Regularization of the Integral Equations for Unsteady Heat Conduction Problems*

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It has been found that the boundary integral equations for steady problems such as those of potential, elasticity, fluid mechanics and so on can be regularized by introducing relative quantities of field functions. This paper describes that fundamental integral equations for unsteady heat conduction problems can also be regularized by applying the same techniques. The regularized integral equations with relative quantity are obtained by superposing a particular solution under the condition of time-independent uniform potential upon the conventional ones. This approach has made it possible to derive the integral equation of potential gradient on a surface point, which has not been given up to now in the conventional formulation due to hyper-singularity. Through two- and three-dimensional numerical investigations, it is verified that the present integral equations give accurate numerical results everywhere over the domain and that they are valid and effective.

Key Words: Boundary Element Method, Computational Mechanics, Unsteady Heat Conduction, Computer-Aided Engineering, Singular Integral Equation, Regular Integral Equation

1. Introduction

The author has developed the regularizing technique using relative quantities⁽¹⁾⁻⁽¹⁰⁾ of field functions. The technique has successfully been applied to several steady problems such as those of potential⁽²⁾⁻⁽⁴⁾, elasticity⁽³⁾⁻⁽⁵⁾, fluid mechanics⁽⁶⁾⁻⁽⁸⁾ and so on. It is natural that such studies have been necessary in order to get over the problems due to singularity since they prevent the boundary element method (BEM) from being more practical and more versatile. This paper intends to extend the regularizing technique to the unsteady heat conduction problem⁽⁹⁾.

While several schemes have been proposed for solving such an unsteady problem by BEM⁽¹¹⁾, the time-stepping scheme using a fundamental solution depending on time is employed here since this scheme seems to have some advantages. Namely, there is no problem of singularity due to time integration since inte-

gration with respect to time can be performed analytically and there merely remains the singularity due to distance of the same order as in the steady problem after the time integration. Therefore, the techniques used in the steady problems can be applied to the unsteady problems almost as it is.

As discussed in the previous papers⁽²⁾⁻⁽⁹⁾, introducing relative quantity is equivalent to superposing an appropriate particular solution. It is found that the suitable particular solution, in this case, is the condition of time-independent uniform potential for both of two- and three-dimensional problems. The singularity in the usual integral equations is cancelled by the relative quantity of potential and the regularized equations are derived. Owing to such fruits, the integral equation of potential gradient at a point on the surface is readily obtained by directly differentiating the equation of potential. It should be noted that it has not been given until now.

Through two- and three-dimensional numerical investigations, it is verified that the present integral equations give accurate numerical results everywhere over the domain and that they are valid and effective. The present results indicate that the analysis system

* Received 10th January, 1996. Japanese original: Trans. Jpn. Soc. Mech. Eng., Vol. 61, No. 586, A (1995), pp. 1363-1369. (Received 15th August, 1994)

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using BEM is expected to become more practical and more versatile system.

2. Formulations with Direct Method

2.1 Conventional formulation

The unsteady heat conduction problem is governed within a given domain as shown in Fig. 1 by the following differential equation (no heat generation is assumed for simplicity) :

$$\rho c \frac{\partial u}{\partial t} = k \nabla^2 u \quad (\text{inside } V). \quad (1)$$

Where, V is considered domain, u is temperature (potential), ρ and c represent the density and specific heat, respectively, and k is thermal conductivity. Boundary conditions are typically given by

Dirichlet Condition :

$$u(Q, t) = \bar{u}(Q, t) \quad (\text{on } S_1), \quad (2)$$

Neumann Condition :

$$q(Q, t) = -k \frac{\partial u}{\partial n} = \bar{q}(Q, t) \quad (\text{on } S_2), \quad (3)$$

Heat Transfer Condition :

$$q(Q, t) = \alpha \{u(Q, t) - u_o\} \quad (\text{on } S_3). \quad (4)$$

Where, $S = (S_1 + S_2 + S_3)$ is the boundary, n is direction cosine on the boundary, q is heat flux, α is heat transfer coefficient, u_o is temperature of surroundings and Q denotes a field point. Initial condition is given at $t = t_0$ by

$$\text{Initial Condition : } u(Q, t_0) = u_0(Q) \quad (\text{inside } V), \quad (5)$$

where, $u_0(Q)$ is initially given distribution of temperature.

