Dynamic Behavior of Thick Plates Due to Oblique Impact*

(Numerical Analysis of Three-Dimensional Stress Waves in an Elastic/Viscoplastic Body)

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A numerical method is presented to analyze three-dimensional stress waves in an elastic/viscoplastic thick plate, when oblique impact is applied to the surface of the thick plate at various impact angles. For the numerical analysis, the finite difference method based on integration along the bicharacteristics is employed. From the numerical results obtained for various loading conditions, the effect of the impact angle on impact fracture is examined. It is also demonstrated that the dynamic behavior of the plate due to oblique impact can be examined in detail using this method.

Key Words: Numerical Analysis, Stress Wave, Oblique Impact, Bicharacteristics, Elastic/Viscoplastic Body

1. Introduction

The finite difference method based on integration along the bicharacteristics is the most suitable method for the analysis of two-dimensional stress wave propagation^{(1)~(7)}. The advantage of this method is attributed to the fact that the characteristic surface coincides with the wave surface. Thus, using this method, the propagation, reflection and interaction of stress waves in a body can be investigated in detail. It is also comparatively easy to derive finite difference equations on the boundary of a simple shape such as a plate or a cylinder. From many studies^{(1)~(9)}, the validity of this method for the analysis of two-dimen-

sional stress wave propagation has been clarified. However, the analysis of three-dimensional stress wave propagation by this method has not been fully studied, because the derivation of differential relations along the bicharacteristics is very complicated and requires a great deal of labor. Regardless of this troublesome task, our purpose is the establishment of a numerical method to simulate the propagation of three-dimensional stress waves in an isotropic body, an inhomogeneous body and an anisotropic body, making use of the merits of this method effectively.

In recent years, studies of impact fracture in connection with speeding up of the machines have become active rapidly. For example, spall fracture is a particular kind of impact fracture which occurs as a planar separation of the material along the direction parallel to the incident and reflected plane wave fronts⁽¹⁰⁾. Therefore, the machine such as a spaceship which faces the danger of high-velocity impact should have counterplans against spall fracture. Particle impact in combustion gas will also cause damage to the ceramic turbine⁽¹¹⁾. This problem must be solved before the ceramic turbine is put into practical use. Therefore, in order to clarify the impact fracture

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mechanism, the establishment of a highly accurate analysis technique for the propagation of stress waves in bodies is necessary.

The purpose of this work is to establish a numerical method for simulating the propagation of three-dimensional stress waves in bodies which have many kinds of properties, such as an inhomogeneous body or an anisotropic body. As a first step, the governing equations for three-dimensional stress wave propagation in an elastic/viscoplastic body are formulated by means of the finite difference method based on integration along the bicharacteristics. A numerical example is presented for the case of three-dimensional stress wave propagation in the elastic/viscoplastic thick plate, when oblique impact is applied to the surface at various impact angles. From the numerical results for various loading conditions, the effect of the impact angle on impact fracture is examined. We also examined the applicability of this numerical method to the analysis of impact fracture which has been simulated by the finite element method.

The practical constitutive equations, which cover a wide range of strain rates for the elastic/viscoplastic body, are used^{(12),(13)}.

2. Theory

2.1 Governing equations

Figure 1 shows an infinite plate having a thickness h subjected to an impulsive load. The impulsive load is assumed to be applied to part of a surface of this plate. The cartesian coordinates (x, y, z) are taken, where the z axis is in the thickness direction. The equations of motion are given as

$$\begin{aligned}\rho \frac{\partial u_x}{\partial t} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \\ \rho \frac{\partial u_y}{\partial t} &= \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \\ \rho \frac{\partial u_z}{\partial t} &= \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z},\end{aligned}\quad (1)$$

where ρ is the density, t is the time, u_x , u_y and u_z are the velocity components, and σ_{xx} , σ_{yy} , σ_{zz} , σ_{xy} , σ_{yz} and σ_{zx} denote stress components.

The constitutive equations are assumed as

Fig. 1 Coordinates

$$\begin{aligned}\dot{\epsilon}_{xx} &= \frac{1}{2\mu} \dot{S}_x + \frac{\dot{\sigma}_m}{3K_v} + \langle D \rangle S_x \\ \dot{\epsilon}_{yy} &= \frac{1}{2\mu} \dot{S}_y + \frac{\dot{\sigma}_m}{3K_v} + \langle D \rangle S_y \\ \dot{\epsilon}_{zz} &= \frac{1}{2\mu} \dot{S}_z + \frac{\dot{\sigma}_m}{3K_v} + \langle D \rangle S_z \\ \dot{\epsilon}_{xy} &= \frac{1}{2\mu} \dot{\sigma}_{xy} + \langle D \rangle \sigma_{xy} \\ \dot{\epsilon}_{yz} &= \frac{1}{2\mu} \dot{\sigma}_{yz} + \langle D \rangle \sigma_{yz} \\ \dot{\epsilon}_{zx} &= \frac{1}{2\mu} \dot{\sigma}_{zx} + \langle D \rangle \sigma_{zx},\end{aligned}\quad (2)$$

