although these model show some success, the underlying physical meanings are unclear. In this paper, the fracture strength of ceramics with various defects is evaluated based on the *R*-curve method. The shape of the *R*-curve of silicon nitride is first determined from the relationship between the bending strength and the defect size. The effects of various defect shapes and sizes on the fracture strength of ceramics were evaluated on the basis of the *R*-curve. A Monte Carlo simulation is performed to obtain the probabilistic nature of the bending strength.

2. *R*-Curve Method

2.1 Shape of *R*-curve

The resistance K_R to crack propagation is given by

$$K_R = K_0 + K_m + K_s, \quad (1)$$

where K_0 is the fracture toughness of single crystals, K_m represents the effect of microstructures, microstress or grain structure^{(5),(6)}, and K_s represents crack

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shielding caused by bridging⁽⁷⁾, microcracking⁽⁸⁾ or transformation⁽⁹⁾.

If K_m is disregarded, the R -curve will have a shape such as that shown in Fig. 1. K_R is the constant value K_0 until the crack length reaches c_1 . K_R increases with crack growth, and indicates the constant value K_∞ at crack lengths exceeding c_2 , since the contribution of crack shielding becomes saturated. Mai and Lawn proposed the following R -curve equations to explain bridging:

$$K_R = K_0 + (K_\infty - K_0) \left\{ 1 - \left[1 - \left(\frac{c_2(c^2 - c_1^2)}{c(c_2^2 - c_1^2)} \right)^{1/2} \right]^3 \right\} \quad (2)$$

$$K_R = K_0 + (K_\infty - K_0) \left\{ 1 - \left[1 - \left(\frac{c - c_1}{c_2 - c_1} \right)^{1/2} \right]^3 \right\} \quad (3)$$

Equation (2) is for a halved-penny-shaped crack⁽¹⁰⁾ and Eq. (3) is for a through-crack⁽¹¹⁾.

Figure 2 shows the R -curve shapes of the penny-shaped crack and the through-crack. The material constants used to construct the curve in Fig. 2 were those for alumina reported by Cook et al.⁽¹⁰⁾ and given in the following: the grain size $d = 20 \mu\text{m}$, $c_1 = 40 \mu\text{m}$, $c_2 = 420 \mu\text{m}$, $K_0 = 1.73 \text{ MPa}\sqrt{\text{m}}$, and $K_\infty = 4.08 \text{ MPa}\sqrt{\text{m}}$.

Fig. 1 R -Curve

Fig. 2 Shape of R -curve

2.2 Crack extension

Crack extension occurs when the applied stress intensity factor is larger than K_R . For example, the initial defect without bridging has length t as shown in Fig. 3(a). The minimum stress required to extend the crack to c_1 is σ_1 , and slow crack growth occurs with increasing applied stress. When the increment of applied stress intensity factor K_A is larger than that of the resistance K_R at σ_2 ($dK_R/dc \leq dK_A/dc$), unstable crack growth occurs. The crack extension for defects with bridging is shown in Fig. 3(b). The resistance begins to increase even for an initial defect. The R -curve originates from the initial defect, and σ_1 and σ_2 are determined in the same way as Fig. 3(a).

2.3 Determination of R -curve parameters

The material constants in Eq. (2) and (3) are c_1 , c_2 , K_0 , and K_∞ . The fracture toughness obtained from SEPB (single-edge precracked beam) is used as K_∞ . Since it is difficult to determine the rest of the R -curve parameters by experiment, the optimum values of these are calculated from the experimental results of the bending strength and the equivalent crack length using the simplex method.

The test specimen was pressureless sintered silicon nitride, mechanical properties of which are shown in Table 1. The mean diameter of grain size with an acicular shape was $1.2 \mu\text{m}$. The test specimen for the bending test was 3 mm in height, 4 mm in width and 40

(a) Defect without bridging

(b) Defect with bridging

Fig. 3 Crack extension based on R -curve method

Table 1 Mechanical properties of the test specimen

Young's modulus E (GPa)	Poisson's ratio ν	Bulk density g/cm^3	Fracture toughness K_C MPa $\sqrt{m}$	Bending strength σ_B (MPa)
298	0.268	3.22	6.59	748.5

Fig. 4 Simplex vector

mm in length. The tension side of the test specimen was lapped. The four-point bending test was performed. After bending tests, the fracture origin of specimens was examined with a scanning electron microscope, and its shape was approximated by an ellipse, a half-ellipse or quarter-ellipse. The stress intensity factor was obtained using the equation given in our previous study⁽³⁾, and the equivalent crack length was calculated. The fracture toughness for a long crack was determined using SEPB specimens. The mean value of the bending strength was 748.5 MPa, and that of fracture toughness was 6.59 MPa $\sqrt{m}$ ⁽¹²⁾.

