Study on Accuracy of Finite-Element Solutions in Elastoplastic Large Deformation*

(Effects of Shape Function and Numerical Integration, and Application of Mixed Method)

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We discuss the accuracy of finite-element solutions for metals possessing dominant plasticity, resulting in an incompressible response in a large deformation field. It is known that poor numerical solutions are obtained for the constrained problem due to incompressibility of deformed metals, but they can be improved by selecting an appropriate shape function and numerical integration technique, as well as by applying the mixed method derived from Lagrangian multipliers. Many studies have been made for rigid-plastic finite-element solutions so far, but large-deformation elastoplastic structural analysis is rarely discussed in the literature. In this work, we discuss the advantages of such techniques in large-deformation analysis using the Jaumann stress rate and isotropic hardening hypoelasticity model.

Key Words: Finite-Element Method, Mixed Method, Incompressibility, Numerical Integration, Shape Function, Large-Deformation Analysis

1. Introduction

In order to carry out reasonable design of metal sheet plastic working, high accuracy of numerical solutions such as for stress and deformation are required. The displacement-based finite-element method using the stationary principle of potential energy is widely employed in numerical analysis of elastoplastic deformation, therefore, the present authors have discussed effects of the hypoelastic constitutive equation and plastic hardening laws on numerical simulation⁽¹⁾. It is pointed out that when using the displacement method, the accuracy of numerical solutions, such as for pressure, decreases markedly due to the volume constraint of plastic incompressibility⁽²⁾. This is because the independent degrees of freedom of displacement decrease due to

internal constraints of dilatational deformation in a plastic field when plasticity becomes dominant. In this case, it is known that the internal constraints can be relaxed by adopting an appropriate shape function and numerical integration technique to obtain accurate numerical solutions⁽³⁾. However, elastoplastic large-deformation analyses to study effects of shape function and integration technique on the accuracy of numerical solutions in the case of incompressibility constraint of plasticity are few. Although the mixed method or the penalty method has been proposed, where pressure or dilatational strain is assumed as an unknown quantity in addition to displacement velocity⁽⁴⁾⁻⁽⁷⁾, most of the analyses are focused on the rigid-plastic finite-element method.

The objectives of the present study are to discuss the effects of shape functions, including typical triangular and quadrilateral isoparametric elements, and full- or reduced-integration techniques on numerical simulations by the displacement method for a generalized shear problem in a plane-strain field, to formulate the mixed method based on the mixed variational principle using displacement- and pressure-rate^{(8),(9)}, and also to compare numerical differences between

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the displacement method and the mixed method.

2. Effects of Shape Function and Numerical Integration

2.1 Displacement-based finite-element method

The updated Lagrange formulation based on the current configuration C is employed, where the stationary principle of potential energy is expressed as a function of velocity $\mathbf{v}$ as⁽¹⁾

$$\Pi^p(\mathbf{v}) = \int \left\{ U_T + \frac{1}{2} \mathbf{T} : (\nabla \mathbf{v} \cdot \mathbf{v} \nabla) \right\} dV - \int \rho \dot{\mathbf{a}} \cdot \mathbf{v} dV - \int \dot{\mathbf{t}} \cdot \mathbf{v} dS, \quad (1)$$

where $\mathbf{T}$ is the Cauchy true stress tensor, ∇ is the gradient operator, ρ is the density of material, $\dot{\mathbf{a}}$ is the body force rate, and $\dot{\mathbf{t}}$ is the surface force rate. The notation $(\dot{})$ indicates the material time derivative. Moreover, U_T is defined as internal energy due to the Truesdell stress rate $\dot{\mathbf{T}}_T$ as

$$\delta U_T = \dot{\mathbf{T}}_T : \delta \mathbf{D} \quad (2.a)$$

$$\dot{\mathbf{T}}_T \equiv \dot{\mathbf{T}} - \mathbf{L} \cdot \mathbf{T} - \mathbf{T} \cdot \mathbf{L}^t + (\text{tr } \mathbf{D}) \mathbf{T}, \quad (2.b)$$

where $\mathbf{L}$ is the velocity gradient tensor, $\mathbf{D}$ is the stretching tensor, and the notations $()^t$ and $\text{tr}()$ mean the transpose of the tensor and the inner product between the identity tensor of second order $\mathbf{I}$ and the tensor in the parentheses, respectively. Furthermore, numerical solutions calculated in the present paper employ the hypoelastoplastic constitutive equation using the Jaumann rate and isotropic plastic hardening laws⁽¹⁾. Moreover, we consider the experimental results of Swift⁽¹⁰⁾ for mild steel to approximate the uniaxial stress-strain curve as

