Thermal Residual Stresses in Bonded Dissimilar Materials and Their Singularities*

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Thermal residual stress distributions on the interface and in the vicinity of the intersections of the interface and the free surfaces of bonded dissimilar materials are calculated using the boundary element method. Thermoelastic constant stress terms are calculated using Airy's stress function. The thermal residual stresses, when the thermoelastic constant stress terms are subtracted, show free-edge stress singularity. The values of the order of thermal residual stress singularity and stress distribution functions agree well with the theoretical ones calculated based on Airy's stress function. It is shown that the thermal stress singularity disappears for certain ranges of wedge angles of a pair of materials, as predicted.

Key Words: Boundary Element Method, Composite Material, Bonded Dissimilar Materials, Residual Stress, Airy's Stress Function, Stress Singularity

1. Introduction

Thermal residual stresses develop in bonded dissimilar materials during the cooling-down process in manufacture due to the difference in coefficients of thermal expansion of these materials. These stresses may lower the strength of bonded dissimilar materials, and sometimes give rise to damage in them. It is therefore important to evaluate thermal residual stresses developing in bonded dissimilar materials. So far, numerical analyses using the finite element method and the boundary element method, and experimental measurement of thermal residual stresses have been made⁽¹⁾⁻⁽⁷⁾. Because of the difficulties in measuring residual stress and the unestablished nature of free edge stress singularities under thermal stress loading, the method of evaluating thermal residual stress

which develops in bonded dissimilar materials has not been established. The behavior of free edge stress singularities under thermal stress loading remains to be clarified. Some researchers claim that logarithmic singularities develop under thermal stress loading^{(2),(7)}, while others claim that the order of singularity developing under thermal stress loading is the same as that under mechanical loading^{(4),(5)}. Free edge stress singularities may or may not develop under the geometrical condition of no free edge stress singularity under thermal stress loading.

In this study, thermal residual stress distribution on the interface and near the intersection of the interface and the free surface are calculated using the boundary element method^{(7),(8)}. Free edge stress singularities under thermal stress loading are discussed theoretically and numerically, and constant stress terms on the interface and in the bonded dissimilar materials are calculated using Airy's stress function.

2. Thermoelastic Constant Stress Term

Suppose that isotropic materials 1 and 2 having apex angles θ_1 and θ_2 , respectively, are bonded on a straight interface as shown in Fig. 1. For material

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j ($j=1, 2$), shear modulus, Poisson's ratio and coefficient of thermal expansion are denoted by μ_j , ν_j and α_j , respectively.

When bonded dissimilar materials shown in Fig. 1 are cooled by a temperature difference ΔT , distribution of stress σ near the intersection of the interface and the free surface can be written as

$$\sigma = Kr^{-p} + \sigma_0, \quad (1)$$

where r is the distance from the intersection of the interface and the free surface, and p , σ_0 and K are, respectively, the exponent of stress singularity, a thermoelastic constant stress term and the stress intensity factor. Constant stress term σ_0 was calculated by Mizuno et al.⁽⁴⁾ and Munz and Yung⁽⁵⁾ for $\theta_1 = \theta_2 = 90^\circ$, and stress distribution on the interface is written as

$$\sigma_{yy} - i\tau_{xy} = \{(K_1 + iK_2)/\sqrt{2\pi}\}r^{-p} + \sigma_0,$$

where σ_{yy} and τ_{xy} are normal stress and shear stress on the interface, respectively, and $K_1 + iK_2$ is the complex stress intensity factor. The thermoelastic constant term σ_0 is written as

$$\sigma_0 = -\Delta\alpha\Delta E\Delta T \quad (2)$$

$$\Delta\alpha = \begin{cases} \alpha_1 - \alpha_2 & \text{for plane stress} \\ (1 + \nu_1)\alpha_1 - (1 + \nu_2)\alpha_2 & \text{for plane strain} \end{cases}$$

$$\Delta E = [\{ (K_1 - 3)/\mu_1 - (K_2 - 3)/\mu_2 \} / 8]^{-1}$$

$$k_j = \begin{cases} (3 - \nu_j)/(1 + \nu_j) & \text{for plane stress} \\ 3 - 4\nu_j & \text{for plane strain.} \end{cases}$$

