Generation Method of Distributed Data for FEM Analysis*

Yoshihiro OCHIAI**

Many automatic mesh generation methods for the finite element method (FEM) have been reported. However, for the case of complicated heat generation, a large number of data depending on the position must be added to the mesh data. Other examples, which also need a large number of data depending on the position, are functionally gradient material and biomechanics. In these cases, it is difficult to prepare the distributed data. This paper shows that these problems can be solved by using an improved multiple-reciprocity boundary element method. In this method, contour lines of distribution are used and these distributions are assumed to satisfy the Poisson equation approximately.

Key Words : Finite-Element Method, Automatic Mesh Generation, Boundary Element Method, Computational Mechanics, Numerical Analysis, Polyharmonic Function

1. Introduction

Many automatic mesh generation methods for the finite element method have been reported⁽¹⁾⁻⁽³⁾. The analysis using the finite element method is easy, because the finite element analysis program on the market includes an excellent preprocessor. However, for a heat conduction analysis with complicated heat generation and the analysis of biomaterials, nonhomogeneous materials and functionally gradient materials, much labor is required to prepare the distributed data. Thus, it is important to establish a method which generates the distributed data in as short a time as possible, because the analysis of biomaterials, nonhomogeneous materials and functionally gradient materials is conducted frequently. When perfect distributed data are obtained by experiment and can be used in analysis, the present method

is not necessary. Several interpolations must be used, if the assumed distribution is not expressed by an easy function. When number of data obtained from experiments are small, the value at an arbitrary point must be interpolated.

There are several interpolation methods available, an example of which is the spline function. However, it is difficult to interpolate the value in an arbitrary domain having only a small number of data at arbitrary points. Therefore, this paper presents an automatic data generation method for the distributed data, which is also effective even in the case of an arbitrary shape and a small number of data available. This method is effective in the case of smooth distribution of data, and is not effective in the case that data hardly change in a small region. The distributed data can be approximated by contour lines in a two-dimensional problem and it can be given by an equivalent-value surface in a three-dimensional problem. The value at an arbitrary point of an element, which is obtained by automatic mesh generation, is interpolated from these data. In the present paper, the value at an arbitrary point is obtained by a boundary integral equation using a polyharmonic function. The present

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** Department of Mechanical Engineering, Faculty of Science and Technology, Kinki University, 3-4-1 Kowakae, Higashi-Osaka 577, Japan

method is an improvement of the multiple-reciprocity boundary element method for heat conduction analysis with heat generation⁽⁴⁾. The method, which is used to obtain the total amount of the distributed value, the average value and the gradient of distribution, is shown, because the total amount of distributed value is known or given in some problems dealing with heat generation.

2. Theory

2.1 Interpolation using polyharmonic function

The interpolation method, which uses the polyharmonic function without domain integral, is shown. For ease of understanding, the case of a two-dimensional problem is considered. Distributed data $W_{s1}(x, y)$ within the region are assumed to satisfy the Poisson equation approximately:

$$\nabla^2 W_{s1} = -W_{s2} - W_{L2} - W_{P2} - W_{D2} \quad (1)$$

where W_{P2} represents a value only at a point, W_{L2} represents a value only on a line, and W_{s2} represents the value on a surface. On the other hand, W_{D2} corresponds to a double layer in the potential theory. Generally, we obtain

$$\nabla^2 W_{sf} = -W_{sf+1} - W_{Lf+1} - W_{Pf+1} - W_{Df+1}. \quad (2)$$

$$\begin{aligned} cW_{s1}(P) = & -\sum_{f=1}^F (-1)^f \int_{\Gamma} \{T_f(P, Q) \frac{\partial W_{sf}(Q)}{\partial n} - \frac{\partial T_f(P, Q)}{\partial n} W_{sf}(Q)\} d\Gamma - \sum_{f=1}^F (-1)^f \sum_{m=1}^M T_f W_{Pmf+1} \\ & - \sum_{f=1}^F (-1)^f \int_{\Gamma_L} T_f W_{Lf+1} d\Gamma_L - \sum_{f=1}^F (-1)^f \int_{\Gamma_D} \frac{\partial T_f}{\partial n} W_{Df+1} d\Gamma_D \end{aligned} \quad (8)$$

where $c=0.5$ on the smooth boundary, and $c=1$ in the domain. Moreover, in Eq.(8) is similarly given by

$$\begin{aligned} cW_{sf}(P) = & \sum_{e=f}^F (-1)^{e-f} \int_{\Gamma} \left\{ T_{e-f+1}(P, Q) \frac{\partial W_{se}(Q)}{\partial n} - \frac{\partial T_{e-f+1}(P, Q)}{\partial n} W_{se}(Q) \right\} d\Gamma \\ & + \sum_{e=f}^F (-1)^{e-f} \sum_{m=1}^M T_{e-f+1} W_{Pmf+1} + \sum_{e=f}^F (-1)^{e-f} \int_{\Gamma_L} T_{e-f+1} W_{Lf+1} d\Gamma_L + \sum_{e=f}^F (-1)^{e-f} \int_{\Gamma_D} \frac{\partial T_{e-f+1}}{\partial n} W_{Df+1} d\Gamma_D. \end{aligned} \quad (9)$$

