Optimum Design of Transient Heat Conduction Fields Using Boundary Element Inverse Analysis*

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In this paper we present a method of optimum design for transient heat conduction fields using inverse analysis with the boundary element method and the Kalman filter algorithm. It is expected that the inverse analysis method can be applied to optimization problems of determining unknown parameters by minimizing the cost function. The optimization problem under consideration is such that the boundary conditions on part of the boundary are to be determined so that requirements of responses on the remainder of the boundary are satisfied. A few sample problems of optimization are computed by means of the inverse analysis method to demonstrate its potential usefulness.

Key Words: Inverse Problem, Optimum Design, Boundary Element Method, Extended Kalman Filter, Transient Heat Conduction, Temperature Control

1. Introduction

Inverse problems related to heat conduction problems are, for instance, estimation problems in which distributions of temperature or heat flux of the domain or boundary are estimated using some measured values of temperature, identification problems in which unknown defects or phase boundary shape is to be identified using measured data, and determination problems of unknown physical properties. They are important subjects which are widely spread in engineering fields, so many investigations of them have been performed[1]–[8]. It is considered that the optimum design problem is also an inverse problem in which boundary conditions or domain geometries should be defined to satisfy some required specifications, such as desired distribution of temperature or heat flux.

Our group has already proposed inverse analysis methods using boundary element analysis combined with the extended Kalman filter algorithm or the optimization method[8]–[10]. The proposed methods were developed for identification of unknowns, for example, internal defects or unknown boundary conditions, while it can be considered that they are also applicable to analysis of optimum design problems. In this study, we investigate the applicability of the inverse analysis method to optimum design in transient heat conduction fields. In this paper, we show an outline of the inverse analysis method using the boundary element analysis for transient heat conduction problems combined with the extended Kalman filter algorithm, and then we explain how to apply the inverse analysis method to the optimum design problem. Some results are also shown for examples of optimum design in which variation of control temperature is determined in order to maintain a required distribution of temperature on another part of the boundary.
2. Optimum Design of Transient Heat Conduction Field

We shall consider the transient heat conduction problem in an isotropic, homogeneous region. Under the assumption of no internal heat source, the governing equation can be expressed as the following diffusion equation:

\[ k \nabla^2 u(x, T) = \frac{\partial u(x, T)}{\partial T}, \]

where \( u(x, T) \) denotes temperature at point \( x \) in the domain at a time \( T \), while \( \nabla^2 \) is the Laplacian. \( k \) is a thermal diffusivity which is expressed as

\[ k = \frac{\lambda}{c \rho}, \]

where \( \lambda \) denotes thermal conductivity, \( c \) specific heat, and \( \rho \) mass density. Moreover, it is assumed that \( k \) is independent of space and time.

We consider an optimum design problem in the above-mentioned heat conduction field. The whole boundary of the field under Eq.(1) is assumed to be composed of a boundary \( \Gamma_b \) of prescribed temperature, a boundary \( \Gamma_r \) of prescribed heat flux, a boundary \( \Gamma_f \) in which distribution of boundary values is required and a boundary \( \Gamma_e \) in which distribution of boundary values is determined to satisfy the requirements on \( \Gamma_f \) shown in Fig. 1. When \( \Gamma_b \) is subjected to a known heat load \( u(x, T) \) and unknown heat flux \( q(x, T) \) is prescribed on \( \Gamma_b \), the optimization problem to be considered is that the boundary conditions \( u(x, T) \) or \( q(x, T) \) on \( \Gamma_e \) should be determined to achieve some requirements of the response \( u_e(x, T) \) or \( q_e(x, T) \) on the specific boundary \( \Gamma_e \). The optimum design of this type is expected to be applied to temperature control problems, such as in the optimum design of a cooling/heating process for a precision injection mold, and cooling control design for heat shocks.

3. Inverse Analysis Method for Optimum Design Using Boundary Element Analysis Combined with Extended Kalman Filter Algorithm

The temperature control problem mentioned above corresponds to an optimization problem in which the variation of the optimum control value is determined as a function of design variables. In this study, we must identify unknown parameters with respect to the design variables using inverse analysis based on the boundary element method combined with the extended Kalman filter algorithm. Unknown temperature \( u(x, T) \) or unknown heat flux \( q(x, T) \) is approximated by B-spline functions, and we treat the unknown coefficients of the function as the design parameters.