In this study, consider the case of a smooth free surface, i.e., $\theta_1 + \theta_2 = 180^\circ$, for the bonded dissimilar wedge shown in Fig. 1. For material j , Airy's stress function is expressed in terms of a complex eigenvalue $\lambda^{(n)}$ ($n=1, 2, \dots$) using polar coordinates (r , θ) with the origin at the intersection of the interface and the free surface, as^{(9),(10)}

$$\phi_j = \sum_{\lambda^{(n)} \neq 1} r^{\lambda^{(n)}+1} F_j(\lambda^{(n)}, \theta) + r^2 G_{jo}(\theta), \quad (3)$$

where

$$F_j(\lambda^{(n)}, \theta) = A_j \sin(\lambda^{(n)} + 1)\theta + B_j \cos(\lambda^{(n)} + 1)\theta + C_j \sin(\lambda^{(n)} - 1)\theta + D_j \cos(\lambda^{(n)} - 1)\theta \quad (4)$$

Fig. 1 Edge-bonded dissimilar wedges

$$G_{jo}(\theta) = A_{jo} \sin 2\theta + B_{jo} \cos 2\theta + C_{jo}\theta + D_{jo}. \quad (5)$$

When stress function is expressed by Eq. (3), stress and strain are expressed in terms of $r^{\lambda^{(n)}-1}$ ($n=1, 2, \dots$). When the smallest eigenvalue in Eq. (4) is denoted by λ , stress is proportional to $r^{\lambda-1}$ in the vicinity of the intersection of the interface and the free surface. The relationship between λ in Eq. (3) and ρ in Eq. (1) is written as

$$\lambda - 1 = -\rho. \quad (6)$$

When thermal strain in material j is written by ϵ_{jo} , the relationship between ϵ_{jo} and elastic displacement parallel to the interface u_{jr} ($j=1, 2$) on interface is given as

$$\epsilon_{10}r + u_{1r} = \epsilon_{20}r + u_{2r}. \quad (7)$$

Since ϵ_{10} and ϵ_{20} are independent of r , the term G_{jo} only, giving linear displacement and constant strain in Eq. (3), should be taken into consideration for satisfying Eq. (7).

The condition of continuity of displacement and traction on interface and the traction-free condition on the free surface are reduced to 8 linear equations for A_{jo} , B_{jo} , C_{jo} , and D_{jo} ($j=1, 2$). A_{jo} , B_{jo} , C_{jo} and D_{jo} are determined by solving the 8 equations simultaneously, and thermoelastic constant stress terms $\sigma_{\theta 0}$, $\sigma_{r 0}$ and $\sigma_{r\theta 0}$ are given as

$$\sigma_{\theta 0} = 2\epsilon^* \Delta T (s_1 \sin 2\theta + c_1 \cos 2\theta - 1)/\delta \quad (8)$$

$$\sigma_{r 0} = 2\epsilon^* \Delta T (-s_1 \sin 2\theta - c_1 \cos 2\theta - 1)/\delta \quad (9)$$

$$\sigma_{r\theta 0} = 2\epsilon^* \Delta T (-s_1 \cos 2\theta + c_1 \sin 2\theta)/\delta, \quad (10)$$

where

$$\delta = (c_1 + 1 - 2\nu_1^*)/\mu_1 - (c_2 + 1 - 2\nu_2^*)/\mu_2 \quad (11)$$

$$s_j = \sin 2\theta_j, c_j = \cos 2\theta_j \quad (j=1, 2)$$

$$\epsilon^* = \begin{cases} \alpha_1 - \alpha_2 & \text{for plane stress} \\ (1 + \nu_1)\alpha_1 - (1 + \nu_2)\alpha_2 & \text{for plane strain} \end{cases}$$

$$\nu_j^* = \begin{cases} \nu_j/(1 + \nu_j) & \text{for plane stress} \\ \nu_j & \text{for plane strain.} \end{cases}$$

Thermoelastic constant stress terms given in Eqs. (8), (9) and (10) approach infinity when $\delta = 0$. From Eq. (11), the condition of $\delta = 0$ is given as

$$(c_1 + 1 - 2\nu_1^*)/\mu_1 = (c_2 + 1 - 2\nu_2^*)/\mu_2. \quad (12)$$

For the case of $\theta_1 + \theta_2 = 180^\circ$, the condition of $\delta = 0$ is reduced to

$$2 \sin^2 \theta_1 = [(\mu_2/\mu_1)(\chi_1 + 1) - (\chi_2 + 1)]/[2(\mu_2/\mu_1 - 1)]. \quad (13)$$