$$T_{11} = 179 + 345 \{1 - \exp(-3.40\xi)\} \quad [\text{MPa}].$$

The yield function $f(\xi)$ of isotropic hardening is assumed to be

$$f(\xi) = 1 + \sum_{i=1}^n a_i \{1 - \exp(-\gamma_i \xi)\}.$$

Thus, the material constants are obtained as

$$T_y = 179 [\text{MPa}], \quad a_1 = 1.93, \quad \gamma_1 = 3.40.$$

2.2 Constraints due to plastic deformation

Error estimates in finite-element analysis in an elastic field are usually made using Sobolev norms. If the displacement $\mathbf{u}$ is in Hilbert space of $(k+1)$ order, that is, if $\mathbf{u} \in H^{k+1}$, the error $\mathbf{e}$ between approximate and exact solutions is bounded by

$$\|\mathbf{e}\|_m \leq c h^{k+1-m} \|\mathbf{u}\|_{k+1}, \quad (3)$$

where m is the order of the highest derivatives appearing in the energy expression, k is the degree of polynomial completeness of the shape functions and h is the mesh parameter defined as the diameter of the smallest circle that contains the largest element of the mesh, while c is a constant independent of k and m . With increasing k or decreasing h , the order of convergence will be increased when m is assumed to be constant in each element, and accurate solutions can

be expected when employing high-order elements or fine-mesh discretization. Moreover, decreasing m increases the order of convergence when k is assumed to be constant for all elements, which indicates that the displacement is more accurate than the strain for the same mesh.

However, constrained deformation on an element in large-deformation elastoplastic analysis occurs due to the incompressibility of metals in plastic fields. For the above reason, increasing the number of elements by adopting a fine mesh does not always increase the degrees of nonconstraint freedom, which is different from the estimation using Eq. (3). The internal constraints depend not only on the employed shape function but also on the element configurations. In the present study, we will use the typical triangular and quadrilateral isoparametric elements, widely used in finite element method plane problem to discuss the effects of four kinds of shape functions which approximate displacement rate in linear or nonlinear fields. We also consider the effects of different mesh configurations of triangular element.

Table 1 shows the relationship between degrees of freedom and constraints for the displacement method for four kinds of elements identified by Nagtegaal et al.⁽³⁾ and Hughes⁽¹¹⁾, as well as for the mixed method for two kinds of elements discussed in a later section. The black points in the table denote the displacement freedom, while the white points express the pressure freedom. The ratio between degrees of freedom and constraint shown in the right-hand column of Table 1 shows that the independent degrees of freedom for displacements are divided by those of constraints in an element for dilatation deformations; thus, the optimal behavior is obtained for $r=2$, and severe locking is expected for $r \leq 1$. Moreover, for 3-node and 6-node triangular element, the upper and lower figures represent r values for bisection pattern and cross-diagonal pattern mesh, respectively. The values in Table 1 correspond to fine mesh, where $p \equiv \text{constraints/element}$ on the left side is the number of constraints for one element, $q \equiv \text{nodes/element}$ in the center is the number of nodes for one element, and $r \equiv \text{freedom/constraint}$ on the right side is the degrees of freedom for one constraint. Since the degrees of freedom for one node equals 2 in the plane problem, r can be defined by $2q/p$. For example, for the bisection pattern of the triangular constant-strain element, r equals 1 because $p=1$ and $q=1/2$. However, p equals $3/4$ for the cross-diagonal pattern, since incompressibility constraints among the four elements in a quadrilateral are not independent. On the other hand, for the mixed method, the number of dilatation constraints is

Table 1 Relationship between degrees of freedom and constraint for employed element in analysis

element type		constraints element	nodes element	freedom constraint
displacement method		1	1/2	1
	constant strain triangle	3/4	1/2	4/3
	3	1	2/3
	linear strain triangle	3	2	4/3
		11/4	2	16/11
mixed method		6	3	1
	u-bilinear p-constant	1	1	2
	u-serendipity p-bilinear	4	3	3/2

consistent with the number of pressure nodes, since the pressure is regarded as an unknown independent degree of freedom, where $p=1$ for a 4-node element, and $p=4$ for an 8-node element.