Next, we describe the method of determining unknown parameters using the boundary element method combined with the extended Kalman filter algorithm. The observation vector \( y_t \) at time \( t \) is expressed by the following observation equation with respect to the unknown parameter \( x_t \):

\[ y_t = h_t(x_t) + v_t, \]

where \( v_t \) denotes the observation noise.

At the time of applying the extended Kalman filter algorithm to the optimum design problems, the observation vector corresponds to the required boundary values on \( \Gamma_r \), while the unknown parameters correspond to the coefficients of B-spline functions which describe the variation of the control boundary values determined by means of obtaining the required values. In this case, it can be considered that \( t \) denotes the number of iterations in place of time. Hence the state equation expresses the steady condition of the parameters as

\[ x_{t+1} = Jx_t, \]

where \( J \) is the unit matrix.

An estimated value of the unknown parameter \( \hat{x}_{t+1} \) is determined by using iterative computation of the filter equations

\[ \hat{x}_{t+1} = \hat{x}_{t+1} + K_t[y_t - H_t \hat{x}_{t+1}], \]

\[ K_t = P_{t+1}H_t^T[H_tP_{t+1}H_t^T + R_t]^{-1}, \]

\[ P_t = P_{t+1} - K_tH_tP_{t+1}, \]

where \( K_t \) is the Kalman gain and \( H_t \) the observation matrix. \( P_t \) and \( R_t \) are the covariance vectors of estimation errors and observation errors, respectively.

The observation matrix \( H_t \) in Eq.(5) is given as

\[ H_t = \frac{\partial h_t}{\partial \hat{x}_{t+1}} \]

The partial derivative in Eq.(8) is calculated by a finite difference approximation using the boundary value \( h_t(\hat{x}_{t+1}) \) which can be computed by the boundary element method. For the boundary element formulation of transient heat conduction problems we
use the Laplace transform method based on the regularized boundary integral equations.

The initial values of iterative computation are given as
\[ \hat{x}_{0} = x, \quad (9) \]
\[ P_{0} = \Sigma, \quad (10) \]
As the convergence criteria of iterative computation, we consider the following two conditions. In the first one, it is regarded that the iterative computation is converged when the change of the nondimensional residual \( Z \) with respect to the required value and the calculated one is sufficiently small. This condition can be written as
\[ |Z_{n} - Z_{n-1}| < \varepsilon, \quad (11) \]
where \( \varepsilon \) is a small positive number and \( Z \) is defined below. Some evaluation points are placed on \( \Gamma_{s} \). We define a cost function with respect to the residual between the calculated boundary values of these evaluation points for the estimated variation on \( \Gamma_{s} \) and the corresponding required values. Here, for the case that temperatures are prescribed as required values on \( \Gamma_{s} \), the cost function \( W \) can be written as
\[ W = \sum_{i} \sum_{j} \left( \frac{u(x_{i}, T_{d}) - u_{r}(x_{i}, T_{d})}{u_{r}(x_{i}, T_{d})} \right)^{2}, \quad (12) \]
where \( u(x_{i}, T_{d}) \) describes the calculated temperature at a time \( T_{d} \) and an evaluation point \( x_{i} \), which can be obtained from the boundary element analysis, while \( u_{r}(x_{i}, T_{d}) \) is the required temperature at the corresponding time and point. \( L \) and \( D \) denote the numbers of evaluation points in the direction of space and time, respectively. Then \( Z \) is expressed as
\[ Z = \frac{1}{2} \log_{10} W. \quad (13) \]
For the second criterion, we consider changes of the estimated parameters. When all changes are sufficiently smaller than given tolerance limits \( \xi_{n} \), the iterative computation is regarded as converged. Namely, the second criterion can be written as
\[ |\hat{x}_{n} - \hat{x}_{n-1}| < \xi_{n}, \quad n = 1, 2, \ldots, N, \quad (14) \]
where \( N \) denotes the total number of unknown parameters. In this study, it is assumed that the iteration is regarded as converged when either of the above two criterions is satisfied.