where, ϵ_{xx} , ϵ_{yy} , ϵ_{zz} , ϵ_{xy} , ϵ_{yz} and ϵ_{zx} are strain components, μ is the shear modulus, K_v is the volume modulus, S_x , S_y and S_z are the deviatoric stress, and σ_m is the mean normal stress. The dot denotes the time derivative. The symbol $\langle D \rangle$ is

$$\langle D \rangle = \begin{cases} \frac{\eta^*}{\sqrt{J_2}} \exp\left\{\frac{-D'}{\sqrt{J_2} - \tau^*}\right\} & (\sqrt{J_2} > \tau^*) \\ 0 & (\sqrt{J_2} \leq \tau^*) \end{cases}\quad (3)$$

where J_2 is the second invariant of the stress deviator, τ^* is the back stress due to the work hardening, and η^* and D' are material parameters.

From Eqs.(1) and (2), the governing equations in the dimensionless form may be written in a matrix form:

$$L(\mathbf{W}) = \mathbf{A}^t \mathbf{W}_t + \mathbf{A}^x \mathbf{W}_x + \mathbf{A}^y \mathbf{W}_y + \mathbf{A}^z \mathbf{W}_z + \mathbf{B} = 0\quad (4)$$

where the subscripts of the vector $\mathbf{W}$ denote the partial derivatives with respect to t , x , y and z . The vectors $\mathbf{W}$ and $\mathbf{B}$ and matrices $\mathbf{A}^t$, $\mathbf{A}^x$, $\mathbf{A}^y$ and $\mathbf{A}^z$ are

$$\mathbf{W} = \begin{bmatrix} u \\ v \\ w \\ (l+m+n)/(3\xi) \\ k_c(l-m)/2 \\ k_c(l+m-2n)/2\sqrt{3} \\ k_cp \\ k_cs \\ k_cq \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \langle D \rangle (l-m)/k_c \\ \langle D \rangle (l+m-2n)/(\sqrt{3} k_c) \\ 2\langle D \rangle p/k_c \\ 2\langle D \rangle s/k_c \\ 2\langle D \rangle q/k_c \end{bmatrix}$$

$$\begin{aligned}
\mathbf{A}^t &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
\mathbf{A}^x &= \begin{bmatrix} 0 & 0 & 0 & -\xi & -\eta & -\zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\eta & 0 \\ -\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\eta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\mathbf{A}^y &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\eta & 0 & 0 \\ 0 & 0 & 0 & -\xi & \eta & -\zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\eta \\ 0 & -\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\eta & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\mathbf{A}^z &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\eta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\eta \\ 0 & 0 & 0 & -\xi & 0 & 2\zeta & 0 & 0 & 0 \\ 0 & 0 & -\xi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\zeta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned} \quad (5)$$

where

$$\begin{aligned}
t &= \frac{c_1}{b} \hat{t}, \quad x = \frac{\hat{x}}{b}, \quad y = \frac{\hat{y}}{b}, \quad z = \frac{\hat{z}}{b} \\
u &= \frac{\hat{u}_x}{c_1}, \quad v = \frac{\hat{u}_y}{c_1}, \quad w = \frac{\hat{u}_z}{c_1} \\
l &= \frac{\hat{\sigma}_{xx}}{\rho c_1^2}, \quad m = \frac{\hat{\sigma}_{yy}}{\rho c_1^2}, \quad n = \frac{\hat{\sigma}_{zz}}{\rho c_1^2} \\
p &= \frac{\hat{\sigma}_{xy}}{\rho c_1^2}, \quad q = \frac{\hat{\sigma}_{yz}}{\rho c_1^2}, \quad s = \frac{\hat{\sigma}_{xz}}{\rho c_1^2} \\
k_c &= c_1/c_2, \quad c_1 = (K_v/\rho + 4\mu/3\rho)^{1/2}, \quad c_2 = (\mu/\rho)^{1/2} \\
\xi &= \{1 - 4/(3k_c^2)\}^{1/2}, \quad \eta = 1/k_c, \quad \zeta = 1/\sqrt{3} k_c
\end{aligned}$$

and b is the representative length and the hat symbols denote the corresponding dimensional quantities.

2.2 Differential relations satisfied along the bicharacteristics⁽⁸⁾

The system of Eq. (4) is a system of semilinear hyperbolic partial differential equations with constant coefficients. This system has certain characteristic properties. The governing equations can be solved using the same method as that of Recker⁽⁸⁾. The outline of the derivation is described below.

