Three-Dimensional Numerical Analysis of Dynamic Thermo-Elastic/Viscoplastic Problem by the Method of Characteristics*

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This paper is concerned with a numerical technique based on the method of characteristics for three-dimensional dynamic thermo-elastic/viscoplastic problems. A constitutive model covering a wide range of strain rates and a wide range of temperatures, proposed by Tanimura, is used. As a numerical example, the three-dimensional stress wave propagation in a thermo-elastic/viscoplastic bar of square cross section subjected to both an impact loading and a thermal shock is presented. The stability and convergence of these numerical solutions are examined by checking the error in the total energy of the system.

Key Words: Computational Mechanics, Numerical Analysis, Stress Wave, Bicharacteristics, Thermo-Elastic/Viscoplastic

1. Introduction

The earliest investigation of a thermal shock problem was apparently that by Danilovskaya⁽¹⁾, who determined the dynamic thermal stresses induced in an elastic half-space by a sudden uniform heating of the entire boundary. Subsequently, some exact solutions for such dynamic thermoelastic problems were obtained⁽²⁾. To analyze dynamic thermo-elastic/viscoplastic problems, it is necessary to use a numerical method. Among the numerical procedures, the method of characteristics is the most suitable for analyzing dynamic thermal shock problems⁽³⁾. The advantage of this method is attributed to the fact that the propagation, reflection and interaction of thermal stress waves, in a body can be investigated in detail. However, the application of this method in numerical analysis of thermal shock problems is mainly restricted to one-dimensional problems⁽⁴⁾.

The present paper deals with a numerical technique for obtaining the numerical solution of three-dimensional dynamic thermo-elastic/viscoplastic problems. The numerical technique is basically equivalent to the method of numerical integration along bicharacteristics⁽⁵⁾⁻⁽⁷⁾. The dynamic response of a thermo-elastic/viscoplastic bar of square cross section subjected to both a uniform velocity impact and a thermal shock on its end surface is calculated. For the constitutive equation, a constitutive model covering a wide range of strain rates and a wide range of temperatures, proposed by Tanimura⁽⁸⁾, is used. In order to evaluate the stability and convergence of this numerical technique, we examined the time variation of the energy error for different numerical conditions.

2. Derivation of the Basic Equations

The three-dimensional independent temperature problem is governed by the heat-conduction equation

$$T_{,ii} = \frac{1}{\kappa} \dot{T}, \quad (1)$$

where T is the temperature and κ the thermal diffusivity.

The equations governing the dynamic deformation

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of an isotropic and homogenous solid may be stated in terms of Cartesian co-ordinates as

$$\sigma_{ij,j} = \rho \dot{\nu}_i \quad (2)$$

where σ_{ij} denote the components of the stress tensor, ρ the mass density, and ν_i the particle velocity. The constitutive equation for the simple model covering a wide range of strain and temperature proposed by Tanimura⁽⁸⁾ is expressed as

$$\dot{e}_{ij} = \frac{1}{2G} \dot{S}_{ij} + \dot{\gamma}_s^p \exp \left\{ \frac{\sqrt{J_2} - \tau_s}{c_s/\sqrt{3}} \right\} \frac{S_{ij}}{\sqrt{J_2}} + a_T \dot{T} \delta_{ij} \quad (\tau_s \geq \sqrt{J_2}) \quad (3.1)$$

$$\dot{e}_{ij} = \frac{1}{2G} \dot{S}_{ij} + a_T \dot{T} \delta_{ij} \quad (\tau_s < \sqrt{J_2}) \quad (3.2)$$

where e_{ij} are the components of the strain, S_{ij} the deviatoric stress tensor, a_T the coefficient of expansion, G the modulus of rigidity, c_s the parameter which exhibits the strain rate sensitivity, J_2 the second invariant of the deviatoric stress tensor, τ_s the yield stress in simple shear, and $\dot{\gamma}_s^p$ the plastic strain rate. Under the assumption of small deformation, the components of the strain are given by

$$\dot{e}_{ij} = (\nu_{i,j} + \nu_{j,i})/2. \quad (4)$$

In order to simplify the equations and subsequent procedures, we introduce the following dimensionless variables:

$$\begin{aligned} \tau &= \frac{c_1}{b} t, \quad X_1 = \frac{x_1}{b}, \quad X_2 = \frac{x_2}{b}, \quad X_3 = \frac{x_3}{b}, \\ U_1 &= \frac{\nu_1}{c_1}, \quad U_2 = \frac{\nu_2}{c_1}, \quad U_3 = \frac{\nu_3}{c_1}, \\ U_4 &= \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3\gamma_0 \rho c_1^2}, \quad U_5 = \frac{\gamma(\sigma_{11} - \sigma_{22})}{2\rho c_1^2}, \\ U_6 &= \frac{\gamma(\sigma_{11} + \sigma_{22} - 2\sigma_{33})}{2\sqrt{3}\rho c_1^2}, \\ U_7 &= \frac{\gamma\sigma_{12}}{\rho c_1^2}, \quad U_8 = \frac{\gamma\sigma_{13}}{\rho c_1^2}, \quad U_9 = \frac{\gamma\sigma_{23}}{\rho c_1^2}, \\ D &= \begin{cases} \frac{2\rho c_1 b}{\gamma^2} \left[\dot{\gamma}_s^p \exp \left\{ \frac{\sqrt{J_2} - \tau_s}{c_s/\sqrt{3}} \right\} \frac{1}{\sqrt{J_2}} \right] & \text{for } \tau_s \geq \sqrt{J_2} \\ 0 & \text{for } \tau_s < \sqrt{J_2} \end{cases} \quad (5) \end{aligned}$$

where b is a representative length, and

$$\gamma = c_1/c_2, \quad \gamma_0 = [1 - 4/3\gamma^2]^{\frac{1}{2}},$$

where c_1 and c_2 are, respectively, the dilatational and equivoluminal wave speeds, defined by