Equation (13) is rewritten in terms of Dunders' parameters α and β as

$$\theta_1 = \sin^{-1} [\pm \sqrt{\alpha/2(\alpha - \beta)}], \quad (14)$$

where

$$\alpha = [(\mu_2/\mu_1)(\chi_1 + 1) - (\chi_2 + 1)] / [(\mu_2/\mu_1)(\chi_1 + 1) + (\chi_2 + 1)]$$

$$\beta = [(\mu_2/\mu_1)(\chi_1 - 1) - (\chi_2 - 1)] / [(\mu_2/\mu_1)(\chi_1 + 1) + (\chi_2 + 1)].$$

Equation (14) corresponds to the borderline condition

between the appearance and disappearance of free edge stress singularity, which was obtained by Kubo and Ohji⁽⁹⁾. The thermoelastic problem can be converted to an elastostatic problem under surface traction. It was claimed that logarithmic free edge stress singularity appears under surface traction^{(11),(12)}. When the geometrical condition (14) is fulfilled, it is predicted that logarithmic free edge stress singularity appears under thermal stress loading for the condition of Eq.(14).

3. Stress Distribution on Interface and in Materials

Thermal residual stress distributions on the interface and in materials are calculated using the boundary element method^{(7),(8)} for a plate shown in Fig. 2. Poisson's ratio and thermal expansion coefficients are kept constant; $\nu_1 = \nu_2 = 0.3$ and $\alpha_1 = 3.0 \times 10^{-6}/K$, $\alpha_2 = 12.0 \times 10^{-6}/K$. Stress distributions are calculated for the case of $\theta_1 = \theta_2 = 90^\circ$, and for the case of $\theta_1 + \theta_2 = 180^\circ$.

3.1 The case of $\theta_1 = \theta_2 = 90^\circ$

For the plate shown in Fig. 2, stress distributions are calculated for $\theta_1 = \theta_2 = 90^\circ$. Ratio of Young's moduli is set at $E_1/E_2 = 2, 5$ and 10 . Figure 3 shows stress distribution on the interface ($\theta = 0$) and on lines of $\theta = \pm 45^\circ$ for $E_1/E_2 = 10$ on a log-log diagram.

Fig. 2 Shape of plate for boundary element analyses

Fig. 3 Thermal residual stress distribution for $\theta_1 = \theta_2 = 90^\circ$ (log-log diagram)

Ordinate and abscissa of the figure show stress normalized by $E_1(\alpha_1 - \alpha_2)\Delta T$, and distance r from intersection of interface and free surface normalized by half width of the interface W , respectively. It is difficult to determine whether free edge stress singularity appears or not from the $r - \sigma$ diagram shown in Fig. 3. The same results of boundary element analyses are shown on a semi-log diagram in Fig. 4. It is difficult to determine what type of free edge stress singularity prevails. However, the absolute values of the slopes of stress distributions increase with decreasing r ; thus, it is concluded that free edge stress singularity appears.

Thermoelastic constant stress terms $\sigma_{\theta 0}$ and $\theta_{r\theta 0}$ are subtracted from σ_θ and $\sigma_{r\theta}$. Subtracted stresses $(\sigma_\theta - \sigma_{\theta 0})$ and $(\sigma_{r\theta} - \sigma_{r\theta 0})$ are shown in Fig. 5 on a log-log diagram. The distributions of absolute values of $(\sigma_\theta - \sigma_{\theta 0})$ and $(\sigma_{r\theta} - \sigma_{r\theta 0})$ increase linearly with decreasing r . Therefore, it is concluded that $(\sigma_\theta - \sigma_{\theta 0})$ and $(\sigma_{r\theta} - \sigma_{r\theta 0})$ have r^{-p} -type free edge stress singularity on the interface and in materials. The exponent of free edge stress singularity p is determined from the stress distributions on the interface in the range of 3.0

Fig. 4 Thermal residual stress distribution for $\theta_1 = \theta_2 = 90^\circ$ (semi-log diagram)

Fig. 5 Subtracted stress distribution for $\theta_1 = \theta_2 = 90^\circ$

$\times 10^{-4} \leq r/W \leq 10^{-2}$ using the least squares method, and is shown in Fig. 6 with the theoretical one^{(9),(10)} obtained using Airy's stress function. The exponent of stress singularity ρ determined from the boundary element results agrees well with the theoretical one.

The distributions of subtracted stresses ($\sigma_\theta - \sigma_{\theta 0}$) and ($\sigma_{r\theta} - \sigma_{r\theta 0}$) on circles with radii of $r = 0.0005W$ and $r = 0.01W$ are calculated using the boundary element method. For the case of $E_1/E_2 = 10$, stress distributions are shown in Fig. 7. Solid lines in Fig. 7 show theoretical distributions calculated using stress function and stress distribution on the interface⁽¹⁰⁾. Boundary element results agree well with the theoretical ones.