The optimum values of the other R -curve parameters c_1 , c_2 and K_0 were determined from the results of bending tests of lapped specimens using the simplex method⁽¹³⁾. In this paper, Eq. (2) for a three-dimensional crack was used as R -curve equation, and a semicircular surface crack in a beam subjected to bending was used for the calculation of the stress intensity factor. The K_∞ value was equal to the fracture toughness obtained using SEPB specimens. Initial simplex vectors $V_i (i=1\sim 4)$ were made from c_1 , c_2 and K_0 , as

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1+f & 1 & 1 \\ 1 & 1+f & 1 \\ 1 & 1 & 1+f \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ K_0 \end{bmatrix}, \quad (4)$$

where $f=0.3$ was used in this study. (a) Error R_i between the experimental result and the R -curve given by V_i is calculated. (b) V_i are sorted from V_1 to V_4 according to the magnitude of error. (c) V_1 having the largest error is exchanged with a new vector on the line which intersects the center of gravity of the triangle of V_2 , V_3 and V_4 . A better vector V'_1 is obtained (see Fig. 4) as

$$V'_1 = V_1 + \beta(V_G - V_1). \quad (5)$$

Errors are calculated for the cases of $\beta=0.75, 1.5$,

Fig. 5 Parameters determined by simplex method

2.0 and 2.5, and β giving the minimum error is selected. V'_1 obtained by this procedure is regarded as a new V_1 , the procedures (a)–(c) are repeated, and R -curve parameters converge to optimum values.

In the present study, the following values were obtained: $K_0=0.228$ MPa $\sqrt{m}$, $K_\infty=6.59$ MPa $\sqrt{m}$, $c_1=4.98$ μm , and $c_2=17.6$ μm . The calculated R -curve using these parameters is shown in Fig. 5. The solid symbols in Fig. 5 represent the experimental results, and the curve is the result of prediction for a semicircular surface crack obtained using the above R -curve parameters. The prediction of the bending strength by the R -curve method agrees well with experimental results. In the next section, the fracture strength of ceramics with various types of defects will be estimated on the basis of the R -curve method.

3. Prediction of Fracture Strength

3.1 Fracture from penny-shaped crack

The stress intensity factor for a penny-shaped crack in an infinite body under uniform tensile stress σ is

$$K = \sigma \sqrt{\pi t} F, \quad (6)$$

where the correction factor F is $2/\pi$. The equivalent defect size is given by the following.

$$K = \sigma \sqrt{\pi t_{eq}} = \sigma \sqrt{\pi t} F \quad (7)$$

$$t_{eq} = t \cdot F^2 \quad (8)$$

The stress intensity factor for an unstable fracture, K_C , is

$$K_C = \sigma_B \sqrt{\pi t} F, \quad (9)$$

where σ_B is the bending strength of the material. The relationship between the fracture strength and the equivalent defect length t_{eq} is normalized by the grain size $d=1.2$ μm is shown in Fig. 6. In this figure, the bending strength was calculated for both cases of defects with and without bridging⁽¹⁴⁾. The fracture strength from small defects with bridging is larger

Fig. 6 Fracture strength for penny-shaped crack

Fig. 8 Fracture of elliptic crack

Fig. 7 Effect of c_2 on fracture strength

than that without bridging due to the effect of bridging. The value of K_c becomes smaller than K_∞ in the region of small equivalent defect lengths. This prediction agrees with the fracture strength of ceramics. In the case of defects with bridging, the fracture strength from small defects is nearly constant, since unstable fracture occurs simultaneously with the initiation of crack growth.

Figure 7 shows the calculated results for several values of c_2 . The initial defect without bridging was used in this calculation. For larger c_2 values, the fracture strength from small defects becomes smaller and the crack length when K_c approaches K_∞ is larger.