The residual weighted form of Eq. (1) can be transformed through integration by parts to the following integral equations, which give the potential at a boundary point P and an interior point b , respectively, when the moment is t_f :

$$\begin{aligned} C(P)u(P, t_f) &= \int_{t_{f-1}}^{t_f} \int_S \{Q^*(P, Q; t_f, t)u(Q, t) \\ &\quad - U^*(P, Q; t_f, t)q(Q, t)\} ds dt \\ &\quad + \rho c \int_V U^*(P, Q; t_f, t_{f-1})u(Q, t_{f-1}) dv, \quad (6) \end{aligned}$$

$$\begin{aligned} u(b, t_f) &= \int_{t_{f-1}}^{t_f} \int_S \{Q^*(b, Q; t_f, t)u(Q, t) \\ &\quad - U^*(b, Q; t_f, t)q(Q, t)\} ds dt \\ &\quad + \rho c \int_V U^*(b, Q; t_f, t_{f-1})u(Q, t_{f-1}) dv. \quad (7) \end{aligned}$$

Equation (7) can be differentiated with respect to b . Then the potential gradient at b is obtained as

$$\begin{aligned} u_{,i}(b, t_f) &= \int_{t_{f-1}}^{t_f} \int_S \{Q_{,i}^*(b, Q; t_f, t)u(Q, t) \\ &\quad - U_{,i}^*(b, Q; t_f, t)q(Q, t)\} ds dt \\ &\quad + \rho c \int_V U_{,i}^*(b, Q; t_f, t_{f-1})u(Q, t_{f-1}) dv. \quad (8) \end{aligned}$$

Where, C is the position constant, subscript i denotes the coordinate component and subscript f represents

Fig. 1 Object under Heat Conduction Problem

time step counter. The kernel functions U^* , Q^* and their derivatives are fundamental solutions depending on time. Their expressions are shown in Appendix 1. It is noted that the integral equation of potential gradient at P is not given here. This is because Eq. (6) cannot be differentiated directly due to two obstacles⁽⁴⁾. One is discontinuity of Eq. (6) and Eq. (7) and the other is strong singularity of kernel functions as shown below.

According to the Appendix 1, it can be seen, at a glance, that U^* and Q^* contain high order of singularity not due to distance r but due to time. However, the time integration is convergent and it is possible to perform it prior to space integration. And yet, if the changes of u and q along the time axis are expressed in terms of usual time elements, the time integration can be performed analytically. In this study, constant element and linear element are prepared for time interpolation. Expressions after time integration are shown in Appendix 2. Concerning the singularity, it is enough to consider up to the first order of t at most noting that higher order of t in interpolation has no influence on the singularity. It is found from those that the singularity due to t vanished and the singularity due to r appeared instead. Moreover, its order is the same as one in the steady problems. It follows that the regularizing methods used in the steady problems are also applicable to the unsteady problems.

2.2 Formulation using relative quantity

According to the series of the present studies, the regularized integral equations are described using relative quantity of field functions. It has been shown that the relative quantities can generally be introduced by superposing a particular solution. On the present occasion, the particular solution which might be called as the condition of time-independent uniform potential is considered to be appropriate. It is evident that such a particular solution is expressed by Eq. (6). Noting that no flux is induced anywhere under this condition and that value of the uniform potential is arbitrary, this condition can be written,

$$- \int_S q(Q, t_f) U_c^*(p, Q) ds + \rho c \int_V \{u(Q, t_{f-1}) - u(P, t_f)\} U_c^*(p, Q) dv + u(P, t_f), \quad (15)$$