The condition that a surface $\psi(t, x, y, z) = \text{const.}$ is a characteristic surface of Eq. (4) is equivalent to the condition that the determinant of the characteristic matrix $\mathbf{A}$ be zero:

$$\text{Det } \mathbf{A} = 0 \quad (6)$$

where

$$\mathbf{A} = \mathbf{A}^t \phi_t + \mathbf{A}^x \phi_x + \mathbf{A}^y \phi_y + \mathbf{A}^z \phi_z. \quad (7)$$

Equation (6) is equivalent to

$$\{\phi_t^2 - (\phi_x^2 + \phi_y^2 + \phi_z^2)\} \{\phi_t^2 - (\phi_x^2 + \phi_y^2 + \phi_z^2)/k_c^2\} = 0 \quad (8)$$

where the quantities in brackets correspond to characteristic surfaces with wave speeds 1 and $1/k_c$, respectively.

The solution of Eq. (8) is

$$\phi = c(t - t_0) - (x - x_0)\alpha_1 - (y - y_0)\alpha_2 - (z - z_0)\alpha_3 \quad (9)$$

which contains the line

$$(x - x_0)\alpha_1 + (y - y_0)\alpha_2 + (z - z_0)\alpha_3 = 0 \quad (10)$$

subject to the condition

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \quad (11)$$

on the plane $t = t_0$. The corresponding characteristic cone through the point (t_0, x_0, y_0, z_0) , determined by the envelopes of characteristic planes, is given by

$$c^2(t - t_0)^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \quad (12)$$

$(c = \pm 1, \pm 1/k_c).$

The bicharacteristics for Eq. (12) are

$$\begin{aligned}
x - x_0 &= c(t - t_0)\alpha_1, \quad y - y_0 = c(t - t_0)\alpha_2 \\
z - z_0 &= c(t - t_0)\alpha_3.
\end{aligned} \quad (13)$$

The bicharacteristic strips associated with the bicharacteristic lines (13) are

$$\begin{aligned}
\phi_t &= c, \quad \phi_x = -\alpha_1, \quad \phi_y = -\alpha_2, \quad \phi_z = -\alpha_3 \\
(c &= 1, 1/k_c).
\end{aligned} \quad (14)$$

In order to determine the equations along the bicharacteristics, the null vectors associated with the system (4) must be determined. The null vectors $\mathbf{l} = [\mathbf{l}^1, \mathbf{l}^2, \mathbf{l}^3, \mathbf{l}^4]$ are the solutions of the following equation:

$$\mathbf{l} \mathbf{A} = 0. \quad (15)$$

From Eqs. (7) and (15), the null vectors are given by

$$\begin{aligned}
\mathbf{l}^1 &= [-\alpha_1, -\alpha_2, -\alpha_3, \xi, (\alpha_1^2 - \alpha_2^2)/k_c, (1 - 3\alpha_3^2)/(\sqrt{3} k_c), 2\alpha_1\alpha_2/k_c, 2\alpha_1\alpha_3/k_c, 2\alpha_2\alpha_3/k_c] \\
\mathbf{l}^2 &= [0, \alpha_3, -\alpha_2, 0, \alpha_2\alpha_3, -\sqrt{3}\alpha_2\alpha_3, -\alpha_1\alpha_3, \alpha_1\alpha_2, \alpha_2^2 - \alpha_3^2] \\
\mathbf{l}^3 &= [\alpha_3, 0, -\alpha_1, 0, -\alpha_1\alpha_3, -\sqrt{3}\alpha_1\alpha_3, -\alpha_2\alpha_3, -\alpha_2^2 + \alpha_1^2, \alpha_1\alpha_2] \\
\mathbf{l}^4 &= [\alpha_2, -\alpha_1, 0, 0, -2\alpha_1\alpha_2, 0, -\alpha_2^2 + \alpha_1^2, -\alpha_2\alpha_3, \alpha_1\alpha_3].
\end{aligned} \quad (16)$$

The desired differential equations along the bicharacteristics are obtained from

$$\mathbf{I}^i \cdot \mathbf{L}(\mathbf{W}) = 0 \quad (i=1, 2, 3, 4) \quad (17)$$

where the dot denotes the inner product. The derivative of a function $\mathbf{W}$ in a bicharacteristic direction is given by

$$\frac{d\mathbf{W}}{dt} = \mathbf{W}_t + \mathbf{W}_x \frac{dx}{dt} + \mathbf{W}_y \frac{dy}{dt} + \mathbf{W}_z \frac{dz}{dt} \quad (18)$$

where dx_i/dt ($i=1, 2, 3$) along the bicharacteristics are obtained by the differentiation of Eq. (13). By substituting Eqs. (16) and (18) into Eq. (17), we have

$$a_1 du + a_2 dv + a_3 dw + a_1^2 dl + a_2^2 dm + a_3^2 dn + 2a_1 a_2 dp + 2a_1 a_3 ds + 2a_2 a_3 dq = -S_1(a_1, a_2, a_3) dt \quad (19)$$

$$-a_3 dv + a_2 dw - k_c a_2 a_3 dm + k_c a_2 a_3 dn - k_c a_1 a_3 dp + k_c a_1 a_2 ds + k_c(a_2^2 - a_3^2) dq = -S_2(a_1, a_2, a_3) dt \quad (20)$$