3.2 Fracture from elliptic defect

The fracture strength of elliptic defects in an infinite body under uniform tension was calculated for various aspect ratios t/l , where t is half of minor axis and l is half of major axis. The calculated results are shown in Fig. 8. The correction factor of the stress intensity factor is $F=1/E(k)$, where $E(k)$ is the

complete elliptic integral of the 2nd kind with $k = \sqrt{1-(t/l)^2}$. The crack extension at the minor and major axes is assumed to be independent of each other. Comparing the stress intensity factor of a crack at the minor axis with that at the major axis, the crack which had the larger stress intensity factor was extended by a small amount. The relationship between σ_B and t_{eq} for $t/l=0.3$ to 1.0 shows almost same tendency as shown in Fig. 8. This supports the use of the equivalent defect length for relating bending strength with defect size.

3.3 Fracture from spherical void

The R -curve method is applied to a fracture from a spherical void. The stress intensity factor for a spherical void with a circumferential crack at its equator was given by Baratta and Parker⁽¹⁵⁾ as

$$K = \sigma \sqrt{\pi c} \left[1.12 - 0.31 \tan^{-1} \left(\frac{c}{t} \right) \right] \times \left\{ 1 + \frac{1}{2(7-5\nu)} [(4-5\nu)p^3 + 9p^5] \right\} \quad (10)$$

$$p = \frac{1}{1 + c/t},$$

where t is the radius of a spherical void and c is the length of the circumferential crack.

The stress intensity factor for a spherical void with a semicircular crack at its equator was given by Evans et al.⁽¹⁶⁾ as

$$K = \sigma \sqrt{\pi c} \frac{2}{\pi} \left[1 + (K_t - 1) \frac{1}{1 + c/t} \right], \quad (11)$$

where K_t is the stress concentration factor and is a function of Poisson's ratio ν as

$$K_t = \frac{27-15\nu}{14-10\nu}. \quad (12)$$

The equivalent defect length for the spherical void is $t_{eq} = tF^2$ ($F=2/\pi$). K_t is 2.03 for $\nu=0.268$.

Figure 9 shows the relationship between the bending strength and the equivalent defect length. Most

Fig. 9 Fracture from void

ceramics fractures originate from voids or pores. When a crack originates from a void, the fracture mode depends on the size of the void. The fracture strength for a small void is estimated from Baratta and Parker's equation, since the critical crack length is large compared with the size of the void. On the other hand, the fracture strength for a large void is estimated from Evans et al.'s equation, since the critical crack length is small compared with the size of the void. For a small spherical void, the fracture strength is close to that for a penny-shaped crack. With an increase in the size of the void, the fracture strength given by Evans et al.'s equation approaches the bending strength of ideal materials divided by the stress concentration factor K_t .

3.4 Fracture from elliptic hole

Fracture from a through-the-thickness elliptic hole in an infinite plate under uniform tension will be derived in a similar way as fracture from a penny-shaped crack. The stress intensity factor for a small crack c formed at the notch tip was calculated by⁽¹⁷⁾

$$K_A = \sigma \sqrt{\pi c} \frac{1.122 K_t}{\sqrt{1 + 4.5c/\rho}}, \quad (13)$$

where ρ is the notch-tip radius. For a large crack, the stress intensity factor was calculated by

$$K_B = \sigma \sqrt{\pi(c+t)}. \quad (14)$$

The minimum crack length at $K_A = K_B$ is denoted by c^* . K_A was used for $c < c^*$ and K_B was used for $c > c^*$. The fracture strength was estimated on the basis of the R -curve method using these stress intensity factors. The relationship between the stress intensity factor and the notch-tip radius is shown in Fig. 10. The stress intensity factor K_p in the figure was calculated from

$$K_p = \sigma_B \sqrt{\pi t}, \quad (15)$$

where the half-length of the elliptic hole t is 10 mm. The value of K_p approaches the fracture toughness with decrease in $\sqrt{\rho/d}$. The result of this prediction

Fig. 10 Fracture from elliptic hole

agrees well with the result of the fracture toughness test of notched specimens^{(1),(18)}. The critical value of the notch-tip radius to obtain the fracture toughness is larger for larger c_2 values.