$$u_i(p, t_f) = \int_S \{u(Q, t_f) - u(P, t_f)\} Q_{ci}^*(p, Q) ds - \int_S q(Q, t_f) U_{ci}^*(p, Q) ds + \rho c \int_V \{u(Q, t_{f-1}) - u(P, t_f)\} U_i^*(p, Q) dv, \quad (16)$$

$$C(P)u_i(P, t_f) = \int_S \{u(Q, t_f) - u(P, t_f)\} Q_{ci}^*(P, Q) ds - \int_S q(Q, t_f) U_{ci}^*(P, Q) ds + \rho c \int_V \{u(Q, t_{f-1}) - u(P, t_f)\} U_i^*(P, Q) dv. \quad (17)$$

The kernel functions appeared in the above equations are shown in Appendix 2.1.

3.2 Time integration using linear element

In using linear element, nodal values of potential (as well as flux) are interpolated as follows:

$$u(Q, t) = \frac{t_f - t}{t_f - t_{f-1}} u(Q, t_{f-1}) + \frac{t - t_{f-1}}{t_f - t_{f-1}} u(Q, t_f). \quad (18)$$

Then, after the time integration, Eqs. (10) and (11) become as follows:

$$0 = \int_S \{u(Q, t_f) - u(P, t_f)\} \tilde{Q}^*(P, Q) ds - \int_S q(Q, t_f) \tilde{U}^*(P, Q) ds + \int_S \{u(Q, t_{f-1}) - u(P, t_f)\} \hat{Q}^*(P, Q) ds - \int_S q(Q, t_{f-1}) \hat{U}^*(P, Q) ds + \rho c \int_V \{u(Q, t_{f-1}) - u(P, t_f)\} U^*(P, Q) dv, \quad (19)$$

$$u(p, t_f) = \int_S \{u(Q, t_f) - u(P, t_f)\} \tilde{Q}^*(p, Q) ds - \int_S q(Q, t_f) \tilde{U}^*(p, Q) ds + \int_S \{u(Q, t_{f-1}) - u(P, t_f)\} \hat{Q}^*(p, Q) ds - \int_S q(Q, t_{f-1}) \hat{U}^*(p, Q) ds + \rho c \int_V \{u(Q, t_{f-1}) - u(P, t_f)\} U^*(p, Q) dv + u(P, t_f). \quad (20)$$

Where, the new kernel functions are defined, referencing Appendix 2.2, as

$$\tilde{U}^* = \frac{t_f}{t_f - t_{f-1}} U_c^* - \frac{1}{t_f - t_{f-1}} U_L^*, \quad (21)$$

$$\hat{Q}^* = \frac{t_f}{t_f - t_{f-1}} Q_c^* - \frac{1}{t_f - t_{f-1}} Q_L^*, \quad (22)$$

$$\tilde{U}^* = \frac{1}{t_f - t_{f-1}} U_L^* - \frac{t_{f-1}}{t_f - t_{f-1}} U_c^*, \quad (23)$$

$$\hat{Q}^* = \frac{1}{t_f - t_{f-1}} Q_L^* - \frac{t_{f-1}}{t_f - t_{f-1}} Q_c^*. \quad (24)$$

Although the integral equations of potential gradient for both of the boundary and the interior can be

written in a similar way, they are omitted here for space saving.

3.3 Discretization of integral equations

After time integration in each step, Eq. (14) or (19) is discretized using space elements to construct a set of linear simultaneous equations similarly with the steady problem. For the i -th source point, a linear equation is obtained by discretizing equation (14) or (19) as follows:

$$0 = h_{i1}(u_1 - u_i) + h_{i2}(u_2 - u_i) + \dots - g_{i1}q_1 - g_{i2}q_2 - \dots + B_i + B'_i + B''_i - (b_{i1} + b_{i2} + \dots + h'_{i1} + h'_{i2} + \dots)u_i \\ \therefore h_{i1}u_1 + \dots - (h_{i1} + \dots + b_{i1} + \dots + h'_{i1} + \dots)u_i + \\ = g_{i1}q_1 + \dots - B_i - B'_i - B''_i. \quad (25)$$