$$-a_3 du + a_1 dw - k_c a_1 a_3 dl + k_c a_1 a_3 dn - k_c a_2 a_3 dp + k_c(a_1^2 - a_2^2) ds + k_c a_1 a_2 dq = -S_3(a_1, a_2, a_3) dt \quad (21)$$

$$-a_2 du + a_1 dv - k_c a_1 a_2 dl + k_c a_1 a_2 dm + k_c(a_1^2 - a_2^2) dp - k_c a_2 a_3 ds + k_c a_1 a_3 dq = -S_4(a_1, a_2, a_3) dt \quad (22)$$

where

$$\begin{aligned} S_1(a_1, a_2, a_3) = & (k_c^2 - 2)\{(a_1^2 - 1)u_x + a_1 a_2 u_y + a_1 a_3 u_z + a_1 a_2 v_x + (a_2^2 - 1)v_y + a_2 a_3 v_z + a_1 a_3 w_x + a_2 a_3 w_y \\ & + (a_3^2 - 1)w_z\}/k_c^2 - a_1(1 - a_1^2)l_x + a_1^2 a_2 l_y + a_1^2 a_3 l_z + a_1 a_2^2 m_x - a_2(1 - a_2^2)m_y + a_2^2 a_3 m_z \\ & + a_2^2 a_1 n_x + a_2 a_3^2 n_y - a_3(1 - a_3^2)n_z - a_2(1 - 2a_1^2)p_x - a_1(1 - 2a_2^2)p_y + 2a_1 a_2 a_3 p_z \\ & - a_3(1 - 2a_1^2)s_x + 2a_1 a_2 a_3 s_y - a_1(1 - 2a_3^2)s_z + 2a_1 a_2 a_3 q_x - a_3(1 - 2a_2^2)q_y - a_2(1 - 2a_3^2)q_z \\ & + 2\langle D \rangle \{(3a_1^2 - 1)l + (3a_2^2 - 1)m + (3a_3^2 - 1)n + 6(a_1 a_2 p + a_1 a_3 s + a_2 a_3 q)\}/3k_c^2 \end{aligned} \quad (23)$$

$$\begin{aligned} S_2(a_1, a_2, a_3) = & (a_1 a_3 u_y - a_1 a_2 u_z + a_2 a_3 v_y - a_2^2 v_z + a_3^2 w_y - a_2 a_3 w_z)/k_c - a_1 a_2 a_3 m_x - a_3(a_2^2 - 1)m_y \\ & - a_2 a_3^2 m_z + a_1 a_2 a_3 n_x + a_2^2 a_3 n_y - a_2(1 - a_3^2)n_z - a_3(a_1^2 - 1)p_x - a_1 a_2 a_3 p_y - a_1 a_3^2 p_z \\ & - a_2(1 - a_1^2)s_x + a_1 a_2^2 s_y + a_1 a_2 a_3 s_z + a_1(a_2^2 - a_3^2)q_x - a_2(1 - a_2^2 + a_3^2)q_y \\ & + a_3(1 + a_2^2 - a_3^2)q_z + 2\langle D \rangle \{(n - m)a_2 a_3 - a_1 a_3 p + a_1 a_2 s + (a_2^2 - a_3^2)q\}/k_c \end{aligned} \quad (24)$$

$$\begin{aligned} S_3(a_1, a_2, a_3) = & (a_1 a_3 u_x - a_1^2 u_z + a_2 a_3 v_x - a_1 a_2 v_z + a_3^2 w_x - a_1 a_3 w_z)/k_c - a_3(a_1^2 - 1)l_x - a_1 a_2 a_3 l_y \\ & - a_1 a_3^2 l_z + a_1^2 a_3 n_x + a_1 a_2 a_3 n_y - a_1(1 - a_3^2)n_z - a_1 a_2 a_3 p_x - a_3(a_2^2 - 1)p_y - a_2 a_3^2 p_z \\ & - a_1(1 - a_1^2 + a_3^2)s_x + a_2(a_1^2 - a_3^2)s_y + a_3(1 + a_1^2 - a_3^2)s_z + a_1^2 a_2 q_x - a_1(1 - a_2^2)q_y \\ & + a_1 a_2 a_3 q_z + 2\langle D \rangle \{-a_1 a_3(l - m) - a_2 a_3 p + (a_1^2 - a_3^2)s + a_1 a_2 q\}/k_c \end{aligned} \quad (25)$$

$$\begin{aligned} S_4(a_1, a_2, a_3) = & (a_1 a_2 u_x - a_1^2 u_y + a_2^2 v_x - a_1 a_2 v_y + a_2 a_3 w_x - a_1 a_3 w_y)/k_c - a_2(a_1^2 - 1)l_x - a_1 a_2^2 l_y \\ & - a_1 a_2 a_3 l_z + a_1^2 a_2 m_x - a_1(1 - a_2^2)m_y - a_1 a_2 a_3 m_z - a_1(1 - a_1^2 + a_2^2)p_x + a_2(1 + a_1^2 - a_2^2)p_y \\ & + a_3(a_1^2 - a_2^2)p_z - a_1 a_2 a_3 s_x - a_2^2 a_3 s_y - a_2(a_3^2 - 1)s_z + a_1^2 a_3 q_x + a_1 a_2 a_3 q_y \\ & - a_1(1 - a_3^2)q_z + 2\langle D \rangle \{a_1 a_2(m - l) + (a_1^2 - a_2^2)p - a_2 a_3 s + a_1 a_3 q\}/k_c. \end{aligned} \quad (26)$$