In the two-dimensional problem, the polyharmonic function T_f , in Eq.(8) is given by

$$T_f = \int_{\Gamma} \frac{1}{r} \left[\int_{\Gamma} T_{f-1} dr \right] dr. \quad (10)$$

The polyharmonic function of order f , T_f and its normal derivative are generally given by

$$T_f = \frac{r^{2(f-1)}}{2\pi[(2f-2)!!]^2} \left[\ln\left(\frac{1}{r}\right) + \operatorname{sgn}(f-1) \sum_{e=1}^{f-1} \frac{1}{e} \right] \quad (11)$$

$$\frac{\partial T_f}{\partial n} = \frac{r^{2f-3}}{2\pi[(2f-2)!!]^2} \frac{\partial r}{\partial n} \left[2(f-1) \ln\left(\frac{1}{r}\right) + 2(f-1) \sum_{e=1}^{f-1} \frac{1}{e} - 1 \right] \quad (12)$$

where $(2f)!! = 2f(2f-2) \cdots 4 \cdot 2^{(7)}$.

In the three-dimensional problem, the volume integral of W_{v2} , which distributes in the three-dimensional domain, is transformed into the surface integral using Green's theorem.

In the case of the three-dimensional problem, the fundamental solution of T_1 and its normal derivative are given by

$$T_1(P, Q) = \frac{1}{4\pi r} \quad (13)$$

$$\frac{\partial T_1(P, Q)}{\partial n} = \frac{-1}{4\pi r^2} \frac{\partial r}{\partial n}. \quad (14)$$

The polyharmonic function T_f in the three-dimensional problem is obtained as

$$T_f = \int_{\Gamma} \frac{1}{r^2} \left[\int_{\Gamma} T_{f-1} dr \right] dr. \quad (15)$$

The polyharmonic function T_f in the three-dimensional problem and its normal derivative are generally given

Moreover, in the case of $f=F$, we assume $W_{SF+1}=0$ and obtain

$$\nabla^2 W_{SF} = -W_{LF+1} - W_{PF+1} - W_{DF+1}. \quad (3)$$

Equations (1)-(3) can be expressed as follows:

$$\begin{aligned} \nabla^{2F} W_1 = & (-)^F [W_{LF+1} + W_{PF+1} + W_{DF+1}] \\ & + \operatorname{sgn}(F-1) \sum_{f=1}^{F-1} (-1)^{F+f} \\ & \times \nabla^{2f} [W_{LF-f+1} + W_{PF-f+1} + W_{DF-f+1}] \end{aligned} \quad (4)$$

where $\operatorname{sgn}(\)$ is the sign function. Fundamental solution $T_1(P, Q)$ for the Laplace equation in the two-dimensional problem and its derivative are given by^{(5),(6)}

$$T_1(P, Q) = \frac{1}{2\pi} \ln\left(\frac{1}{r}\right) \quad (5)$$

$$\frac{\partial T_1(P, Q)}{\partial n} = \frac{-1}{2\pi r} \frac{\partial r}{\partial n} \quad (6)$$

where r is the distance between the two points P and Q . On the other hand, the polyharmonic function T_f is defined by

$$\nabla^2 T_f = T_{f-1}. \quad (7)$$

Denoting the number of point sources W_{Pf} as M , the shapes of line source W_{Lf} and the double layer source W_{Df} as Γ_L and Γ_D , and the boundary of region Ω as Γ , the following equation is obtained using Eqs. (1)~(6) and Green's theorem

by

$$T_f = \frac{r^{2f-3}}{4\pi(2f-2)!!(2f-3)!!} \quad (16)$$

$$\frac{\partial T_f}{\partial n} = \frac{r^{2f-4}}{4\pi(2f-2)!!(2f-5)!!} \frac{\partial r}{\partial n} \quad (17)$$

where $0!! = (-1)!! = 1$. Moreover, the polyharmonic function in two-dimensional and three-dimensional problems has the following property :

$$\lim_{f \rightarrow \infty} \frac{T_{f+1}}{T_f} = 0. \quad (18)$$

Therefore it becomes clear that the interpolation is possible using the boundary integral equation (8), if the distributed data W_{s1} satisfy Eq. (4).