Where, the coefficients h, g, b, h', B, B' and B'' represent the corresponding integrated values, respectively. In case of constant time element, let $h' = B' = B'' = 0$. Equation (25) is composed, similarly with the steady problem, in the following matrix form:

$$Hu = Gq + B. \quad (26)$$

After solving the above simultaneous equations, potentials at interior nodes are calculated using Eq. (15) or (20) for next step's computation. It is found, from Eq. (25), that the diagonal terms in the H matrix are determined not by direct singular integration but just from sum of the other coefficients. In this manner, singular integration can naturally be avoided according to the present formulation.

4. Numerical Investigations

Two and three-dimensional examples are presented in order to examine validity and accuracy of the present integral equations with relative quantity. The quadratic element is used for space interpolation. Throughout this investigation, SI units are used in the definition of the examples.

4.1 Two-dimensional example with uni-axis heat flow

As the first example, a square domain model with uni-axis heat flow as shown in Fig. 3 are adopted to avoid the errors due to discretization. The sides $y = 0$ and $y = 1$ are prescribed to the adiabatic condition and the sides $x = 0$ and $x = 1$ are prescribed to the temperature condition. The physical properties are

Fig. 3 Analysis Model for Uni-Axis Heat Flow

Fig. 4 Temperature Distribution at Several Moments (Model: Fig. 3)

given by the imaginary values as shown in the figure. The boundary is divided into 20 elements and the domain is divided into 25 cells. The initial condition is given as $u=0$ in the whole domain. At the moment $t=t_0$, the temperature on the side $x=0$ suddenly rises up to $u=1$ and this temperature is held as it is. Subsequent changes of the temperature distribution are calculated and the numerical results are compared with the theoretical values⁽¹²⁾.

Numerical results of temperature distribution at several moments, in case of constant time element and 0.01 sec of time step size, are graphically shown in Fig. 4 and are compared with theoretical values⁽¹²⁾. It is shown that they agree well except those of the first step. It is suggested that it may be hard to express the sudden change of boundary values at the first step by usual space element. Table 1 shows comparison of numerical results from both of the constant and the linear time elements with theoretical values along the line of $y/L=0.5$, when $t=0.1$. The data in the Table indicate that the results from constant element are quite accurate, while those from linear element are much more accurate.

Effectiveness of relative quantity can be seen in Fig. 5 (in using constant element) and Fig. 6 (in using linear element). These figures describe behavior of the solutions near the boundary (along the dotted line in Fig. 3). In both figures, the solutions from the usual equations without relative quantity vibrate heavily as the computing point approaches the boundary. This is due to the quasi-singularity as seen also in the steady problems. In contrast with it, those with relative quantity never yield such phenomena, i.e., they are accurate everywhere in the domain.

4.2 Two-dimensional example with curved boundary

As the example with curved boundary line, the problem with circle domain as shown in Fig. 7 is employed (this is simplified model of the column with infinite length). Two cases of analysis condition are designated. One is that uniform temperature of $u=0$ is initially given in the whole domain and temperature

Table 1 Numerical Comparison of Results (Model: Fig. 3, $t=0.1$ sec)

x/L	Theore.	Const.	Error %	Linear	Error %
0.1	0.82304	0.82013	-0.35	0.82179	-0.15
0.2	0.65467	0.65177	-0.44	0.65333	-0.20
0.3	0.50219	0.50004	-0.43	0.50115	-0.21
0.4	0.37075	0.36941	-0.36	0.37024	-0.14
0.5	0.26276	0.26211	-0.25	0.26280	0.02
0.6	0.17797	0.17774	-0.13	0.17837	0.22
0.7	0.11387	0.11368	-0.17	0.11435	0.42
0.8	0.06635	0.06579	-0.84	0.06661	0.39
0.9	0.03027	0.02930	-3.20	0.03015	-0.40

Fig. 5 Behavior of Solutions near the Boundary (Constant Time Element, Model: Fig. 3)

Fig. 6 Behavior of Solutions near the Boundary (Linear Time Element, Model: Fig. 3)

on the circumference changes in accordance with $u=3t$. The other is that uniform temperature of $u=500$ is initially given in the whole domain. Then, at the moment $t=t_0$, temperature on the circumference suddenly drops to $u=0$ and this temperature is held as it is. The physical properties in both cases are given as shown in the figure. The boundary is divided into 16 elements and the domain is divided into 27 cells. Time step size is decided as 0.02 sec.