Equation (19) is the differential relation along the bicharacteristics on the dilatational wave surface (i.e., $c=1$), while Eqs. (20), (21) and (22) are the differential relations along the bicharacteristics on the shear wave surface (i.e., $c=1/k_c$).

Integration of Eqs. (19) - (22) along the bicharacteristics from the point (t_0, x_0, y_0, z_0) to the point of intersection of the bicharacteristics with the plane $t=t_0-K_t$ gives

$$\begin{aligned} a_1 \delta u + a_2 \delta v + a_3 \delta w + a_1^2 \delta l + a_2^2 \delta m + a_3^2 \delta n + 2a_1 a_2 \delta p + 2a_1 a_3 \delta s + 2a_2 a_3 \delta q \\ = -K_t [(S_1(a_1, a_2, a_3))_i + (S_1(a_1, a_2, a_3))^\circ]/2 - W_1(a_1, a_2, a_3) + 0(K_t^3) \end{aligned} \quad (27)$$

$$\begin{aligned} -a_3 \delta v + a_2 \delta w - k_c a_2 a_3 \delta m + k_c a_2 a_3 \delta n - k_c a_1 a_3 \delta p + k_c a_1 a_2 \delta s + k_c(a_2^2 - a_3^2) \delta q \\ = -K_t [(S_2(a_1, a_2, a_3))_i + (S_2(a_1, a_2, a_3))^\circ]/2 - W_2(a_1, a_2, a_3) + 0(K_t^3) \end{aligned} \quad (28)$$

$$\begin{aligned} -a_3 \delta u + a_1 \delta w - k_c a_1 a_3 \delta l + k_c a_1 a_3 \delta n - k_c a_2 a_3 \delta p + k_c(a_1^2 - a_2^2) \delta s + k_c a_1 a_2 \delta q \\ = -K_t [(S_3(a_1, a_2, a_3))_i + (S_3(a_1, a_2, a_3))^\circ]/2 - W_3(a_1, a_2, a_3) + 0(K_t^3) \end{aligned} \quad (29)$$

$$\begin{aligned} -a_2 \delta u + a_1 \delta v - k_c a_1 a_2 \delta l + k_c a_1 a_2 \delta m + k_c(a_1^2 - a_2^2) \delta p - k_c a_2 a_3 \delta s + k_c a_1 a_3 \delta q \\ = -K_t [(S_4(a_1, a_2, a_3))_i + (S_4(a_1, a_2, a_3))^\circ]/2 - W_4(a_1, a_2, a_3) + 0(K_t^3) \end{aligned} \quad (30)$$

$$\begin{aligned} W_1(a_1, a_2, a_3) = & [a_1 u + a_2 v + a_3 w + a_1^2 l + a_2^2 m + a_3^2 n + 2a_1 a_2 p + 2a_1 a_3 s + 2a_2 a_3 q]_i \\ & - [a_1 u + a_2 v + a_3 w + a_1^2 l + a_2^2 m + a_3^2 n + 2a_1 a_2 p + 2a_1 a_3 s + 2a_2 a_3 q]_i \end{aligned} \quad (31)$$

$$\begin{aligned} W_2(a_1, a_2, a_3) = & [-a_3 v + a_2 w - k_c a_2 a_3 m + k_c a_2 a_3 n - k_c a_1 a_3 p + k_c a_1 a_2 s + k_c(a_2^2 - a_3^2)q]_i \\ & - [-a_3 v + a_2 w - k_c a_2 a_3 m + k_c a_2 a_3 n - k_c a_1 a_3 p + k_c a_1 a_2 s + k_c(a_2^2 - a_3^2)q]_i \end{aligned} \quad (32)$$

$$\begin{aligned} W_3(a_1, a_2, a_3) = & [-a_3 u + a_1 w - k_c a_1 a_3 l + k_c a_1 a_3 n - k_c a_2 a_3 p + k_c(a_1^2 - a_2^2)s + k_c a_1 a_2 q]_i \\ & - [-a_3 u + a_1 w - k_c a_1 a_3 l + k_c a_1 a_3 n - k_c a_2 a_3 p + k_c(a_1^2 - a_2^2)s + k_c a_1 a_2 q]_i \end{aligned} \quad (33)$$