$$c_1 = [(\lambda + 2G)/\rho]^{\frac{1}{2}}, \quad c_2 = (G/\rho)^{\frac{1}{2}},$$

where λ is Lamé's constant. Substituting Eqs. (5) into the previous system of Eqs. (2), (3) and (4), we obtain a system of the governing equations in matrix notation

$$L[U] \equiv U_\tau + A^i U_{X_i} + B = 0, \quad (6)$$

where U and B are given by

$$U^T = [U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_9],$$

$$B^T = [0, 0, 0, 3a_T \dot{T}, DU_5, DU_6, DU_7, DU_8, DU_9].$$

$A^i (i=1, 2, 3)$ denote symmetric matrix with the following nonzero components:

$$\begin{aligned} A_{14}^1 &= -\gamma_0, \quad A_{15}^1 = A_{27}^1 = A_{38}^1 = -1/\gamma, \\ A_{16}^1 &= -1/(\sqrt{3}\gamma), \quad A_{17}^2 = A_{39}^2 = -A_{25}^2 = -1/\gamma, \\ A_{24}^2 &= -\gamma_0, \quad A_{26}^2 = -1/(\sqrt{3}\gamma), \\ A_{18}^3 &= A_{29}^3 = -1/\gamma_0, \quad A_{34}^3 = \gamma, \quad A_{36}^3 = 2/(\sqrt{3}\gamma) \end{aligned}$$

in which the subscripts refer to the row and column of the component.

3. Characteristic Properties of Governing Equations

The temperature distribution which is the solution of Eq. (1) may be solved by the difference method⁽⁹⁾. Using the numerical result for temperature, the solution of Eq. (6) may be obtained by using the method of finite differences along bicharacteristics. Let a surface $\Phi(\tau, X_1, X_2, X_3) = \text{const.}$ be a characteristic surface of Eq. (6). Then, the following characteristic condition is satisfied.

$$\text{Det } A = 0 \quad (7)$$

where the characteristic matrix A is given by

$$A = \Phi_\tau + A^i \Phi_{X_i}. \quad (8)$$

Equation (7) is equivalent to

$$[\Phi_\tau^2 - \Phi_{X_i} \Phi_{X_i}] [\Phi_\tau^2 - \Phi_{X_i} \Phi_{X_i} / \gamma^2] \Phi_\tau^3 = 0. \quad (9)$$

The bracketed quantities in Eq. (9) correspond to characteristic surfaces with dimensionless wave velocities c equal to ± 1 and $\pm 1/\gamma$, respectively. The extra factor Φ_τ^3 corresponds to a characteristic surface with zero velocity; it is a result of introducing the additional dependent variables by choosing stresses and velocities as the dependent variables.

The solutions of Eq. (9) are given by the families of characteristic planes through the point $(\tau_0, X_1^0, X_2^0, X_3^0)$ which is an arbitrary point in space-time,

$$\Phi \equiv c(\tau - \tau_0) + (X_i - X_i^0) \quad (c = \pm 1, \pm 1/\gamma), \quad (10)$$

where λ_i is the direction cosine of the projection of the characteristic planes on the plane $\tau = \text{const.}$ with respect to the X_i co-ordinate

$$\lambda_i = \cos \alpha_i = \frac{X_i}{\sqrt{X_j X_j}} \quad (0 \leq \alpha_i \leq 2\pi) \quad (11)$$

and subject to the condition

$$\lambda_i \lambda_i = 1. \quad (12)$$

The corresponding characteristic cones determined by the envelopes of these two-parameter families of characteristic planes are given by

$$c^2(\tau - \tau_0)^2 = (X_i - X_i^0)(X_i - X_i^0) \quad (c = \pm 1, \pm 1/\gamma) \quad (13)$$

and are also characteristic surfaces. The characteristic cones have two parts extending forward and extending backward in time. The point $(\tau_0, X_1^0, X_2^0, X_3^0)$ is the vertex of the cones. It is convenient to consider only the cones extending backward in time.

Generators of the backward cones (which also are the lines of contact of the one-parameter families of planes with their envelopes) are

$$L_i \equiv \lambda_i c(\tau - \tau_0) + (X_i - X_i^0) = 0$$

$$(c=1, 1/\gamma, i=1, 2, 3) \quad (14)$$

which are the bicharacteristics for Eq. (6). The lines of contact given by Eq. (14) in combination with the directions normal to the characteristic surfaces containing the bicharacteristics are:

$$\begin{cases} \Phi_\tau = c \\ \Phi_{X_i} = \lambda_i \end{cases} \quad (c=1, 1/\gamma, i=1, 2, 3) \quad (15)$$

which define the bicharacteristic strips through the point $(\tau_0, X_1^0, X_2^0, X_3^0)$. In order to derive the equation along bicharacteristics, the null vectors associated with Eq. (6) must be determined. The nine linear independent null vectors $l^m (m=1, \dots, 9)$ are the solutions of the equations

$$l^m A = 0 \quad (m=1, \dots, 9) \quad (16)$$

where A is the characteristic matrix defined by Eq. (8). It is clear that l^m must correspond to the roots of Eq. (9) which are a simple root for $c=\pm 1$, a repeated root for $c=\pm 1/\gamma$, and a triple root for $c=0$ (the sum of the roots is 9). The null vectors l^m associated with the bicharacteristic strips defined by Eqs. (14) and (15) for $c=1, 1/\gamma$ are

$$l^1 = [\lambda_1, \lambda_2, \lambda_3, \gamma_0, (\lambda_1^2 - \lambda_2^2)/\gamma, (1 - 3\lambda_3^2)/(\sqrt{3}\gamma), 2\lambda_1\lambda_2/\gamma, 2\lambda_1\lambda_3/\gamma, 2\lambda_2\lambda_3/\gamma]^{-1} \text{ for } c=1, \quad (17.a)$$