The accuracy for a 4-node element decreases due to the small value of $r=2/3$ for the displacement method; however, for the cross-diagonal pattern of 3-node and 6-node elements, accurate solutions can be expected since constraint ratio $r=4/3=1.33$ and $r=16/11=1.45$, respectively. The constraint ratio r in the mixed method is improved relative to that in the displacement method.

2.3 Effects of shape function

The generalized shear problem of the rectangular block with aspect ratio of 2 analyzed by the present authors⁽¹⁾ is taken as an example in the present study. To discretize the model with nearly the same nodes for various kinds of elements, the mesh is discretized with 281 nodes for both the constant-strain (3-node) and linear-strain (6-node) triangular elements, with 325 nodes for the quadrilateral (4-node) element, and with 253 nodes for the serendipity (8-node) element.

Fig. 1 Effects on load-deformation curve of shape function using full-integration technique

Fig. 2 Effects of shape function on deformed mesh configuration and contour of plastic strain using full integration

The full-integration technique is employed in the calculation for all the elements.

Figure 1 shows the load-deformation curve, where shear force F and axial force N , which are the resultant forces occurred on the top surface of the block, are plotted as a function of shear deformation e/a . The displacement e is the horizontal deformation on the left top, and a is the width of the rectangular block. The deformed mesh configuration and the plastic strain contour at $e/a=2$ for all kinds of elements are shown in Fig. 2. The points in the deformed mesh configuration are the Gaussian points and indicate the state of loading of plasticity.

It is observed in Fig. 1 that for the 4-node quadrilateral element, the shear force F is almost the same as F for other elements, but the axial force N is

higher than N for others. This is due to the locking phenomenon which can be inferred from the deformed mesh configuration as well as the plastic strain contour in Fig. 2. These results correspond to those in Table 1 for the 4-node element where constraint r is less than 1. The initial part of the load-deformation curve in the 3-node element is rigid compared to other elements, because the stiffness in the elastic field is estimated to be high for a constant-strain element, while the solutions converge for other elements like the 6-node or 8-node element in the dominantly plastic field.

2.4 Effects of reduced integration techniques

The numerical solutions using the 4-node quadrilateral element with full integration show rigid results, as was indicated in section 2.3. It is known that the reduced-integration technique may improve the accuracy of solutions in the case of locking phenomena, compared to those using the full-integration technique. According to Hughes' proposal⁽¹¹⁾, the optimum number of Gaussian reduced-integration points can be calculated as

$$n_G = \bar{k} + k - 2m, \quad (4)$$

where $\bar{k}$ denotes the order of the highest-order monomial in the shape function. Consequently, the numbers of Gaussian reduced-integration points for various kinds of elements are calculated as follows.

4-node element: $n_G = 2 + 1 - 2 = 1$

6-node element: $n_G = 2 + 2 - 2 = 2$

8-node element: $n_G = 3 + 2 - 2 = 3$

The number of reduced-integration points is consistent with the number of full-integration points for 3-node constant-strain element. In the present paper, for the sake of convenience, we employ the Gauss integration techniques using one point for 3-node and 4-node elements, three points for the 6-node element and 2×2 points for the 8-node element.

Figure 3 shows the load-deformation curve, where each of the four kinds of elements exhibits the

Fig. 3 Effects on load-deformation curve of shape function using reduced-integration technique

same behavior. However, the hourglass deformation, appearing in the free boundary of both sides, especially the top-left and bottom-right parts, is observed on the deformed mesh configuration and plastic strain contour of Fig. 4 for the 4-node element.

The comparisons of axial force N between the full-integration and reduced-integration techniques is shown in Fig. 5. For the 4-node element, the results obtained by the full-integration technique differs significantly from those by other models, while the same difference is observed even for the 8-node element.

2.5 Effects of element configuration

The ratio r of degrees of freedom vs constraint depends on mesh arrangement even if the same triangular element is adopted. For example, in the case of the 3-node linear element, the constraint ratio is improved from 1 in the case of the bisection pattern to

Fig. 4 Effects of shape function on deformed mesh configuration and contour of plastic strain using reduced integration

Fig. 5 Comparison between full- and reduced-integration techniques on load-deformation curve

4/3 in the case of the cross-diagonal pattern. These patterns are shown in Fig. 6.