$$\begin{aligned} W_4(a_1, a_2, a_3) = & [-a_2 u + a_1 v - k_c a_1 a_2 l + k_c a_1 a_2 m + k_c(a_1^2 - a_2^2)p - k_c a_2 a_3 s + k_c a_1 a_3 q]_i \\ & - [-a_2 u + a_1 v - k_c a_1 a_2 l + k_c a_1 a_2 m + k_c(a_1^2 - a_2^2)p - k_c a_2 a_3 s + k_c a_1 a_3 q]_i \end{aligned} \quad (34)$$

where the operator δ is defined by $\delta \mathbf{W} = \mathbf{W}^\circ(t_0, x_0, y_0, z_0) - \mathbf{W}_\circ(t_0 - K_t, x_0, y_0, z_0)$. The superscript " $\circ$ " denotes evaluation of the function at the point (t_0, x_0, y_0, z_0) and the subscript " $\circ$ " denotes evaluation of the function at

the point $(t_0 - K_t, x_0, y_0, z_0)$. The subscripts "i" and "i'" denote evaluation of the function at the points $(t_0 - K_t, x_0 + K_t\alpha_1, y_0 + K_t\alpha_2, z_0 + K_t\alpha_3)$ and $(t_0 - K_t, x_0 + K_tk_c\alpha_1, y_0 + K_tk_c\alpha_2, z_0 + K_tk_c\alpha_3)$, respectively, where the bicharacteristics intersect the plane $t = t_0 - K_t$.

Nine additional differential equations, obtained by integrating Eq. (4) along the time axis, are

$$\delta u = K_t[l_{x_0} + p_{y_0} + s_{z_0} + l_x^0 + p_y^0 + s_z^0]/2 \quad (35)$$

$$\delta v = K_t[p_{x_0} + m_{y_0} + q_{z_0} + p_x^0 + m_y^0 + q_z^0]/2 \quad (36)$$

$$\delta w = K_t[s_{x_0} + q_{y_0} + n_{z_0} + s_x^0 + q_y^0 + n_z^0]/2 \quad (37)$$

$$(\delta l + \delta m + \delta n)/3\xi = K_t\xi[u_{x_0} + v_{y_0} + w_{z_0} + u_x^0 + v_y^0 + w_z^0]/2 \quad (38)$$

$$k_c(\delta l - \delta m)/2 = K_t[-(-u_{x_0} + v_{y_0} - u_x^0 + v_y^0)/k_c - \langle D_0 \rangle(l_0 - m_0)/k_c - \langle D^0 \rangle(l^0 - m^0)/k_c]/2 \quad (39)$$

$$k_c(\delta l + \delta m - 2\delta n)/2\sqrt{3} = K_t[(u_{x_0} + v_{y_0} - 2w_{z_0} + u_x^0 + v_y^0 - 2w_z^0)/3k_c - \langle D_0 \rangle(l_0 + m_0 - 2n_0)/\sqrt{3}k_c - \langle D^0 \rangle(l^0 + m^0 - 2n^0)/\sqrt{3}k_c]/2 \quad (40)$$

$$k_c\delta p = K_t[(v_{x_0} + u_{y_0} + v_x^0 + u_y^0)/k_c - 2(\langle D_0 \rangle p_0 + \langle D^0 \rangle p^0)/k_c]/2 \quad (41)$$

$$k_c\delta s = K_t[(u_{x_0} + w_{y_0} + u_x^0 + w_y^0)/k_c - 2(\langle D_0 \rangle s_0 + \langle D^0 \rangle s^0)/k_c]/2 \quad (42)$$

$$k_c\delta q = K_t[(v_{x_0} + w_{y_0} + v_x^0 + w_y^0)/k_c - 2(\langle D_0 \rangle q_0 + \langle D^0 \rangle q^0)/k_c]/2. \quad (43)$$

The superscripts and subscripts have the same meaning as in Eqs. (27) - (30).

For simplification, we denote Eq. (27) corresponding to the particular values of α_1, α_2 and α_3 by $a_j(\alpha_1, \alpha_2, \alpha_3)$ ($j=1, 2, 3, 4, 5, 6$), and Eqs. (28), (29) and (30), for the particular values of α_1, α_2 and α_3 , respectively, by $b_k(\alpha_1, \alpha_2, \alpha_3)$, $c_k(\alpha_1, \alpha_2, \alpha_3)$ and $d_k(\alpha_1, \alpha_2, \alpha_3)$ ($k=1, 2, 3, 4$).