3.2 The case of $\theta_1 + \theta_2 = 180^\circ$

For the plate shown in Fig. 2, stress distributions are calculated by changing the value of the apex angle θ_1 and keeping $\theta_1 + \theta_2 = 180^\circ$ for a ratio of Young's moduli $E_1/E_2 = 10$. Boundary element results for $\theta_1 = 110^\circ$ and 125° are shown in Figs. 8(a) and 8(b), respectively. Free edge stress singularity appears for $\theta_1 = 110^\circ$, and disappears for $\theta_1 = 125^\circ$, theoretically^{(9),(10)}. It is difficult to determine whether free

edge stress singularity appears or not from the stress distributions on log-log diagrams shown in Figs. 8(a) and 8(b). The same results of boundary element analyses are shown on semi-log diagrams in Figs. 9(a) and 9(b). It is difficult to determine what type of singularity develops. However, the absolute values of slopes of stress distributions increase with decreasing r in Fig. 9(a); thus, it is concluded that free edge stress singularity appears for the case of $\theta_1 = 110^\circ$. For the case of $\theta_1 = 125^\circ$, the absolute values of slopes of stress distributions decrease with decreasing r in Fig. 9(b); thus, it is concluded that free edge stress singularity disappears.

Using the boundary element results and thermoelastic constant stress term given as Eqs. (8) and (10), subtracted stresses $|\sigma_\theta - \sigma_{\theta 0}|$ and $|\sigma_{r\theta} - \sigma_{r\theta 0}|$ are calculated and the results are shown in Figs. 10(a) and 10(b) for $\theta_1 = 110^\circ$ and 125° , respectively. For $\theta_1 = 110^\circ$, subtracted stresses $|\sigma_\theta - \sigma_{\theta 0}|$ and $|\sigma_{r\theta} - \sigma_{r\theta 0}|$ increase linearly with decreasing r on a log-log diagram; thus, it is seen that $r^{-\rho}$ -type free edge stress singularity appears. For $\theta_1 = 125^\circ$, the subtracted

Fig. 6 Exponent of singularity for $\theta_1 = \theta_2 = 90^\circ$

Fig. 7 Subtracted stress distribution in the vicinity of intersection of interface and free surface for $\theta_1 = \theta_2 = 90^\circ$

(a) $\theta_1 = 110^\circ$

(b) $\theta_1 = 125^\circ$

Fig. 8 Thermal residual stress distribution for $\theta_1 + \theta_2 = 180^\circ$ (log-log diagram)

(a) $\theta_1 = 110^\circ$ (b) $\theta_1 = 125^\circ$ Fig. 9 Thermal residual stress distribution for $\theta_1 + \theta_2 = 180^\circ$ (semi-log diagram)

stresses decrease with decreasing r ; thus, free edge stress distribution disappears. For the cases where the free edge singularity appears, the exponent of singularity p is determined from the boundary element results using the least squares method, and is shown with the theoretical one^{(9),(10)} in Fig. 11. It is found that the exponent of singularity from boundary element results agrees well with the theoretical one.

For the cases of $\theta_1 = 110^\circ$ and 125° , distributions of subtracted stresses $(\sigma_\theta - \sigma_{\theta 0})$ and $(\sigma_{r\theta} - \sigma_{r\theta 0})$ on circles with radii of $r = 0.0005W$ and $r = 0.01W$ are calculated using the boundary element method, and the results are shown in Figs. 12(a) and 12(b), respectively. Solid lines in Fig. 12(a) show theoretical distributions calculated based on stress function and stress distribution on the interface. Boundary element results agree well with the theoretical ones⁽¹⁰⁾.

The condition in which the thermoelastic constant term approaches infinity corresponds to the borderline condition (Eq.(14)) between the appearance and disappearance of r^{-p} -type free edge stress singularity, which was obtained by Kubo and Ohji⁽⁹⁾. When ratio

(a) $\theta_1 = 110^\circ$ (b) $\theta_1 = 125^\circ$ Fig. 10 Subtracted stress distribution for $\theta_1 + \theta_2 = 180^\circ$ Fig. 11 Exponent of singularity for $\theta_1 + \theta_2 = 180^\circ$

of Young's moduli E_1/E_2 is 10 and Poisson's ratios ν_1 and ν_2 are 0.3, the apex angle θ_1 which satisfies Eq. (14) is 118.7° . For $\theta_1 = 118.7^\circ$, distributions of σ_θ and $\sigma_{r\theta}$ are shown in Fig. 13 on a semi-log diagram. Distributions of stresses σ_θ and $\sigma_{r\theta}$ increase linearly with decreasing r on a semi-log diagram; thus, it is found that logarithmic singularity develops for $\theta_1 = 118.7^\circ$.