$$l^2 = [0, -\lambda_3, \lambda_2, 0, \lambda_2, \lambda_3, -\sqrt{3}\lambda_2\lambda_3, -\lambda_1\lambda_3, \lambda_1\lambda_2, \lambda_2^2 - \lambda_3^2]^{-1} \text{ for } c=1/\gamma, \quad (17.b)$$

$$l^3 = [-\lambda_3, 0, \lambda_1, 0, -\lambda_1\lambda_3, -\sqrt{3}\lambda_1\lambda_3, -\lambda_2\lambda_3, \lambda_1^2 - \lambda_2^2, \lambda_1\lambda_2]^{-1} \text{ for } c=1/\gamma, \quad (17.c)$$

$$l^4 = [-\lambda_2, \lambda_1, 0, 0, -2\lambda_1\lambda_3, 0, \lambda_1^2 - \lambda_2^2, -\lambda_2\lambda_3, \lambda_1\lambda_3]^{-1} \text{ for } c=1/\gamma. \quad (17.d)$$

It must be remarked that Eqs. (17) are limited by the conditions that for Eqs. (17.b), (17.c) and (17.d), λ_2 and λ_3 , λ_1 and λ_3 , and λ_1 and λ_2 respectively must not both be zero. The desired differential equations along bicharacteristics are obtained from

$$l^m \cdot L[U] = 0 \quad (m=1, \dots, 4) \quad (18)$$

in which the dot denotes the inner product. Since each of Eqs. (18) involves no derivatives normal to the surface Φ , it can at each point be written in terms of derivatives in directions internal to the surface Φ . The derivative of the components of the vector U in a bicharacteristic direction is given by

$$\frac{dU_m}{d\tau} = (U_m)_\tau + (U_m)_{X_i} \frac{dX_i}{d\tau} \quad (m=1, \dots, 9) \quad (19)$$

where $dX_i/d\tau (i=1, 2, 3)$ along bicharacteristics are obtained by differentiation of Eq. (15).

Substituting Eqs. (17) into Eqs. (18), and eliminating partial derivatives with respect to τ by use of Eqs. (19), we find that the equations $l^m \cdot L[U] = 0$ involve derivatives along bicharacteristics, and derivatives in planes $\tau = \text{const.}$ Representing the co-ordinates, (X_1, X_2, X_3) by (X, Y, Z) , the transformed Eqs. (18) can be written in the following differential forms:

$$\begin{aligned} &\lambda_1 dU_1 + \lambda_2 dU_2 + \lambda_3 dU_3 + \gamma_0 dU_4 + \gamma_1(\lambda_1^2 - \lambda_2^2) dU_5 \\ &+ \gamma_1 \sqrt{3}(1 - \lambda_3^2) dU_6 + \gamma_1 \lambda_1 \lambda_2 dU_7 + \gamma_1 \lambda_1 \lambda_3 dU_8 \\ &+ \gamma_1 \lambda_2 \lambda_3 dU_9 + 3\gamma_0 a_7 dT = -S_1(\lambda_1, \lambda_2, \lambda_3) d\tau \end{aligned} \quad (20.a)$$

$$\begin{aligned} &-\lambda_2 dU_1 + \lambda_1 dU_2 - 2\lambda_1 \lambda_2 dU_5 + (\lambda_1^2 - \lambda_2^2) dU_7 \\ &= -S_2(\lambda_1, \lambda_2, \lambda_3) d\tau \end{aligned} \quad (20.b)$$

$$\begin{aligned} &-\lambda_3 dU_1 + \lambda_1 dU_3 - \lambda_1 \lambda_3 dU_5 - 3\lambda_1 \lambda_3 dU_6 + (\lambda_1^2 - \lambda_3^2) dU_8 \\ &= -S_3(\lambda_1, \lambda_2, \lambda_3) d\tau \end{aligned} \quad (20.c)$$

$$\begin{aligned} &-\lambda_3 dU_2 + \lambda_2 dU_3 + \lambda_2 \lambda_3 dU_5 - \lambda_2 \lambda_3 dU_6 + (\lambda_2^2 - \lambda_3^2) dU_9 \\ &= -S_4(\lambda_1, \lambda_2, \lambda_3) d\tau \end{aligned} \quad (20.d)$$

where the $S_i (i=1, \dots, 4)$ are given by

$$\begin{aligned} S_1(\lambda_1, \lambda_2, \lambda_3) = &[\gamma_3(U_{2X} + U_{1Y}) + 2\gamma_1(U_{9X} + U_{8Y} + U_{7Z})\lambda_3 + 2\gamma_1 DU_7]\lambda_2 \\ &\times [\gamma_3(U_{2X} + U_{1Y}) + 2\gamma_1 DU_8]\lambda_3 + \gamma_1(\lambda_3^2 - 2)U_{5X} \\ &+ \gamma_1(1 + 2\lambda_2^2)U_{7Y} + \gamma_1(1 - 2\lambda_3^2)U_{8Z}]\lambda_1 \\ &+ [\gamma_3(U_{3X} + U_{2Z}) + 2\gamma_1 DU_9]\lambda_3 + \gamma_1(2\lambda_3^2 - 1)U_{9Z} \\ &+ \gamma_1(2\lambda_3^2 - \lambda_2^2 - 1)U_{7X} + \gamma_1(1 + \lambda_1^2)U_{5Y}]\lambda_2 \\ &- \{\gamma_1(\sqrt{3}\lambda_3^2 - 1/\sqrt{3})DU_6 \\ &+ \gamma_1(\lambda_1 U_{6X} + \lambda_2 U_{6Y})\lambda_3^2 + \gamma_1(\lambda_3^2 - 1)U_{6Z}\lambda_3\} \\ &- \gamma_3\{(1 - \lambda_1^2)U_{1X} + (1 - \lambda_2^2)U_{2Y} + (1 - \lambda_3^2)U_{3Z}\} \\ &+ \gamma_1(\lambda_1^2 - \lambda_2^2)DU_5 + \gamma_1\lambda_3\{(\lambda_1^2 - \lambda_2^2)U_{5Z} \\ &+ (2\lambda_3^2 - 1)U_{8X} + (2\lambda_2^2 - 1)U_{9Y}\}, \end{aligned}$$