When the inverse analysis method is applied to optimum design, the observation vector \( \gamma_{r} \) in the extended Kalman filter is assigned to the required value which should be achieved on \( \Gamma_{s} \). In this case, it can be considered that the errors of the observation vector do not exist. Then the covariance vector of errors \( \Sigma_{r} \) becomes zero, provided that a very small value, for example, \( 1.0 \times 10^{-30} \) is assigned to \( \Sigma_{r} \) in the computer program. For the case that \( \Sigma_{r} \) equals zero and the state equation expresses the steady state of the system, as shown by Eq. (4), it is shown in Ref. (11) that the set of Eqs. (5)–(7) in the extended Kalman filter is equivalent to the Gauss–Newton method.

4. Examples of Optimum Design and Discussions

We show several examples of optimum design in a transient heat conduction field in which distributions and histories of temperature on the control boundary should be determined to achieve the required variations of temperature on the other part of the boundary.

As shown in Fig. 2, we consider a two-dimensional square region ABCD with two circular holes. It is assumed that the boundary AB has known temperature \( u(x, T) \). We must determine the boundary conditions of two holes to satisfy the specified variation of the temperature on the side CD. The target temperature on the side CD is assumed to be distributed uniformly at 20°C throughout the direction of time and space. Seven and 30 evaluation points are assigned uniformly on the side CD and in the time direction, respectively. The outer boundary of the square region is discretized into 12 isoparametric quadratic boundary elements. Each boundary of two holes is discretized into 4 elements. The number of Laplace transform parameters considered for boundary element analysis is assumed as 30, while the interval of the analysis is 30 seconds.

The first example (Example 1) is that for the variation and history of temperature prescribed on the side AB given as
\[ u(T) = 200 \sin \left( \frac{T}{15} \right) \quad (0 \leq T \leq 15 \text{ s}), \quad u(T) = 0 \quad (15 \leq T \leq 30 \text{ s}). \quad (15) \]
Figure 3 illustrates the prescribed temperature on the side AB which is expressed by Eq. (15). In this example, it is assumed that boundary conditions of temperature on holes 1 and 2 are of the same value. The boundary values on the holes are approximated by B-spline functions. The number of unknown parameters...
is assumed as 8. At the beginning of iterative computation, we also assume that the distribution of temperature is uniform at 20°C (0 s ≤ T ≤ 30 s) on the boundaries of the two holes.

The temperature response on the side CD without optimization is shown in Fig. 4. Figure 5 shows the optimized state of boundary conditions of the two holes obtained by using the inverse analysis. Figure 6 shows the temperature response on the side CD when the optimum control temperature is prescribed on the boundary of the two holes. The difference between the required temperature and that obtained is sufficiently small, so it can be concluded that the optimization is successful.

As the second example (Example 2), we shall analyze the case that the prescribed temperature on the side AB is given as

$$u(T) = f(T) \sin \frac{\pi T}{30} \quad (0 \leq T \leq 30 \text{ s})$$

$$f(T) = \frac{5}{3}x + 100 \quad (0 \text{ mm} \leq x \leq 60 \text{ mm})$$

(16)

In this example, we consider that the boundary conditions of temperature on the boundaries of holes 1 and 2 are different; then we take 8 unknown parameters at each boundary of the holes. Therefore we treat a total of 16 unknown parameters. The initial values of the iterative computation are given similarly to those in Example 1.