All the terms on the right-hand side of Eqs. (27) - (34) and (35) - (43) can be evaluated from data on the plane $t = t_0 - K_t$, except those terms having the superscript "°". All the terms involving partial derivatives at the point (t_0, x_0, y_0, z_0) can be eliminated by forming linear combinations of Eqs. (27) - (34) and (35) - (43). Consequently, we can obtain nine equations which determine the unknown increments $\delta u, \delta v, \delta w, \delta l, \delta m, \delta n, \delta p, \delta q$ and δs .

$$a_1 - a_2 - c_3 + c_4 - d_3 + d_4 - 4(32)|2\delta u \quad (44)$$

$$a_3 - a_4 - b_3 + b_4 + d_1 - d_2 - 4(33)|2\delta v \quad (45)$$

$$a_5 - a_6 + b_1 - b_2 + c_1 - c_2 - 4(34)|2\delta w \quad (46)$$

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 - 4(k_c^2 - 2)(35)/k_c^2|2(\delta l + \delta m + \delta n)/\xi^2/3 \quad (47)$$

$$a_1 + a_2 - a_3 - a_4 + 2(k_c^2 - 2) \times k_c(36)/k_c^2|k_c^2(\delta l + \delta m) \quad (48)$$

$$[a_1 + a_2 + a_3 + a_4]/2 - a_5 - a_6 + \sqrt{3}(k_c^2 - 2)k_c(37)/k_c^2|k_c^2(\delta l + \delta m - 2\delta n)/2 \quad (49)$$

$$d_1 + d_2 - d_3 - d_4 - 2(38)|2k_cp \quad (50)$$

$$b_1 + b_2 - b_3 - b_4 - 2(40)|2k_cq \quad (51)$$

$$c_1 + c_2 - c_3 - c_4 - 2(39)|2k_cs \quad (52)$$

where the terms on the right side of the line "|" are the unknown terms that occur in the equation resulting from the corresponding combination. Though the equations include the unknown quantity D^0 , the system is solved by successive approximations.

In order to convert Eqs. (44) - (52) to difference equations, the partial derivatives with respect to x, y and z are replaced by the corresponding difference equations. For example, the following equations are used:

$$Wx = \{W(x+h, y, z) - W(x-h, y, z)\}/2h$$

$$Wxx = \{W(x+h, y, z) + W(x-h, y, z) - 2W(x, y, z)\}/h^2$$

$$Wxy = \{W(x+h, y+h, z) + W(x-h, y-h, z) - W(x+h, y-h, z) - W(x-h, y+h, z)\}/4h^2 \quad (53)$$

where h is the spatial interval.

2.3 Boundary conditions

For points on the boundary or the corner, we can make use of boundary conditions instead of relations along the bicharacteristics involving data outside the region of interest. The boundary conditions are as follows:

(1) Normal impact

Impact region: $w = W(t), q = 0, s = 0$;

Top surface of the plate excluding the impact region:

$$n = 0, q = 0, s = 0;$$

Bottom face of the plate: $n = 0, q = 0, s = 0$.

$$(54)$$

(2) Oblique impact

Impact region: $u = U(t), v = V(t), w = W(t)$;

Top surface of the plate excluding the impact region:

$$n = 0, q = 0, s = 0;$$

Bottom face of the plate: $n = 0, q = 0, s = 0$.

$$(55)$$

As the initial condition, it is assumed that the stress components and the particle velocity component are all zero at $t = 0$.

3. Numerical Results and Discussions

Numerical calculations have been made using the following parameters: thickness of the plate $h = 20$ mm and the representative length $b = 20$ mm. Since the material is assumed to be mild steel, the parameters^{(11),(12)} are chosen as

$$\rho = 7.86 \times 10^3 \text{ kg/m}^3, c_1 = 5940 \text{ m/s},$$

$$c_2 = 3175 \text{ m/s},$$

$$\eta^* = 1437 \text{ s}^{-1}, \tau^* = 275.8 \text{ MPa},$$

$$D' = 248.2 \text{ MPa}.$$

In order to express the oblique impact velocity as a function of direction and time, we divide the particle velocity $V(t)$ (as shown in Fig. 2) into three components u, v and w in the x, y - and z -directions, respectively. The symbols ϕ and θ are, respectively, the angle between $V(t)$ and the x -axis, and that between $V(t)$ and the z -axis (x and z are both positive). In the numerical calculation, the impact angle ϕ and θ are chosen as $\theta = 0^\circ, 45^\circ$ and 60° , and $\phi = 45^\circ$.

The function of the impact velocity $W(t)$ is assumed as

$$W(t) = \begin{cases} 0 & \text{for } t < 0, \\ W_m t/t_1 & \text{for } 0 \leq t < t_1, \\ -W_m(t-t_2)/(t_2-t_1) & \text{for } t_1 \leq t < t_2, \\ 0 & \text{for } t \geq t_2, \end{cases} \quad (56)$$

where $W_m = W_0/c_1$, W_0 is the maximum impact velocity in the z -direction and t_1 is a coefficient corresponding to the nondimensional rising time. In this calculation, $t_1 = 0.45$, $t_2 = 0.9$ and $W_0 = 60 \text{ m/s}$. For the oblique impact, the functions of the impact velocity in the x - and y -directions are of the same type as those shown in Eq. (56). The impact velocity is assumed as

$$\sqrt{U_0^2 + V_0^2 + W_0^2} = 60 \text{ m/s} \quad (57)$$

where U_0 and V_0 are the maximum impact velocities in the x - and y -directions.