$$\begin{aligned} S_2(\lambda_1, \lambda_2, \lambda_3) = &-\{\gamma_5(U_{1X} - U_{2Y}) + 2DU_5\}\lambda_2 \\ &+ (\gamma_1 + \lambda_2^2)U_{7X} + (-\gamma_1 + 2\lambda_2^2)U_{5Y} + \gamma_1 U_{9Z}]\lambda_1 \\ &- \lambda_2\{(-\gamma_1 + 2\lambda_1^2)U_{5X} - (\gamma_1 + \lambda_1^2)U_{7Z}\} \\ &+ (\lambda_1^2 - \lambda_2^2)DU_7 + (\gamma_4\lambda_1^2 + \gamma_1\lambda_2^2)U_{2X} \\ &- (\gamma_1\lambda_1^2 + \gamma_4\lambda_2^2)U_{1Y} + \gamma_1\lambda_2 U_{8Z} + \lambda_1^3 U_{7X} - \lambda_2^3 U_{7Y} \\ &- \gamma_0(\lambda_1 U_{4Y} - \lambda_2 U_{4X}) - \gamma_1(\lambda_1 U_{6Y} - \lambda_2 U_{6X})/\sqrt{3}, \end{aligned}$$

$$\begin{aligned} S_3(\lambda_1, \lambda_2, \lambda_3) = &-\{\gamma_5(U_{1X} - U_{3Z}) + DU_5\}\lambda_3 \\ &+ (\gamma_1 + \lambda_3^2)U_{8X} + \lambda_3^2 U_{5Z} + \gamma_1 U_{9Y}]\lambda_1 \\ &- \lambda_3(-\gamma_1 + \lambda_1^2)(U_{5X} - U_{8Z}) + (\lambda_1^2 - \lambda_3^2)DU_8 + (\gamma_4\lambda_1^2 \\ &+ \gamma_1\lambda_3^2)U_{3X} - (\gamma_1\lambda_1^2 + \gamma_4\lambda_3^2)U_{1Z} + \lambda_3 U_{7Y} \\ &- \lambda_1^3 U_{8X} - \lambda_3^3 U_{8Z} - 3(\lambda_3 DU_6 + \lambda_3 U_{5Z} - 2U_{6Z}/3)\lambda_1 \\ &- \gamma_0(\lambda_1 U_{4Z} - \lambda_3 U_{4X}) - \lambda_3(\sqrt{3}\lambda_1^2 - 1/\sqrt{3})U_{6X}, \end{aligned}$$

$$\begin{aligned} S_4(\lambda_1, \lambda_2, \lambda_3) = &-\{\gamma_5(U_{2Y} - U_{3Z}) - DU_5\}\lambda_3 \\ &+ (\gamma_1 + \lambda_3^2)U_{9Y} + \lambda_3 U_{5Z} + \gamma_1 U_{8X}]\lambda_2 \\ &+ \lambda_3(-\gamma_1 + \lambda_2^2)U_{5Y} + \lambda_3(\lambda_2^2 + \gamma_1 + \lambda_3^2)U_{9Z} \\ &+ (\lambda_2^2 - \lambda_3^2)DU_9 + (\gamma_1\lambda_3^2 + \gamma_4\lambda_2^2)U_{3Y} \\ &- (\gamma_1\lambda_3^2 + \gamma_3\lambda_2^2)U_{2Z} + \gamma_1 U_{7X} + \lambda_2^3 U_{9Y} \\ &- 3(\lambda_3 DU_6 + \lambda_3^2 U_{6Z} - 2\gamma_1 U_{6Z}/3)\lambda_2 \\ &- \gamma_0(\lambda_2 U_{4Z} - \lambda_3 U_{4Y}) + \lambda_3(\sqrt{3}\lambda_2^2 - 1/\sqrt{3})U_{6Y}, \end{aligned}$$

where $\gamma_1 = 1/\gamma$, $\gamma_2 = 1/(\sqrt{3}\gamma)$, $\gamma_3 = (\gamma^2 - 2)/\gamma^2$, $\gamma_4 = (\gamma - 1)/\gamma$ and $\gamma_5 = (\gamma - 5)/\gamma$.

Equation (20.a) is the desired differential relation along bicharacteristics on the dilatational wave ($c=1$) surface, while Eqs. (20.b), (20.c), and (20.d) are the desired differential relations along bicharacteristics on the equivoluminal wave ($c=1/\gamma$) surface.