Effects of the triangular element arrangement are shown in Fig. 7 and Fig. 8 through numerical solutions of the load-deformation curve, deformed configuration and plastic strain contour. In the case of the bisection pattern, the direction of the diagonal significantly affects the solutions. On the right bisection pattern, locking phenomenon occurs, as is observed in the 4-node quadrilateral element with full integration, while the solution of the left bisection pattern is completely different from the others.

2.6 Remarks

The small amount of computer time required is an advantage of using the 4-node quadrilateral element when the model is discretized with the same degrees of freedom as other models. However, plastic locking is caused in the full integration, and the hourglass deformation appears in the reduced integration. In order to suppress the hourglass deformation, the mixed method, the selective reduced-integration method, utilization of incompatible elements and the hourglass stabilization stiffness method have been proposed⁽¹¹⁾⁻⁽¹³⁾. In the present study, we will show that the locking phenomenon and the hourglass mode

can be suppressed by utilizing the mixed method.

In the case of equal dilatation constraints of plasticity, highly accurate solutions are obtained using the 3-node element with a cross-diagonal pattern and the 6-node or 8-node element with the reduced-integration technique. The cross-diagonal arrangement of triangular elements can improve the effects of constraint, and the direction of the diagonal greatly affects the numerical solution for the bisection pattern.

In the case of 3-node elements with bisection pattern and 4-node elements with the full-integration technique, the elements adjacent to the free boundary of both right and left sides should be perpendicular to the boundary after deformation, since the shear force is not applied at the boundary. However, due to constraint, such is not the case.

3. Application of Mixed Method

3.1 Mixed variational principle

By applying the Lagrange-multiplier method for the stationary principle of potential energy used as a basis of the displacement method, we derive the mixed variational principle where pressure stress increment is unknown. The mixed variational principle using the displacement and pressure-rate proposed by Watanabe^{(8),(9)} is formulated in terms of the current quantities of Cauchy stress $\mathbf{T}$ by employing the updated Kirchhoff stress $\mathbf{T}_{(uK)}$ (which is convenient to describe the constitutive equation of Jaumann type), and the variation of internal energy U_I of Eq. (1) may be separated into deviatoric and dilatation components of

$$\delta U_I = \delta U_I' + \delta U_{Im} \quad (5.a)$$

(a) bisection (right) (b) bisection (left) (c) cross-diagonal

Fig. 6 Configurations of three kinds of elements

Fig. 7 Effects on load deformation curve of different configurations

Deformed mesh configuration at $e/a=2.0$

Contour of plastic strain at $e/a=2.0$
right (12x24) left (12x24) cross-diagonal (8x16)

Fig. 8 Deformed mesh configuration and contour of plastic strain for different configurations

$$\delta U'_t = \mathbf{T}'_t : \delta \mathbf{D}' \quad (5.b)$$

$$\delta U_{Tm} = 3 \dot{\mathbf{T}}_{Tm} \delta D_m, \quad (5.c)$$

where the notations ()' and ()_m signify deviatoric and dilatational components, respectively. The deviatoric and dilatational components of Truesdell's stress rate can be expressed as follows using the deviatoric rate component $\dot{\mathbf{S}}_{(uK)}$ and dilatational rate component $\dot{T}_{(uK)m}$ of updated Kirchhoff stress $\mathbf{T}_{(uK)}$:

$$\dot{\mathbf{T}}_{Tm} = \dot{T}_{(uK)m} - \frac{2}{3}(\mathbf{S} : \mathbf{D}') - 4 T_m D_m \quad (6.a)$$

$$\mathbf{T}'_T = \dot{\mathbf{S}}_{(uK)} - (\mathbf{L} \cdot \mathbf{T})' - (\mathbf{T} \cdot \mathbf{L}')', \quad (6.b)$$

where

$$\dot{T}_{(uK)m} = \dot{T}_m + 3 T_m D_m, \quad \dot{\mathbf{S}}_{(uK)} = \dot{\mathbf{S}} + 3 S D_m. \quad (7)$$

The tensor $\mathbf{S}$ is the deviatoric component of $\mathbf{T}$. Equation (6.b) is rearranged using Jaumann's stress rate as

$$\begin{aligned} \mathbf{T}'_T &= \dot{\mathbf{S}}_{(uK)J} - \mathbf{D}' \cdot \mathbf{S} - \mathbf{S} \cdot \mathbf{D}' - 2 D_m \mathbf{S} \\ &\quad - 2 T_m \mathbf{D}' + \frac{2}{3}(\mathbf{S} : \mathbf{D}') \mathbf{I}, \end{aligned} \quad (8)$$

where Jaumann's stress rate is given by

$$\dot{\mathbf{S}}_{(uK)J} = \dot{\mathbf{S}}_{(uK)J} - \mathbf{W} \cdot \mathbf{S} - \mathbf{S} \cdot \mathbf{W}. \quad (9)$$