2.2 Sum of distributed quantity and gradient

Next, the total amount of distributed value, for example, the distribution of heat generation, and the average value of data are obtained using a boundary integral equation. If the distributed data W_{s1} satisfy Eq. (4), the next equation is obtained using Green's theorem.

$$\begin{aligned} \int_{\Omega} W_{s1} d\Omega = & - \sum_{f=1}^F (-1)^f \int_{\Gamma} \left\{ \psi_f(P, Q) \frac{\partial W_{sf}(Q)}{\partial n} - \frac{\partial \psi_f(P, Q)}{\partial n} W_{sf}(Q) \right\} d\Gamma(Q) \\ & + \sum_{f=1}^F (-1)^f \int_{\Gamma_L} \psi_f W_{Lf+1} d\Gamma_L + \sum_{f=1}^F (-1)^f \int_{\Gamma_D} \frac{\partial \psi_f}{\partial n} W_{Df+1} d\Gamma_D + \sum_{f=1}^F (-1)^f \sum_{m=1}^M \psi_f W_{Pmf+1} \end{aligned} \quad (19)$$

where the function ψ_f and its normal derivative are given by

$$\psi_f = \frac{r^{2f}}{[(2f)!!]^2} \quad (20)$$

$$\frac{\partial \psi_f}{\partial n} = \frac{r^{2f-1}}{(2f)!!(2f-2)!!} \frac{\partial r}{\partial n}. \quad (21)$$

Using Eq.(21), the area of domain Ω can be obtained by

$$\int_{\Omega} d\Omega = \int_{\Omega} \frac{\partial \psi_1}{\partial n} d\Gamma. \quad (22)$$

The average value can be obtained using Eqs.(19) and (22). Similarly, the values of the three-dimensional problem can be calculated and the function ψ_f and its normal derivative for the three-dimensional problem are given by

$$\psi_f = \frac{r^{2f}}{(2f+1)!!(2f)!!} \quad (23)$$

$$\frac{\partial \psi_f}{\partial n} = \frac{r^{2f-1}}{(2f+1)!!(2f-2)!!} \frac{\partial r}{\partial n}. \quad (24)$$

On the other hand, differentiating Eq. (8) with respect to x_i gives the gradient of the distributed data as follows :

$$\begin{aligned} \frac{\partial W_{s1}(P)}{\partial x_i} = & - \sum_{f=1}^F (-1)^f \int_{\Gamma} \left\{ \frac{\partial T_f(P, Q)}{\partial x_i} \frac{\partial W_{sf}(Q)}{\partial n} - \frac{\partial^2 T_f(P, Q)}{\partial x_i \partial n} W_{sf}(Q) \right\} d\Gamma - \sum_{f=1}^F (-1)^f \sum_{m=1}^M \frac{\partial T_f}{\partial x_i} W_{Pmf+1} \\ & - \sum_{f=1}^F (-1)^f \int_{\Gamma_L} \frac{\partial T_f}{\partial x_i} W_{Lf+1} d\Gamma_L - \sum_{f=1}^F (-1)^f \int_{\Gamma_D} \frac{\partial^2 T_f}{\partial x_i \partial n} W_{Df+1} d\Gamma_D \end{aligned} \quad (25)$$

where

$$\frac{\partial T_f}{\partial x_i} = \frac{r^{2f-3} r_{,i}}{2\pi[(2f-2)!!]^2} \left[2(f-1) \ln\left(\frac{1}{r}\right) + 2(f-1) \sum_{e=1}^{f-1} \frac{1}{e} - 1 \right] \quad (26)$$

$$\frac{\partial^2 T_f}{\partial x_i \partial n} = \frac{r^{2f-4}}{2\pi[(2f-2)!!]^2} \left\{ \left[n_i + 2(f-2) r_{,i} \frac{\partial r}{\partial n} \right] \left[2(f-1) \ln\left(\frac{1}{r}\right) + 2(f-1) \sum_{e=1}^{f-1} \frac{1}{e} - 1 \right] - 2(f-1) r_{,i} \frac{\partial r}{\partial n} \right\} \quad (27)$$

and $r_{,i} = \partial r / \partial x_i$. The functions $\partial T_f / \partial n$ and $\partial^2 T_f / \partial x_i \partial n$ in the case of the three-dimensional problem are generally given by

$$\frac{\partial T_f}{\partial x_i} = \frac{r_{,i} r^{2f-4}}{4\pi(2f-2)!!(2f-5)!!} \quad (28)$$

$$\begin{aligned} \frac{\partial^2 T_f}{\partial x_i \partial n} = & \frac{r^{2f-5}}{4\pi(2f-2)!!(2f-5)!!} \\ & \times \left[n_i + (2f-5) r_{,i} \frac{\partial r}{\partial n} \right]. \end{aligned} \quad (29)$$

2.3 Example of application of integral equation

In practice, the distributed data W_{s1} are given, but $W_{s2} \sim W_{sf}$ and W_{Lf} , W_{Pf} in Eqs.(8) and (9) are not.