Fig. 2 Impact velocity

Fig. 3 Distribution of the stress $n(=\sigma_z)$ in the thick plate

By carrying out the numerical calculation which involves changing the ratio of the dimensionless time interval k to the dimensionless spatial interval h , we can confirm the accuracy of the numerical solutions. Accuracy of computation is evaluated by the energy method in which the relative energy error $[(E_{\text{inp}} - E_{\text{int}})/E_{\text{inp}}]$ is examined, using both input energy E_{inp} from the impact end and energy E_{int} possessed by the cylinder. In this study, we take $k = 0.05$ and $h = 0.1$, which agree with those in the study concerning the stability of difference equations⁽⁸⁾. Relative errors are within 4.0% for our case.

Figure 3 shows the stress variation $n(=\sigma_z)$ along the z axis, when normal impact is applied to the thick plate. The compressive stress wave travels out in the plate from the top surface (where $z=0$) and is reflected at the free bottom face of the plate (where $z=10$). A large tensile stress is generated at $z=8$ by the reflected tensile stress wave, when $t=1.65$. This tensile stress is considered to be a source of spall fracture⁽¹⁴⁾. The effect of viscoplasticity on the stress distribution is also found. The stress n becomes zero at $z=0$ in Fig. 3, because the impact velocity disappears when $t=0.9$.

When an impulsive load is applied to a body made of brittle materials which are weak in resisting the large tensile stress, it is considered that the position of fracture is closely related to the position of the highest tensile stress. In addition, even if the material is not brittle, it is considered that the position of fracture caused by repeated impulsive loads closely corresponds to the position where the tensile stress is generated. In this regard, we investigate the maximum value of the tensile stress generated in the plate.

In the first place, we investigate the generation of spall fracture by examining the tensile stress $n(=\sigma_z)$. Figure 4 shows positions where the maximum tensile stress n_{max} is generated in the elastic plate. The symbols $\bullet$, $\times$ and $\blacktriangle$ show the positions where the maximum tensile stress is generated when $\theta = 0^\circ, 45^\circ$ and 60° , respectively. The part subjected to the impulsive load is hatched in the figure. The position where the maximum tensile stress is generated in the plate shifts along the direction of oblique impact. Figures 5, 6 and 7 show distributions of tensile stresses on the z -plane, including the position where the maximum tensile stress n_{max} is generated, when $\theta = 0^\circ, 45^\circ$ and 60° , respectively. Figure 5 shows that the large tensile stress is centered at the origin (where $x=0$ and $y=0$). If spall fracture is generated by the large tensile stress, this point is considered to be the position where spall fracture will be generated. Figures 6 and 7 show that the position of the maximum tensile stress shifts along the direction of oblique impact, and that the

Fig. 4 Position of the maximum value of the tensile stress

Fig. 5 Distribution of the tensile stress ($n_{\max}$, $\theta=0^\circ$) Fig. 6 Distribution of the tensile stress ($n_{\max}$, $\theta=45^\circ$) Fig. 7 Distribution of the tensile stress ($n_{\max}$, $\theta=60^\circ$)

Fig. 8 Position of the maximum value of the tensile principal stress

values of the maximum tensile stresses become smaller as the impact angle θ increases. Spall fracture is not generated when the impact angle between the direction of the impact velocity and the plate becomes acute.

In order to clarify impact fracture of thick plates in more detail, we investigate the maximum value of the tensile principal stress $\sigma_{\max}$ generated at various points in the plate. In Fig. 8, the symbols ●, × and ▲ also show the positions where the maximum tensile stress is generated when $\theta=0^\circ$, 45° and 60° , respectively. The part subjected to the impulsive load is hatched in the figure. The maximum principal stress is generated on the bottom free face of the plate when $\theta=0^\circ$.

In this case, it is predicted that the plate will crack at this point. When $\theta = 45^\circ$ and 60° , maximum tensile stresses are generated on the surface subjected to the impulsive load. In these cases, the formation of surface cracks is predicted. Therefore, when the impact angle between the direction of the impact velocity and the plate becomes acute, impact fracture on the surface is generated and spall fracture is not.

From these numerical results, it is confirmed that the propagation of three-dimensional stress waves can be investigated with high accuracy using this numerical method. With the introduction of the theoretical model for the formation of the crack and crack propagation, this method may be applied to the numerical simulation of a complicated phenomenon such as impact damage.

4. Conclusions

Three-dimensional stress wave propagation in the elastic/viscoplastic body was analyzed using the finite difference method based on integration along the bicharacteristics. A numerical example was presented for the case of three-dimensional stress wave propagation in the elastic/viscoplastic thick plate, when oblique impact was applied to the surface at various impact angles. From the numerical results for various loading conditions, it was found that impact fracture is generated in the free bottom face of the plate and the region near the bottom in the case of normal impact, and that impact fracture is generated on the top surface when the impact angle between the direction of the impact velocity and the plate becomes acute. From these results, it is also found that stress wave propagation due to oblique impact can be investigated with high accuracy using this numerical method.

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