Based on the plastic constitutive equation of the internal time theory⁽¹⁾, the constitutive equation of the deviatoric component can be given as

$$\dot{\mathbf{S}}_{(uK)J} = 2 \mu_0 \left[\mathbf{D}' - \frac{(\mathbf{S} - \mathbf{r})}{C(S_{yf}^0)^2} \langle (\mathbf{S} - \mathbf{r}) : \mathbf{D}' \rangle \right]. \quad (10)$$

The constitutive equation for the dilatational component can be assumed to have the linear form of

$$\dot{T}_{(uK)m} = 3 K_0 D_m. \quad (11)$$

The Hellinger-Reissner mixed variational principle can be obtained when complementary energy due to the dilatational stress rate is substituted into the stationary principle of the potential energy of Eq. (1), where the dilatation work is introduced as an additional term through the Legendre transformation.

$$\begin{aligned} \Pi^{HR}(\mathbf{v}, \dot{T}_{(uK)m}) &= \int \left\{ U_J(\mathbf{D}') - \mathbf{S} : \mathbf{D}'^2 - 2 D_m (\mathbf{S} : \mathbf{D}') \right. \\ &\quad - T_m (\mathbf{D}' : \mathbf{D}') - 6 T_m D_m^2 + \frac{1}{2} \mathbf{T} : (\nabla \mathbf{v} \cdot \nabla \mathbf{v}) \\ &\quad - \frac{1}{2 K_0} (\dot{T}_{(uK)m})^2 + 3 \dot{T}_{(uK)m} D_m \left. \right\} dV \\ &\quad - \int \rho \dot{\mathbf{a}} \cdot \mathbf{v} dV - \int \dot{\mathbf{t}} \cdot \mathbf{v} dS_t - \int \dot{\mathbf{t}} \cdot (\mathbf{v} - \bar{\mathbf{v}}) dS_v \end{aligned} \quad (12)$$

The additional condition is $D_m = (\nabla \mathbf{v})_m$. Moreover, $U_J(\mathbf{D}')$ is the internal work of deviatoric component of stress, expressed as

$$\begin{aligned} \delta \Pi^{HR} &= \int \left[\left\{ \frac{\partial U_J}{\partial \mathbf{D}'} - 2 \mathbf{S} \cdot \mathbf{D}' - 2 D_m \mathbf{S} - 2 T_m \mathbf{D}' \right\} : \delta \mathbf{D}' \right. \\ &\quad + \{ 3 \dot{T}_{(uK)m} - 2 (\mathbf{S} : \mathbf{D}') - 6 T_m D_m \} \delta D_m \\ &\quad + \mathbf{T} : (\nabla \mathbf{v} \cdot \delta \mathbf{v}) - \left(\frac{1}{K_0} \dot{T}_{(uK)m} \right. \\ &\quad \left. \left. - 3 D_m \right) \delta \dot{T}_{(uK)m} \right] dV - \int \rho \dot{\mathbf{a}} \cdot \delta \mathbf{v} dV \end{aligned}$$

$$- \int \dot{\mathbf{t}} \cdot \delta \mathbf{v} dS_t - \int \{ \delta \dot{\mathbf{t}} \cdot (\mathbf{v} - \bar{\mathbf{v}}) + \dot{\mathbf{t}} \cdot \delta \mathbf{v} \} dS_v, \quad (13)$$

where

$$\begin{aligned} U_J(\mathbf{D}') &= \frac{1}{2} \dot{\mathbf{S}}_{(uK)J} : \mathbf{D}' \\ &= \mu_0 \left[(\mathbf{D}' : \mathbf{D}') - \frac{\langle (\mathbf{S} - \mathbf{r}) : \mathbf{D}' \rangle^2}{C(S_{yf}^0)^2} \right]. \end{aligned} \quad (14)$$