Therefore, an example for obtaining unknowns W_{sf} ($F > 1$) and W_{Lf} is shown. For ease of understanding, $F=2$ is assumed and constant elements are used for the boundary and the shape of Γ_L . Replacing W_{sf} and $\partial W_{sf} / \partial n$ by vectors $\mathbf{W}_f$ and $\mathbf{V}_f$, respectively, and discretizing Eq.(8), we obtain

$$\mathbf{H}_1 \mathbf{W}_1 = \mathbf{G}_1 \mathbf{V}_1 + \mathbf{H}_2 \mathbf{W}_2 - \mathbf{G}_2 \mathbf{V}_2 - \mathbf{G}_{L2} \mathbf{W}_{L3} \quad (30)$$

where $\mathbf{H}_1$, $\mathbf{G}_1$, $\mathbf{H}_2$, $\mathbf{G}_2$ and $\mathbf{G}_{L2}$ are the matrices with the following elements :

$$H_{1ij} = \frac{1}{2} \delta_{ij} + \int_{\Gamma_j} \frac{\partial T_i(P, Q)}{\partial n} d\Gamma_j \quad (31)$$

$$G_{1ij} = \int_{\Gamma_j} T_1(P, Q) d\Gamma_j \quad (32)$$

$$H_{2ij} = \int_{\Gamma_j} \frac{\partial T_2(P, Q)}{\partial n} d\Gamma_j \quad (33)$$

$$G_{2ij} = \int_{\Gamma_j} T_2(P, Q) d\Gamma_j \quad (34)$$

$$G_{L2ij} = \int_{\Gamma_{Lj}} T_2(P, q_L) d\Gamma_{Lj} \quad (35)$$

When the case of $F=2$ is considered, W_{s2} is obtained by Eq.(9) as follows:

$$\mathbf{H}_1 \mathbf{W}_2 = \mathbf{G}_1 \mathbf{V}_2 + \mathbf{G}_{L1} \mathbf{W}_{L3} \quad (36)$$

where $\mathbf{G}_{L1}$ is the matrix with the following element:

$$G_{L1ij} = \int_{\Gamma_{Lj}} T_1(P, q_L) d\Gamma_{Lj} \quad (37)$$

Moreover, making use of the contour lines for the quantity $W(p_L)$, we obtain by Eq.(8)

$$\begin{aligned} W(p_L) = & -\mathbf{H}_3 \mathbf{W}_1 + \mathbf{G}_3 \mathbf{V}_1 + \mathbf{H}_4 \mathbf{W}_2 \\ & -\mathbf{G}_4 \mathbf{V}_2 - \mathbf{G}_{L3} \mathbf{W}_{L3}, \end{aligned} \quad (38)$$

where $\mathbf{H}_3$, $\mathbf{G}_3$, $\mathbf{H}_4$, $\mathbf{G}_4$ and $\mathbf{G}_{L3}$ are the matrices with the following elements:

$$H_{3ij} = \int_{\Gamma_j} \frac{\partial T_1(p_L, Q)}{\partial n} d\Gamma_j \quad (39)$$

$$G_{3ij} = \int_{\Gamma_j} T_1(p_L, Q) d\Gamma_j \quad (40)$$

$$H_{4ij} = \int_{\Gamma_j} \frac{\partial T_2(p_L, Q)}{\partial n} d\Gamma_j \quad (41)$$

$$G_{4ij} = \int_{\Gamma_j} T_2(p_L, Q) d\Gamma_j \quad (42)$$

$$G_{L3ij} = \int_{\Gamma_{Lj}} T_2(p_L, q_L) d\Gamma_{Lj} \quad (43)$$

Assuming that $\mathbf{W}_2 = \mathbf{0}$, the following equation is obtained using Eqs. (30), (36) and (38):

$$\begin{bmatrix} \mathbf{G}_1 & -\mathbf{G}_2 & -\mathbf{G}_{L2} \\ \mathbf{0} & \mathbf{G}_1 & \mathbf{G}_{L1} \\ \mathbf{G}_3 & -\mathbf{G}_4 & -\mathbf{G}_{L3} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{W}_{L3} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_1 \mathbf{W}_1 \\ \mathbf{0} \\ \mathbf{H}_3 \mathbf{W}_1 + \mathbf{W}(p_L) \end{bmatrix} \quad (44)$$