4. Monte Carlo Simulation

4.1 Procedure of simulation

Fracture simulation of silicon nitride based on the R -curve method was conducted using the technique developed by Taniguchi et al.⁽¹⁹⁾ In this study, we assumed the shape of initial flaws to be circular, and the cumulative distribution function $G(t)$ ⁽²⁰⁾ to be expressed by

$$G(t) = 1 - \left(\frac{t}{t_0} \right)^{-w}, \quad (16)$$

where t is the radius of the circular flaw. Random numbers from 0 to 1 were generated, and the radius t corresponding to the random number was determined using Eq. (16). Parameters w , t_0 and the defect density q were given empirically, since there was no result of or method for the distribution of defects. We set $t_0 = 7 \mu\text{m}$, $w = 4$ and $q = 5/\text{mm}^3$ in this simulation. The coordinates of the defect position, x , y , z , were given by random numbers in the space of the specimen which had 20 mm length, 3 mm height and 4 mm width.

After approximating the initial defect for the penny-shaped crack on the surface, below the surface or at the corner, the stress intensity factor K was calculated from equations used in the previous study⁽³⁾.

The initial defect without bridging was assumed. For surface cracks and corner cracks shown in Fig. 11, the bending strength at A was compared with that at B. A small value of crack extension was assigned to the position with smaller bending strength. The crack aspect ratio was changed with stable crack

growth. For cracks below the surface, the bending strength at A was compared with that at B, where it was found that the crack grew in the direction of smaller bending strength. The maximum bending strength is the actual fracture strength obtained in the bending test. The bending strength was calculated for all defects in the specimen. We assume that the actual fracture starts from the defect which has the smallest bending strength.

Fig. 11 Procedure of crack growth in simulation

Fig. 12 An example of Monte Carlo simulation

For R -curve parameters of silicon nitride, K_{∞} is the fracture toughness obtained from SEPB specimens, and c_1 , c_2 and K_0 were determined using the simplex method, as described above.

4.2 Simulation results and discussion

The distribution of defects and the result of the simulation are shown in Fig. 12. Defects were projected onto a plane perpendicular to the stress direction. Defects in the compression side are indicated by dots. The circles drawn by solid lines indicate initial defects and circles drawn by broken lines indicate crack shapes at unstable fracture. Defects were magnified 2.2 times to facilitate visualization. In the specimen, a fracture originated from the internal defect near the surface. The crack propagated stably to become a surface crack, and then the specimen fractured.

Figure 13 presents simulated data of the relationship between bending strength and equivalent defect length for 25 specimens. The curve in the figure indicates the relationship predicted using optimized parameters of the R -curve. The simulated result, the experimental result, and the predicted result agree well.

The Weibull distribution of the simulated result and experimental result is shown in Fig. 14. Open symbols indicate the simulated results and solid symbols the experimental results. In the simulated result, the shape parameter m is 6.41 and the scale parameter α is 866 MPa. Except for the smallest value of the bending strength, m becomes 10.2 and α becomes 860 MPa. In the experiment, m is 10.7 and α is 779 MPa.

By Itoh et al.⁽²¹⁾, there are samples which have very small bending strengths as compared with the mean value, and their plots do not lie on the Weibull distribution line. The smallest bending strength from the simulated result is similar to Itoh et al.'s experimental result.

In the fracture mode in this simulation, we found

Fig. 13 Bending test simulated by Monte Carlo method

Fig. 14 Weibull plots simulated by Monte Carlo method

that the number of samples with surface fractures was 13, and that with surface fracture after stable crack growth from internal defects 5, and that with inside fracture 7.

5. Conclusions

Fracture of ceramics with various kinds of defects was examined on the basis of the *R*-curve method and the results are summarized as follows.

(1) Fracture toughness was constant for large penny-shaped cracks, while fracture strength was constant for small defects. The bending strength of fracture from the elliptical defects is independent of the aspect ratio between 0.3 and 1.0. Therefore, the equivalent crack length which is obtained by approximating a defect with an ellipse shape is an appropriate parameter for predicting fracture strength.

(2) Fracture strength from a small spherical void is nearly the same as that from a penny-shaped crack. For a large void, the fracture strength gradually approaches the value of the fracture strength divided by the elastic stress concentration factor. The fracture toughness of a through-the-thickness elliptic hole is constant when the notch-tip radius is small.

(3) Monte Carlo simulation of bending fracture was performed using *R*-curve parameters determined by the simplex method. The relationship between the bending strength and the equivalent crack length, and the Weibull distribution of the simulation result were successfully derived.

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