3.2 Discretization

The variables of the mixed variational principle in Eq. (13) are the displacement rate field and the pressure rate field, and different shape functions are applied for those fields⁽¹¹⁾. The mean stress field $\dot{T}_{(uK)m}$ is discretized by nodal pressure rate $\dot{p}_c$, and the displacement rate field $\mathbf{v}_i$ on a finite element is discretized by nodal displacement rate $\dot{q}_a$.

$$v_i = v_{ia} \dot{q}_a, \quad (a=1, 2, \dots) \quad (15.a)$$

$$\dot{T}_{(uK)m} = T_{(uK)mc} \dot{p}_c, \quad (c=1, 2, \dots) \quad (15.b)$$

Here, v_{ia} and $T_{(uK)mc}$ are shape functions. The other quantities can be calculated using v_{ia} as follows.

$$v_{i,j} = v_{ia,j} \dot{q}_a \equiv v_{ija} \dot{q}_a \quad (16.a)$$

$$D_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}) = \frac{1}{2} (v_{ija} + v_{jia}) \dot{q}_a \equiv D_{ija} \dot{q}_a \quad (16.b)$$

$$D_m = \frac{v_{i,i}}{3} = \frac{v_{iia}}{3} \dot{q}_a \equiv D_{ma} \dot{q}_a \quad (16.c)$$

$$D'_{ij} = D_{ij} - D_m \delta_{ij} = (D_{ija} - D_{ma} \delta_{ij}) \dot{q}_a \equiv D'_{ija} \dot{q}_a \quad (16.d)$$

The stiffness equation obtained from Eq. (14) is

$$\begin{bmatrix} K_{ab} & H_{ad} \\ G_{cb} & -M_{cd} \end{bmatrix} \begin{Bmatrix} \dot{q}_b \\ \dot{p}_d \end{Bmatrix} = \begin{Bmatrix} \dot{f}_a \\ 0 \end{Bmatrix}, \quad (17)$$

where matrices K_{ab} and M_{cd} are both symmetric and positive definite, where K_{ab} is comprised of the four matrices as

$$K_{ab} = K_{ab}^{(0)} + K_{ab}^{(1)} + K_{ab}^{(2)} + K_{ab}^{(3)}, \quad (18)$$

where the matrix in Eqs. (17) and (18) is defined as follows.

$$K_{ab}^{(0)} = \int A_{ijkl} D'_{ija} D'_{kjb} dV \quad (19.a)$$

$$K_{ab}^{(1)} = \int T_{ij} v_{kia} v_{kjb} dV \quad (19.b)$$

$$K_{ab}^{(2)} = -2 \int S_{ij} D'_{ika} D'_{kjb} dV \quad (19.c)$$

$$\begin{aligned} K_{ab}^{(3)} &= -2 \int \{ S_{ij} D_{ma} D'_{ijb} + S_{ij} D_{mb} D'_{ijb} \\ &\quad + 3 T_m D_{ma} D_{mb} + T_m D'_{ija} D'_{ijb} \} dV \end{aligned} \quad (19.d)$$

$$H_{ad} = 3 \int T_{(uK)md} D_{ma} dV \quad (19.e)$$

$$G_{cb} = 3 \int T_{(uK)mc} D_{mb} dV = H_{bc} \quad (19.f)$$

$$M_{cd} = \frac{1}{K_0} \int T_{(uK)mc} T_{(uK)md} dV \quad (19.g)$$

$$f_a = \int \rho \dot{\mathbf{a}}_i v_{ia} dV + \int \dot{\mathbf{t}}_i v_{ia} dS_t \quad (19.h)$$

The coefficient tensor A_{ijkl} in Eq. (19.a) is calculated as follows according to Eq. (14).

$$A_{ijkl} = \mu_0 (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}) - 2\mu_0 \frac{(S_{ij} - r_{ij})(S_{kl} - r_{kl})}{C(S_{y/f}^0)^2} \quad (20)$$

The same isoparametric element used in the displacement method is applied for the displacement rate in the mixed method, and the order of shape function for the pressure rate is lower than that for displacement rate⁽¹¹⁾. The pressure shape function for the 4-node mixed element in Table 1 is constant and discontinuous between the adjacent elements. Moreover, the pressure shape function for the 8-node mixed element in Table 1 is bilinear and discontinuous between the adjacent elements. It is called as incompatible element.