The results of the first case are graphically shown in Fig. 8 with theoretical values for comparison. It can be seen that the BEM solutions from both kinds of time element agree well with the theoretical ones. Figure 9 describes the temperature distribution of the second case during early stage in using constant time element and time step size of 0.1 sec as compared with

	case 1	case 2
$t < 0$ (domain)	$u=0$	$u=500$
$t \geq 0$ ($r/a=1$)	$u=3t$	$u=0$
k ($\text{Wm}^{-1}\text{K}^{-1}$)	1	41.26
c ($\text{Jkg}^{-1}\text{K}^{-1}$)	1	0.1146
ρ (kg/m^3)	1	7860

Fig. 7 Circle Domain Model for Two Dimensional Analysis

Fig. 8 Solutions of Temperature Transition (Case 1 Analysis, Model : Fig. 7)

Fig. 9 Solutions of Temperature Transition in Early Stage (Case 2 Analysis, Model : Fig. 7)

theoretical values. The results of the same problem in later stage are shown in Fig. 10. Time step size are decided as 0.5 sec for long period calculation. The data of both figures agree with theoretical values quite well. Figure 11 shows transition of surface temperature gradient calculated by Eq. (13) newly derived when constant time element and time step size of 0.5 sec are used. The solutions agree well with theoretical values.

Fig. 10 Solutions of Temperature Transition in Later Stage (Case 2 Analysis, Model : Fig. 7)

Fig. 11 Transition of Temperature Gradient on Surface (Case 2 Analysis, Model : Fig. 7)

Fig. 12 Three Dimensional Analysis Model

4.3 Three-dimensional example

Three-dimensional cuboid model as shown in Fig. 12 is provided as the last example. Its six surfaces are divided into 40 quadratic plane elements with 8 nodes and the interior domain is divided into 16 cells with 20 nodes. Boundary condition and initial condition are given to be similar with them in two-dimensional case shown in Fig. 3 in order to compare the results with each other. The physical properties designated are imaginary ones and time step size is 0.01 sec.

Three-dimensional results as well as two-dimensional ones of temperature distribution at several moments are shown in Fig. 13. Although three-dimensional solutions are apt to be a little higher than two-dimensional ones, both of them agree well with theoretical ones. Figure 14 shows transition of surface

Fig. 13 Temperature Distribution at Several Moments (Model: Fig. 12)

Fig. 14 Transition of Temperature Gradient on Surface (Model: Fig. 12)

temperature gradient calculated by Eq. (13). The present solutions from both of two- and three dimensional analysis concur well with theoretical ones.

Through the numerical investigations presented in the above, the present integral equations including one for surface temperature gradient are verified to be valid and effective for both of two- and three dimensional problems.

5. Conclusions

The regularizing technique using relative quantity of field functions is extended to the unsteady heat conduction problem. It is shown that, in the unsteady problems, only the singularity with respect to distance remains and its order becomes equal to one in the steady problems. Then, the present regularizing techniques are also applicable to the unsteady problem. The usual integral equations for potential are converted into the regularized ones by superposing a particular solution which might be called as the condition of time independent uniform potential upon the usual integral equations. The fundamental integral equation for potential can be written without position constant, i.e., written by the sole equation. Owing to these fruits, it becomes possible to differentiate directly the boundary integral equation of potential and then the integral equation for surface potential gradient has been readily obtained. Through the numerical investigations, the present integral equations derived

in this paper are verified to be valid and effective for both of two- and three-dimensional problems.