4. Derivation of Difference Equations

Difference equations can be derived for computing the solution at a mesh point (τ_0, X_0, Y_0, Z_0) in the previous section denoted as $(\tau_0, X_1^0, X_2^0, X_3^0)$ from known data at neighbouring mesh points on the plane

$\tau = \tau_0 - k$, where k is the mesh size of τ . These equations are obtained by linear combinations of equations, which result from the integration of Eqs. (20) along bicharacteristics and the integration of Eq. (6) along the line ($X = X_0$, $Y = Y_0$, $Z = Z_0$). Integrating Eqs. (20) along a bicharacteristic from the point (τ_0 , X_0 , Y_0 , Z_0) to the point ($\tau_0 - k$, $X_0 + kc\lambda_1$, $Y_0 + kc\lambda_2$, $Z_0 + kc\lambda_3$) where this bicharacteristic intersects the plane $\tau = \tau_0 - k$, we obtain the following expressions:

$$\begin{aligned} & \lambda_1 \delta U_1 + \lambda_2 \delta U_2 + \lambda_3 \delta U_3 + \gamma_0 \delta U_4 + \gamma_1 \{(\lambda_1^2 - \lambda_2^2) \delta U_5 \\ & + \sqrt{3}(1 - \lambda_3^2) \delta U_6 + \lambda_1 \lambda_2 \delta U_7 + \lambda_1 \lambda_3 \delta U_8 + \lambda_2 \lambda_3 \delta U_9\} \\ & = -k[S_1(\lambda)^0 + S_1(\lambda)_m] - W_1(\lambda) \\ & - 3\gamma_0 \alpha_T \delta T + O(k^3), \end{aligned} \quad (21.a)$$

$$\begin{aligned} & -\lambda_2 \delta U_1 + \lambda_1 \delta U_2 - 2\lambda_1 \lambda_2 \delta U_5 + (\lambda_1^2 - \lambda_2^2) \delta U_7 \\ & = -k[S_2(\lambda)^0 + S_2(\lambda)_n]/2 - W_2(\lambda) + O(k^3), \end{aligned} \quad (21.b)$$

$$\begin{aligned} & -\lambda_3 \delta U_1 + \lambda_1 \delta U_3 - \lambda_1 \lambda_3 \delta U_5 - 3\lambda_1 \lambda_3 \delta U_6 + (\lambda_1^2 - \lambda_3^2) \delta U_8 \\ & = -k[S_3(\lambda)^0 + S_3(\lambda)_n]/2 - W_3(\lambda) + O(k^3) \end{aligned} \quad (21.c)$$

$$\begin{aligned} & -\lambda_3 \delta U_2 + \lambda_2 \delta U_3 + \lambda_2 \lambda_3 \delta U_5 - \lambda_2 \lambda_3 \delta U_6 + (\lambda_2^2 - \lambda_3^2) \delta U_9 \\ & = -k[S_4(\lambda)^0 + S_4(\lambda)_n]/2 - W_4(\lambda) + O(k^3), \end{aligned} \quad (21.d)$$

where the operator δ is defined by $\delta U_m = U_m(\tau_0, X_0, Y_0, Z_0) - U_m(\tau_0 - k, X_0, Y_0, Z_0)$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, and

$$\begin{aligned} W_1(\lambda) &= [\lambda_1 U_1 + \lambda_2 U_2 + \lambda_3 U_3 + \gamma_0 U_4 \\ &+ \gamma_1 \{(\lambda_1^2 - \lambda_2^2) U_5 + \sqrt{3}(1 - \lambda_3^2) U_6 + \lambda_1 \lambda_2 U_7 \\ &+ \lambda_1 \lambda_3 U_8 + \lambda_2 \lambda_3 U_9\} + 3\gamma_0 \alpha_T T]_0 \\ &- [\lambda_1 U_1 + \lambda_2 U_2 + \lambda_3 U_3 + \gamma_0 U_4 \\ &+ \gamma_1 \{(\lambda_1^2 - \lambda_2^2) U_5 + \sqrt{3}(1 - \lambda_3^2) U_6 \\ &+ \lambda_1 \lambda_2 U_7 + \lambda_1 \lambda_3 U_8 + \lambda_2 \lambda_3 U_9\} + 3\gamma_0 \alpha_T T]_m, \\ W_2(\lambda) &= [-\lambda_2 U_1 + \lambda_1 U_2 - 2\lambda_1 \lambda_2 U_5 + (\lambda_1^2 - \lambda_2^2) U_7]_0 \\ &- [-\lambda_2 U_1 + \lambda_1 U_2 - 2\lambda_1 \lambda_2 U_5 + (\lambda_1^2 - \lambda_2^2) U_7]_n, \\ W_3(\lambda) &= [-\lambda_3 U_1 + \lambda_1 U_3 - \lambda_1 \lambda_3 U_5 - 3\lambda_1 \lambda_3 U_6 \\ &+ (\lambda_1^2 - \lambda_3^2) U_8]_0 - [-\lambda_3 U_1 + \lambda_1 U_3 - \lambda_1 \lambda_3 U_5 \\ &- 3\lambda_1 \lambda_3 U_6 + (\lambda_1^2 - \lambda_3^2) U_8]_n, \\ W_4(\lambda) &= [-\lambda_3 U_2 + \lambda_2 U_3 + \lambda_2 \lambda_3 U_5 - \lambda_2 \lambda_3 U_6 \\ &+ (\lambda_2^2 - \lambda_3^2) U_9]_0 - [-\lambda_3 U_2 + \lambda_2 U_3 + \lambda_2 \lambda_3 U_5 \\ &- \lambda_2 \lambda_3 U_6 + (\lambda_2^2 - \lambda_3^2) U_9]_n. \end{aligned}$$

In Eqs. (21), the superscript $( )^o$ and the subscript $( )_o$ denote the associated function evaluated at the points (τ_0, X_0, Y_0, Z_0) and $(\tau_0 - k, X_0, Y_0, Z_0)$, respectively; and the subscripts $( )_m$ and $( )_n$ indicate the associated function evaluated at the point $(\tau_0 - k, X_0 + kc\lambda_1, Y_0 + kc\lambda_2, Z_0 + kc\lambda_3)$ for $c=1$ and $c=1/\gamma$, respectively. It is noted that Eq. (21.a) is associated with the exterior cone, whereas Eqs. (21.b), (21.c) and (21.d) are associated with the interior cone. Nine additional difference equations involving the increments δU_m can be obtained by integration of Eq. (6) along the line $X = X_0$, $Y = Y_0$, $Z = Z_0$ from $\tau = \tau_0 - k$ to $\tau = \tau_0$:

$$\delta U_m = -k(R^0 + R_0)/2 + O(k^3) \quad (22)$$

where

$$R(\tau, X, Y, Z) = R(\tau, X_1, X_2, X_3) = A^i U_{X_i} + B$$

The superscript $( )^o$ and the subscript $( )_o$ have the same meanings as in Eqs. (21). All the terms on

the right-hand side of Eqs. (21) and (22) can be evaluated from known data on the plane $\tau = \tau_0 - k$, except those terms having a superscript $( )^o$.