The full-integration technique is applied in the present mixed method. Gaussian integration points are different due to the different degrees of shape functions for displacement and pressure modes. That is, the integration techniques for both H_{ad} and G_{cb} are taken to be the same as for M_{cd} . Furthermore, if the reduced-integration technique is applied to the matrices for the 4-node mixed element, then the finite-element stiffness matrix obtained from Eq. (14) can be rewritten as follows.

$$(K_{ab} + H_{ad} M_{dc}^{-1} G_{cb}) \dot{q}_b = \dot{f}_a \quad (21)$$

Thus, we find that both the formulation and numerical solution by the mixed method are consistent with those by the displacement method.

3.3 Generalized shear problem

The solutions using the 4-node element show locking or hourglass deformation on adopting full- or reduced-integration techniques, respectively, because in this case the ratio of freedom to constraint is smaller than 1, as discussed at section 2. For this reason, we will use the mixed method to analyze the same generalized shear problem.

The result of the mixed method is shown in Fig. 9 for the load-deformation curve. The shear and axial forces obtained by the mixed method are nearly

Fig. 9 Comparison between displacement and mixed method in load-deformation curve

the same as those obtained by the displacement method with the one-point integration technique.

Comparisons of the deformed configuration and plastic strain contour between the mixed and displacement methods are shown in Fig. 10. The free boundary on the right and left sides become smooth, hourglass deformations are completely suppressed, and a high-precision solution is obtained when the mixed method is applied.

The deformed configuration and plastic strain contour obtained using the 8-node mixed element, as shown in Fig. 11, are consistent with those obtained by the displacement method. No difference is observed between the results obtained using the 4-node and 8-node mixed element with the mixed method.

3.4 Effects of Poisson ratio

Near incompressibility is assumed even though the deformation is elastic, leading to the Poisson ratio of 0.4999. Comparisons of the numerical solutions between the displacement and mixed methods with the 4-node element are made. The load-deformation curve is shown in Fig. 12, where the solutions of the

Fig. 10 Deformed mesh configuration and contour of plastic strain for 4-node element

Fig. 11 Deformed mesh configuration and contour of plastic strain for 8-node mixed element

Fig. 12 Effects of load-deformation curve of Poisson ratio (RI: reduced integration; FI: full integration)

displacement method with the reduced-integration technique and of the mixed method are almost the same, regardless of Poisson ratio. The numerical solution diverges for the displacement method with the full-integration technique for the Poisson ratio of 0.4999. The deformed configuration and plastic strain contour are shown in Fig. 13. The hourglass deformation occurs in the case of the displacement method with the reduced-integration technique; however, deformation is smooth in the case of the mixed method. It can be considered, that the mixed method is an effective method for treating incompressible material with the Poisson ratio $\nu_0 \approx 0.5$, such as rubber.

3.5 Comparison of computational time

It is important to examine computational time as well as the accuracy of solutions. Comparisons of CPU time taken to calculate the generalized shear

Table 2 Comparison of computation time for different elements and methods

element type	displacement method		mixed method
	reduced integration	full integration	
3-node (nodes 281)	1.45		
6-node (nodes 281)	2.98	5.78	
4-node (nodes 325)	1.00	3.52	7.38
8-node (nodes 253)	2.93	7.60	15.19

Fig. 13 Effects of Poisson ratio on deformed mesh configuration and contour of plastic strain

problem in an engineering workstation are shown in Table 2, where the CPU time taken using the 4-node quadrilateral in the displacement method with reduced-integration techniques is taken as a reference, and the results are given as the ratio of time in each case. The time required increases for the mixed method, since a calculation for the pressure field is added.

4. Conclusions

By using the displacement and mixed finite-element methods, the elastoplastic large-deformation simulation was carried out.

(1) It was clearly proven through numerical examples that for metals possessing dominant plasticity, the accuracy of numerical solutions using the displacement method are dependent on shape functions and integration techniques because of the incompressibility constraint, and accuracy increases if an appropriate shape function and integration technique are selected.

(2) Numerical solution is affected by the element configuration. It was indicated numerically that a more accurate solution is obtained for the cross-diagonal mesh subdivision of the 3-node linear element than for bisection mesh subdivision.

(3) The mixed method based on Hellinger-Reissner's variational principle is formulated, when pressure as well as the displacement is unknown. The mixed method is better than the displacement method for incompressible material with Poisson ratio $\nu \approx 0.5$.

(4) In order to increase the accuracy of solutions, the mixed method is preferable, even though the calculation time is the shortest for the displacement method with a one-point integration technique in the case of the same degrees of freedom.

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this support.

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