Appendix 1. Fundamental Solutions Depending on Time

For two dimensional problem:

$$U^* = \frac{1}{4\pi k\tau} \exp\left(-\frac{r^2}{4\lambda\tau}\right), \quad (27)$$

$$Q^* = \frac{r}{8\pi\lambda\tau^2} \frac{\partial r}{\partial n} \exp\left(-\frac{r^2}{4\lambda\tau}\right), \quad (28)$$

$$U_{,i}^* = -\frac{r r_{,i}}{8\pi k\lambda\tau^2} \exp\left(-\frac{r^2}{4\lambda\tau}\right), \quad (29)$$

$$Q_{,i}^* = \frac{1}{8\pi\lambda\tau^2} \left(n_i - \frac{r^2}{2\lambda\tau} r_{,i} \frac{\partial r}{\partial n} \right) \exp\left(-\frac{r^2}{4\lambda\tau}\right). \quad (30)$$

For three dimensional problem:

$$U^* = \frac{1}{\rho c (4\pi\lambda\tau)^{3/2}} \exp\left(-\frac{r^2}{4\lambda\tau}\right), \quad (31)$$

$$Q^* = \frac{r}{16(\pi\lambda)^{3/2} \tau^{5/2}} \frac{\partial r}{\partial n} \exp\left(-\frac{r^2}{4\lambda\tau}\right), \quad (32)$$

$$U_{,i}^* = -\frac{r r_{,i}}{16(\pi\lambda)^{3/2} k \tau^{5/2}} \exp\left(-\frac{r^2}{4\lambda\tau}\right), \quad (33)$$

$$Q_{,i}^* = \frac{1}{16(\pi\lambda)^{3/2} \tau^{5/2}} \left(n_i - \frac{r^2}{2\lambda\tau} r_{,i} \frac{\partial r}{\partial n} \right) \exp\left(-\frac{r^2}{4\lambda\tau}\right). \quad (34)$$

Where, r is distance between P and Q or p and Q , n is direction cosine, k is thermal conductivity, λ is thermal diffusivity ($=k/\rho c$) and $\tau = t_f - t$.

Appendix 2. Time Integration of Fundamental Solutions

A 2.1 Integration of constant term with respect to t (subscript C denotes constant): For two-dimensional problem:

$$U_C^* = \int_{t_{f-1}}^{t_f} U^* dt = \frac{1}{4\pi k} E_1(x_1), \quad (35)$$

$$Q_C^* = \int_{t_{f-1}}^{t_f} Q^* dt = \frac{1}{2\pi r} \frac{\partial r}{\partial n} e^{-x_1}, \quad (36)$$

$$U_{C,i}^* = \int_{t_{f-1}}^{t_f} U_{,i}^* dt = -\frac{r_{,i}}{2\pi k r} e^{-x_1}, \quad (37)$$

$$Q_{C,i}^* = \int_{t_{f-1}}^{t_f} Q_{,i}^* dt = \frac{1}{2\pi r^2} \left(n_i - 2(1+x_1) r_{,i} \frac{\partial r}{\partial n} \right) e^{-x_1}. \quad (38)$$

For three dimensional problem:

$$U_C^* = \int_{t_{f-1}}^{t_f} U^* dt = \frac{1}{4\pi^{3/2} k r} \Gamma\left(\frac{1}{2}, x_1\right), \quad (39)$$

$$Q_C^* = \int_{t_{f-1}}^{t_f} Q^* dt = \frac{1}{2\pi^{3/2} r^2} \frac{\partial r}{\partial n} \Gamma\left(\frac{3}{2}, x_1\right), \quad (40)$$

$$U_{C,i}^* = \int_{t_{f-1}}^{t_f} U_{,i}^* dt = -\frac{r_{,i}}{2\pi^{3/2} k r^2} \Gamma\left(\frac{3}{2}, x_1\right), \quad (41)$$

$$Q_{C,i}^* = \int_{t_{f-1}}^{t_f} Q_{,i}^* dt = \frac{1}{\pi^{3/2} r^3} \left\{ \frac{n_i}{2} \Gamma\left(\frac{3}{2}, x_1\right) - r_{,i} \frac{\partial r}{\partial n} \Gamma\left(\frac{5}{2}, x_1\right) \right\}. \quad (42)$$