Figure 7 illustrates the prescribed variation of temperature on the side AB expressed by Eq.(16). Figure 8 shows the temperature response on the side CD without optimization. Figure 9 shows the optimized histories of the boundary condition on each hole obtained by using the inverse analysis. Figure 10

Fig. 3 Prescribed temperature variation on the side AB (Example 1)

Fig. 4 Temperature response on the side CD without optimization (Example 1)

Fig. 5 Optimum history of temperature on two boundaries of holes 1 and 2 (Example 1)

Fig. 6 Optimized state of temperature response on the side CD (Example 1)
Fig. 7 Prescribed temperature variation on the side AB (Example 2)

Fig. 8 Temperature response on the side CD without optimization (Example 2)

Fig. 9 Optimum history of temperature on each boundary of holes 1 and 2 (Example 2)

Fig. 10 Optimized state of temperature response on the side CD (Example 2)

illustrates the temperature response on the side CD when the optimum history of control temperature is prescribed. In Example 2, the temperature on the side AB varies in the direction of not only time but space; however, the difference between the required temperature and that obtained after optimization is sufficiently small, so it can be said that the optimization is successful.

Next, we analyze Example 2 by applying the alternative inverse analysis method \(^{(T, W)}\) with the conjugate gradient method (CGM) which is one of the standard optimization methods. In this case, the unknown value is also approximated using B-spline functions. It is also assumed that the initial values of iterative computation are distributed uniformly at 20°C (0 ≤ T ≤ 30 s) on the boundaries of two holes.

Figure 11 shows the optimum boundary conditions of temperature for the two holes using the inverse analysis method with CGM. Figure 12 shows the temperature response on the side CD when the optimum boundary conditions are prescribed. We also obtain a good result using CGM as the inverse analysis method.

In Table 1, errors between the obtained temperatures and the required ones at the evaluation points, numbers of iterations to convergence, numbers of boundary element analyses, and the nondimensional values \(Z\) are summarized. In addition to the examples shown in this paper, results obtained using a smaller number of unknown parameters are also shown in this table. The errors in the table are calculated using

\[
E_r = \frac{1}{L + D} \sum_{m=1}^{3} \sum_{n=1}^{3} \left| \frac{u(x, y, T) - u_r(x, y, T)}{u_r(x, y, T)} \right| \times 100\%.
\] (17)
It can be said that a good optimization is obtained for each case in Table 1. If a large number of unknown parameters is taken, the error decreases, but many boundary element analyses and much computing time are needed.

The convergence properties of the nondimensional value $Z$ in Examples 1 and 2 are shown in Figs. 13 and 14, respectively. From these figures, it is found that the convergence of each analysis is good. For Example 2, by comparing the results obtained using the extended Kalman filter algorithm and ones obtained using CGM, we can achieve almost the same distribution as required temperature for optimized cases. On the other hand, the number of the boundary element analyses to achieve convergence using the extended Kalman filter is relatively small compared with the case of using CGM. Consequently, computational time can be reduced by one-quarter. It

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**Table 1** Errors at evaluation points and numbers of boundary element analyses

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<thead>
<tr>
<th></th>
<th>Example 1</th>
<th>Example 2</th>
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<tbody>
<tr>
<td></td>
<td>Kalman</td>
<td>Kalman</td>
</tr>
<tr>
<td>Number of Parameters</td>
<td>3</td>
<td>8</td>
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<tr>
<td>Error $E_r$ %</td>
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<td>0.26</td>
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<tr>
<td>Iterations</td>
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<td>3</td>
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<tr>
<td>Number of BE analyses</td>
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<td>28</td>
</tr>
<tr>
<td>$Z$</td>
<td>-1.25</td>
<td>-2.50</td>
</tr>
</tbody>
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is found that the inverse analysis method using the extended Kalman filter algorithm is more efficient than the method using CGM. In the above examples, we obtain optimum boundary conditions with respect to temperature, but it is also possible to obtain optimum boundary conditions with respect to heat flux.

We treated cases in which the control boundary value is a function of only time direction in this study. It is also possible to analyze cases in which the control boundary conditions are varied in both time and space using the inverse analysis method previously proposed by the authors, in which history and distribution of boundary values in time and space directions are approximated using the spline curved surface.

5. Conclusions

In this study, we applied the inverse analysis method based on the boundary element method combined with the extended Kalman filter algorithm to optimum design problems. Several example problems of optimum design in which the history of control temperature on the boundary should be determined to achieve the required temperature response on the other boundary were analyzed, and we obtained satisfactory results with good convergence. It was demonstrated that the inverse analysis method can be applied effectively to optimum design problems.

References