By taking appropriate linear combinations of the nine Eqs. (22), the six equations obtained by applying Eq. (21.a) to six bicharacteristics which are $L_1(\lambda_1 = \pm 1)$, $L_2(\lambda_2 = \pm 1)$, and $L_3(\lambda_3 = \pm 1)$ (see Eqs. (14)) on the exterior cone ($c=1$), and the twelve equations obtained by applying each of Eqs. (21.b), (21.c), and (21.d) to six bicharacteristics which are $L_1(\lambda_1 = \pm 1)$ for Eqs. (21.c) and (21.d), $L_2(\lambda_2 = \pm 1)$ for Eqs. (21.b) and (21.d), and $L_3(\lambda_3 = \pm 1)$ for Eqs. (21.b) and (21.c) on the interior cone ($c=1/\gamma$), the unknown derivatives having a superscript $( )^o$ can be eliminated, and the following system of nine equations for the nine unknown increments in the dependent variables and the function D^0 is obtained.

$$\begin{aligned} \delta U_1 &= k\gamma_2^2 \{ \sqrt{3} [k(\gamma^2(U_{1XX} + U_{2XY} + U_{3XZ}) \\ &- U_{2XY} - U_{3XZ} - U_{1YY} + U_{1ZZ}) \\ &- \gamma k(D(U_{5X} + U_{7Y} + U_{8Z}) - D_X U_5 \\ &- D_Y U_7 - D_Z U_8 + 2\gamma_6^2 \alpha_T(T_X^0 - T_X)) \\ &+ 2\gamma(U_{5X} + U_{7Y} + U_{8Z})] + \gamma[2\gamma_6 U_{4X} \\ &- k(DU_{6X} + D_X U_6) + 2U_{6X}] \} / 2 + O(k^3), \end{aligned} \quad (23.a)$$

$$\begin{aligned} \delta U_2 &= k\gamma_2^2 \{ \sqrt{3} [k(\gamma^2(U_{1XY} + U_{2YY} + U_{3YZ}) \\ &- U_{2XX} - U_{1XY} - U_{3YZ} + U_{2ZZ}) \\ &- \gamma k(D(U_{7X} + U_{5Y} + U_{9Z}) - D_X U_7 \\ &- D_Y U_5 - D_Z U_9 + 2\gamma_6^2 \alpha_T(T_Y^0 - T_Y)) \\ &+ 2\gamma(U_{7X} + U_{5Y} + U_{9Z})] + \gamma[2\gamma_6 U_{4Y} \\ &- k(DU_{6Y} + D_Y U_6) + 2U_{6Y}] \} / 2 + O(k^3), \end{aligned} \quad (23.b)$$

$$\begin{aligned} \delta U_3 &= k\gamma_2^2 \{ \sqrt{3} [k(\gamma^2(U_{1XZ} + U_{2YZ} + U_{3ZZ}) \\ &- U_{3XX} - U_{1XYZ} - U_{3YY} + U_{2YZ}) \\ &- \gamma k(D(U_{8X} + U_{9Y}) - D_X U_8 - D_Y U_8 \\ &- 2\gamma_6^2 \alpha_T(T_Z^0 - T_Z)) + 2\gamma(U_{8X} + U_{9Y})] \\ &+ 2\gamma[\gamma_1 U_{4Z} + k(DU_{6Z} + D_Z U_6) \\ &+ 2U_{6Z}] \} / 2 + O(k^3), \end{aligned} \quad (23.c)$$

$$\begin{aligned} \delta U_4 &= k\gamma_2^2 \{ \gamma_6 [3k(U_{5XX} + 2(U_{7XY} + U_{8XZ}) \\ &+ U_{9YZ} - U_{5YY}) + \sqrt{3} k(D(U_{6XX} + U_{6YY} - U_{6ZZ}) \\ &+ 6\gamma(U_{1X} + U_{2Y} + U_{3Z}) - 2\gamma_0 \alpha_T \delta T)] \\ &+ \sqrt{3} \gamma_6^2 k(U_{4XX} - U_{4YY} + U_{4ZZ}) \} / 6 + O(k^3), \end{aligned} \quad (23.d)$$

$$\begin{aligned} \delta U_5 &= k\gamma_7 \{ k[\sqrt{3} \gamma_6(U_{4XX} + U_{4XY}) \\ &+ 3(U_{5XX} + U_{8XZ} + U_{5YY} - U_{9YZ}) \\ &+ \sqrt{3}(U_{6XX} - U_{6YY})] + 6\gamma(U_{1X} - U_{2Y}) \\ &- 3\gamma^2 U_5(D + D^0)] / 3 + O(k^3), \end{aligned} \quad (23.e)$$

$$\begin{aligned} \delta U_6 &= k\gamma_8 \{ \sqrt{3} [k(U_{6XX} + U_{6YY} + 4U_{6ZZ}) \\ &- 3\gamma^2 U_6(D + D^0)] + k[\sqrt{3} \gamma_6(U_{4XX} + U_{4YY} - 2U_{4ZZ}) \\ &+ 3(U_{5XX} + 2U_{7XY} - U_{8XZ} - U_{5YY} - U_{9YZ})] \\ &+ 6\gamma(U_{1X} + U_{2Y} - U_{3Z}) \} / 3 + O(k^3), \end{aligned} \quad (23.f)$$