Where, E_1 is integral exponential function, Γ is Gamma function and x_1 is defined as

$$x_1 = \frac{r^2}{4\lambda(t_f - t_{f-1})}. \quad (43)$$

A 2.2 Integration of linear term with respect to t
(subscript L denotes linear) :

For two dimensional problem :

$$U_L^* = \int_{t_{i-1}}^{t_i} U_L^* dt = \frac{1}{4\pi k} \left[t_f E_1(x_1) - \frac{r^2}{4\lambda} \left(\frac{e^{-x_1}}{x_1} - E_1(x_1) \right) \right], \quad (44)$$

$$Q_L^* = \int_{t_{i-1}}^{t_i} Q_L^* dt = \frac{1}{2\pi} \frac{\partial r}{\partial n} \left[\frac{t_f}{r} e^{-x_1} - \frac{r}{4\lambda} E_1(x_1) \right], \quad (45)$$

$$U_{Li}^* = \int_{t_{i-1}}^{t_i} U_L^* dt = \frac{r_{i,i}}{2\pi k} \left[\frac{r}{4\lambda} E_1(x_1) - \frac{t_f}{r} e^{-x_1} \right], \quad (46)$$

$$Q_{Li}^* = \int_{t_{i-1}}^{t_i} Q_L^* dt = \frac{1}{\pi} \left[\frac{t_f}{r^2} \left(\frac{n_i}{2} - r_{i,i} \frac{\partial r}{\partial n} (x_1 + 1) \right) + \frac{r_{i,i}}{4\lambda} \frac{\partial r}{\partial n} \right] e^{-x_1} - \frac{n_i}{8\pi\lambda} E_1(x_1). \quad (47)$$

For three dimensional problem :

$$U_L^* = \int_{t_{i-1}}^{t_i} U_L^* dt = \frac{1}{4\pi^{3/2} k} \left[\frac{t_f}{r} \Gamma\left(\frac{1}{2}, x_1\right) - \frac{r}{4\lambda} x_1^{-3/4} e^{-x_1/2} W_{-3/4, -1/4}(x_1) \right], \quad (48)$$

$$Q_L^* = \int_{t_{i-1}}^{t_i} Q_L^* dt = \frac{1}{2\pi^{3/2}} \frac{\partial r}{\partial n} \left[\frac{t_f}{r^2} \Gamma\left(\frac{3}{2}, x_1\right) - \frac{1}{4\lambda} \Gamma\left(\frac{1}{2}, x_1\right) \right], \quad (49)$$

$$U_{Li}^* = \int_{t_{i-1}}^{t_i} U_L^* dt = \frac{r_{i,i}}{2\pi^{3/2} k} \left[-\frac{1}{4\lambda} \Gamma\left(\frac{1}{2}, x_1\right) - \frac{t_f}{r^2} \Gamma\left(\frac{3}{2}, x_1\right) \right], \quad (50)$$

$$Q_{Li}^* = \int_{t_{i-1}}^{t_i} Q_L^* dt = \frac{1}{\pi^{3/2} r} \left[\left(\frac{t_f n_i}{2r^2} + \frac{r_{i,i}}{4\lambda} \frac{\partial r}{\partial n} \right) \Gamma\left(\frac{3}{2}, x_1\right) - \frac{n_i}{8\lambda} \Gamma\left(\frac{1}{2}, x_1\right) - \frac{t_f r_{i,i}}{r^2} \frac{\partial r}{\partial n} \Gamma\left(\frac{5}{2}, x_1\right) \right]. \quad (51)$$

Where, W is Whittaker function.

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