(a) $\theta_1 = 110^\circ$ (b) $\theta_1 = 125^\circ$

Fig. 12 Subtracted stress distribution in the vicinity of intersection of interface and free surface for $\theta_1 = \theta_2 = 180^\circ$

Fig. 13 Thermal residual stress distribution on interface for $\theta_1 = 118.7^\circ$

4. Conclusions

Distributions of thermal residual stress in bonded dissimilar materials are calculated using the boundary element method. Thermoelastic constant stress terms in bonded dissimilar materials are deduced using

Airy's stress function. It is difficult to determine whether free edge stress singularity appears or not from the distributions of thermal residual stress themselves. However, when thermoelastic constant stress terms are subtracted from thermal residual stress, it is found that thermal residual stress distributions show $r^{-\rho}$ -type singularity which is the same as that observed under mechanical loading. The value of exponent of stress singularity agrees well with the theoretical one. The free edge stress singularity of thermal residual stress disappears for certain ranges of wedge angles of a pair of materials, where free edge stress singularity disappears under mechanical loading. Under the borderline condition between the appearance and disappearance of $r^{-\rho}$ -type free edge stress singularity where thermoelastic constant stress terms approach infinity, $r^{-\rho}$ -type singularity disappears and logarithmic singularity appears.

References

- (1) Yuuki, R., Xu, J.Q. and Schmauder, S., Simple Method to Analyze the Residual Thermal Stress of Dissimilar Material Joints and Some Applications, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 57, No. 536, A (1991), p. 864.
- (2) Yuuki, R., Xu, J.Q., Logarithmic Singularity on Thermal or Residual Stress at the Interface Edge Point of Bonded Dissimilar Materials, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 58, No. 556, A (1992), p. 2394.
- (3) Seo, K., Kusaka, M., Nogata, F., Terasaki, T., Nakao, Y. and Saida, K., Study on the Thermal Stress at Ceramic-Metal Joint, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 55, No. 510, A (1989), p. 312.
- (4) Mizuno, K., Miyazaki, K. and Suga, T., Characterization of Thermal Stresses in Ceramic/Metal-Joint, *Journal of the Faculty of Engineering, The University of Tokyo (B)*, Vol. 39, No. 4, (1988), p. 401.
- (5) Munz, D. and Yang, Y.Y., Stress Singularity at the Interface in Bonded Dissimilar Materials under Mechanical and Thermal Loading, *J. Appl. Mech.*, Vol. 59, No. 4, (1992), p. 857.
- (6) Koguchi, H., Kikuchi, Y., Hino, T. and Yada, T., Relaxation of Thermal Stresses by Varying the Edge Shape of Ceramic-Metal Bonded Structure, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 59, No. 558, A (1993), p. 448.
- (7) Yuuki, R., *Mechanics of Interface*, (1993), Baifuukan.
- (8) Danson, D.J. (ed. Brebbia, C.A.), *Boundary Element Methods*, (1981), p. 105., Springer-Verlag.
- (9) Kubo, S. and Ohji, K., Geometrical Conditions of No Free-Edge Stress Singularities in Edge-Bonded Elastic Dissimilar Wedges, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 57, No. 535, A (1991), p. 632.

- (10) Ohji, K., Kubo, S., Nakai, Y. and Ioka, S., Free-Edge Stress Singularity Dominated Region in Bonded Dissimilar Materials and Its Disappearance, *Journal of the Society of Materials Science, Japan*, Vol. 41, No. 468 (1992), p. 1389.
 - (11) Bogy, D.B., Edge-Bonded Dissimilar Orthogonal Elastic Wedges under Normal and Shear Loading, *J. Appl. Mech.*, Vol. 35., No. 3, (1968), p. 460.
 - (12) Bogy, D.B., On the Problem of Edge Bonded Elastic Quarter Planes Loaded at Boundary, *Int. J. Solids Struct.*, Vol. 6, (1970), p. 1287.
 - (13) Dunders, J., Discussion of (11), *J. Appl. Mech.*, Vol. 36, No. 3 (1969), p. 650.
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