$$\begin{aligned} \delta U_7 &= k\gamma_8 \{ \sqrt{3} [k(U_{7XX} + U_{9XZ} + U_{7YY} + U_{8YZ}) \\ &+ 2\gamma(U_{2X} - U_{1Y}) - \gamma^2 U_7(D + D^0)] \\ &+ 2k(\gamma_6 U_{4XY} + U_{6XY}) \} + O(k^3), \end{aligned} \quad (23.g)$$

$$\begin{aligned} \delta U_8 &= k\gamma_8 \{ \sqrt{3} [k(U_{8XX} + U_{9XY} + U_{5XZ} + U_{7YZ} \\ &+ U_{8ZZ}) + 2\gamma(U_{3X} + U_{1Z}) - \gamma^2 U_8(U + U^0)] \\ &+ 2k(\gamma_6 U_{4XZ} - U_{6XZ}) \} + O(k^3), \end{aligned} \quad (23.h)$$

$$\begin{aligned} \delta U_9 = & k\gamma_8[\sqrt{3}[k(U_{8XY} + U_{7XZ} + U_{9YZ} - U_{5YZ} \\ & + U_{9ZZ}) + 2\gamma(U_{3Y} + U_{2Z}) - \gamma^2 U_9(D + D^0)] \\ & + 2k(\gamma_6 U_{4YZ} - U_{6YZ})] + O(k^3), \end{aligned} \quad (23.i)$$

where $\gamma_6 = \sqrt{3}\gamma^2 - 4$, $\gamma_7 = \gamma^2/(kD^0 + 2)$, $\gamma_8 = \gamma^2/(kD^0 + 2)$.

Since the quantities related to T are given by solving Eq. (1), the quantities on the right-hand side of Eqs. (23) are all known data at the point $(\tau_0 - k, X_0, Y_0, Z_0)$ except the quantity D^0 . The partial derivatives with respect to X , Y and Z are replaced by the corresponding central difference equations having an accuracy $O(h^2)$. Thus, for example, the first and second partial derivatives with respect to X and Y are replaced by the following equations:

$$\begin{aligned} f_X &= [f(X_0 + h, Y_0, Z_0) - f(X_0 - h, Y_0, Z_0)]/(2h) \\ f_{XX} &= [f(X_0 + h, Y_0, Z_0) + f(X_0 - h, Y_0, Z_0) \\ &\quad - 2f(X_0, Y_0, Z_0)]/(2h^2) \\ f_{XY} &= [f(X_0 + h, Y_0 + h, Z_0) + f(X_0 - h, Y_0 - h, Z_0) \\ &\quad - f(X_0 + h, Y_0 - h, Z_0) \\ &\quad - f(X_0 - h, Y_0 + h, Z_0)]/(4h^2) \end{aligned}$$

where h is the spatial interval. Since Eqs. (23) include the unknown quantity $D^0(\tau_0, X_0, Y_0, Z_0)$ which is a function of $U(\tau_0 - k, X_0, Y_0, Z_0)$, the system is solved by successive approximation when the point (τ_0, X_0, Y_0, Z_0) is located in the viscoplastic region.

For the present numerical example, the accuracy of the iterative procedure is determined by the following condition:

$$|D_{j-1}^0 - D_j^0| < 10^{-6}$$

where J is the number of the successive approximations for D^0 . So far, we have only considered difference equations when the point $(\tau_0 - k, X_0, Y_0, Z_0)$ is an interior point of the body. When the point $(\tau_0 - k, X_0, Y_0, Z_0)$ is located on the boundary which is some surface S normal to the plane $\tau = \text{const.}$, since some bicharacteristics on both interior and exterior cones intersect the plane $\tau = \tau_0 - k$ at points outside the region of interest, the relations along those bicharacteristics cannot be used. However, those unused relations are replaced by admissible boundary conditions for the construction of the numerical solution at the point located on the boundary. Then, for approximating derivatives with respect to the co-ordinate normal to the boundary, backward or forward difference equations having accuracy $O(h^2)$ must be used.

5. Numerical Examples and Discussion

In numerical calculations, the following material constants⁽⁸⁾ for mild steel (0.02% C) were used for all computations: Young's modulus, $E = 203.8$ GPa; mass density, $\rho = 7.86 \times 10^3$ kg/m³; shear modulus, $G = 84.1$ GPa. The sensitivity parameter c_s and the static strain hardening parameter r_s are expressed as functions of temperature T and equivalent plastic

strain $\bar{\epsilon}$ as follows:

$$c_s = 28.17 + 10000(\epsilon_{ult} - \bar{\epsilon})/T \quad (T_r \leq T \leq 873 \text{ K})$$

$$\tau_s = \frac{1}{\sqrt{3}} [280.7 \times 338.0 \bar{\epsilon}^{-0.57}] \frac{1.25 T_m - T}{1.25 T_m - T_r} \quad (\text{MPa})$$

where $T_r = 293$ K, denotes the room temperature, $T_m = 1740$ K, the melting point, $\epsilon_{ult} = 0.24$, the nominal strain corresponding to the ultimate strength under tension test at $\dot{\epsilon} = 0.5 \text{ S}^{-1}$ and room temperature.

We consider a dynamic thermo-elastic/viscoplastic problem of a semi-infinite bar with square cross section when a uniform velocity impact and high temperature are imposed suddenly on its end surface. Due to the symmetric property, only the solution for the positive part of the bar is calculated (see fig. 1).