From Eq.(44) we obtain $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{W}_{L3}$. In the above example, $\mathbf{W}_2 = \mathbf{0}$ is assumed, and $\mathbf{V}_2 = \mathbf{0}$ can also be assumed in the same way. If the boundary and contour lines are divided into N_0 and N_1 elements using constant elements, the simultaneous linear algebraic equations with $(FN_0 + N_1)$ unknowns must be solved. Generally many contour lines are used, and accordingly we have to solve the simultaneous equations with a large matrix. Therefore, we have only to divide the given domain into subdomains as shown in Fig. 1, where Eq.(4) is satisfied approximately⁽⁸⁾. Each of these subdomains is called a "boundary-type cell". In particular, these boundary-type cells are effective in

Fig. 1 Boundary-type cell

solving the three-dimensional problem, because the matrix of simultaneous equations of the problem is large. Moreover, these boundary-type cells effectively express the discontinuous values in a composite material.

On the other hand, W_{Lf} is used in Eq.(44), and the point source W_{Pf} can be used in the same way. There is, however, a problem of strong singularity in the boundary integral equation. The singularity becomes weak if a regularized boundary integral equation⁽⁹⁾ is adopted.

3. Procedure of Analysis

After the shape of an object for finite element analysis is given, the mesh is obtained by automatic mesh generation. Next, the value representing physical properties or those of heat generation in each element depending on the position are calculated using the given values. In this calculation, the mesh on the boundary obtained by automatic mesh generation is used. Next, the contour lines in the two-dimensional problem or the equivalent-value surfaces in the three-dimensional problem are given, and these lines or surfaces are divided into elements. If W_{Pf} is used, the value at that point is given. After obtaining the values on the boundary and contour lines using Eq.(44), the value representing physical property or those of heat generation on the mesh, which is generated automatically, are calculated using Eq.(8).

If many contour lines are used, the complicated distribution can be expressed by only $F=2$. If a higher derivative on the boundary is assigned, $F \geq 3$ must be used.

4. Example of Analysis

It is investigated whether the present method can express reasonably the distribution by the given contour lines in the two-dimensional problems. Figure 2 shows the shape and contour lines at intervals of 10,

Fig. 2 Given shape and contour lines

Fig. 3 Obtained distribution for contour lines of Fig. 2

Fig. 4 Detailed contour lines of Fig. 2

Fig. 5 Distribution of gradient

Fig. 6 Given shape and point values

Fig. 7 Obtained distribution

Fig. 3 the obtained distribution and Fig. 4 the detailed contour lines obtained by the calculation. Figure 5 shows the gradient of the obtained distribution in the case of $F=2$, and the average value of the distribution, which is obtained by Eqs.(19) and (20), is 8.53. The obtained surface goes exactly through the midpoints of the divided line in Fig.2, because a constant element is used.

In the next example, the values at an arbitrary point instead of the contour lines are used. Figure 6 shows an example of input data, where values are represented by the length of line segments. Figure 7 shows a distribution under the condition that the boundary is zero. $F=2$ is also used in this example, and the obtained surface of the distribution goes exactly through the given points in Fig.6.

Next, as an example of the three-dimensional problem, the distribution in a cube of sides 10 mm long as shown in Fig. 8 is obtained. In Fig. 9, $W_1=10$ at $x=0$ mm, $W_1=0$ at $x=10$ mm and $\partial W_1/\partial n=0$ on the other surfaces are shown two-dimensionally. The equivalent-value surfaces of $W_1=3$ and $W_1=7$ are

Fig. 8 Given shape and elements

Fig. 9 Given equivalent-value surface

Fig. 10 Obtained distribution of Fig. 9

shown in the top view and front view of Fig. 9, respectively. Figure 10 shows the distributions at heights $z=2$ mm and $z=8$ mm in the case of $F=2$.

5. Conclusion

In biomaterials and functional gradient materials, values representing physical properties are dependent on the position. It is difficult to prepare the data depending on the position in a short time for the finite element analysis. It is shown that it becomes easy to prepare the data, if the distributed data are assumed to satisfy the Poisson equation. In the present method, the distributed data can be obtained from the contour lines in the two-dimensional problem (equivalent-value surface in the three-dimensional problem) or the values at arbitrary points using the improved multiple-reciprocity boundary element method.

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