The initial and boundary conditions are given by

$$\begin{cases} U = 0, \quad T = T_r & \text{for } \tau = 0 \\ U_1 = V(\tau), \quad U_8 = U_9 = 0, \quad T = T(\tau) & \text{for } Z = 0 \\ \gamma\gamma_0 U_4 + U_5 + U_6/\sqrt{3} = 0, \quad U_7 = U_8 = 0, \\ \frac{\partial T}{\partial X} = h_X(T_r - T)b & \text{for } X = 1 \\ \gamma\gamma_0 U_4 - U_5 + U_6/\sqrt{3} = 0, \quad U_7 = U_9 = 0, \\ \frac{\partial T}{\partial Y} = h_Y(T_r - T)b & \text{for } Y = 1 \end{cases}$$

where

$$\begin{aligned} V(\tau) &= \begin{cases} \beta V_0(\tau/\tau_r) & \text{for } \tau \leq \tau_r \\ \beta V_0 & \text{for } \tau_r \leq \tau \end{cases} \\ T(\tau) &= \begin{cases} T_0(\tau/\tau_r) & \text{for } \tau \leq \tau_r \\ T_0 & \text{for } \tau_r \leq \tau \end{cases} \end{aligned}$$

Here $\tau_r = 3$, denotes the rise time, $h_X = h_Y = 0.01$, the relative heat transfer coefficients, $\beta = 3$, the amplification coefficient (for $\beta > 1$, plastic deformation will occur when the bar is subjected to longitudinal impact load only),

$$V_0 = \tau_0/(\rho c_f^2) \text{ and } \tau_0 = \tau_s(T = T_r).$$

The representative length b is taken as 10 mm. From Ref. (9), thermal diffusivity is $\kappa = 14.7 \text{ mm}^2/\text{s}$, and the coefficient of expansion, $\alpha_T = (1.19 + 0.00043T) \times 10^{-5} \text{ }^\circ\text{C}^{-1}$.

Figures 2 and 3, respectively, show the time variations of the stresses σ_z/τ_0 and σ_y/τ_0 at the point $(X=0, Y=0, Z=0)$ and the point $(X=1, Y=0, Z=0)$ for the same impact velocity but different $\theta (= T_0 - T_r)$. From Fig. 2 and Fig. 3, it can be seen that, on the end surface the variations of the stresses are

Fig. 1 Schematic of an example

Fig. 2 Stress variation on the end surface at point ($X=0$, $Y=0$, $Z=0$)

Fig. 4 Stress variation at points ($X=0$, $Y=0$, $Z=1$) and ($X=0$, $Y=0$, $Z=3$)

Fig. 3 Stress variation on the end surface at point ($X=1$, $Y=0$, $Z=0$)

Fig. 5 Stress variation at points ($X=1$, $Y=0$, $Z=1$) and ($X=1$, $Y=0$, $Z=3$)

extremely large, and there is a phenomenon that the higher θ , the larger is the variation of the stresses.

Figure 4 indicates the time variations of the stress σ_z/τ_0 at the points ($X=0$, $Y=0$, $Z=1$) and ($X=0$, $Y=0$, $Z=3$). Figure 5 presents the time variations of the stress σ_z/τ_0 at the points ($X=1$, $Y=0$, $Z=1$) and ($X=1$, $Y=0$, $Z=3$). From Fig. 2 to Fig. 5, we can see that the time variations of the stress σ_z/τ_0 for different θ become the same as the point is far from the end surface of the bar. The time variations of other stress components also show this tendency (the figures are omitted here). This indicates that the effect of the temperature occurs only in the vicinity of the end surface of the bar.

In this study, we examine the stability and convergence of the numerical solutions by checking relative energy error. The relative energy error E_{rr} is defined as

$$E_{rr} = (E_{\text{input}} - E_{\text{total}}) / E_{\text{input}}$$

where the energy input for the present problem is given by

$$E_{\text{input}} = \left| \int_0^\tau \int_0^1 \int_0^1 3a_T \gamma_0 \dot{T} U_4 dX dY d\tau + \int_0^\tau \int_0^1 \int_0^1 \left(\gamma_0 U_4 - \frac{2}{\sqrt{3}} \gamma U_6 \right) dX dY d\tau \right|$$

and the total energy imparted to the bar (i.e. the sum of the kinetic energy, elastic strain energy and viscoplastic strain energy) is given by

$$E_{\text{total}} = \frac{1}{2} \int_0^2 \int_0^1 \int_0^1 \sum_{i=1}^3 U_i^2 dX dY dZ + \frac{1}{2} \int_0^2 \int_0^1 \int_0^1 \sum_{i=4}^9 U_i^2 dX dY dZ + \frac{1}{2} \int_0^\tau \int_0^2 \int_0^1 \int_0^1 \sum_{i=5}^9 U_i D dX dY dZ d\tau.$$

Since the heat dissipation in the present problem is small, the effect of the thermal energy on the relative energy error is ignored here. For the above

Fig. 6 Variation of relative energy error with time

numerical calculations, the spatial interval $h_0=1$ mm ($h=h_0/b=0.1$) and the mesh ratio $k/h=0.25$ are chosen. Figure 6 shows the relative energy errors as a function of time for several values of θ . It is seen that the errors remain small with increasing time for all the values of θ . We also performed some numerical calculations for larger k/h . For $h/k \geq 0.6$ the errors become excessively large within $\tau=4$; for $0.6 > h/k \geq 0.5$, the error variations with time are similar to those shown in Fig. 6, but the errors increase again when $\tau > 9.5$.

6. Conclusions

A numerical technique, based on finite differences along bicharacteristics, has been presented for analyzing three-dimensional thermo-elastic/viscoplastic wave propagation problems. The limitations on the applicability of the technique are that: the material is homogeneous and isotropic; the mass density remains constant and the deformations and rotations are small; and the treatment for general curved boundaries is difficult. The stability and convergence of the numerical results are investigated by evaluating the error in the total energy of the numerical solution. Calculations for the numerical examples presented in

the preceding section were carried out by using a Compaq Prolinea 4/33 personal computer (CPU: 486DX). For example, computation time was about 150 min for the numerical result with $\theta=200$ K shown in Fig. 5 (400 time steps).

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