Applicability of Fracture Mechanics in Strength Evaluation of Functionally Graded Materials*

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Elastic and elastic-plastic analyses of a crack in a particulate-dispersed functionally graded material (FGM) have been carried out using a newly developed finite element method based on Tohgo-Chou-Weng's(1994, 1996) constitutive relation for particulate-reinforced composites. By setting the mechanical properties of particles and a matrix and their content graded in the thickness direction, FGMs and non-FGM are designed. From comparison of the numerical results for the FGMs and non-FGM, the influence of the gradient of the mechanical properties on a stress intensity factor and the crack tip field is discussed. The following conclusions are derived: (1) The stress intensity factor of a crack under constant boundary conditions is considerably affected by the gradient of the mechanical properties. (2) The elastic and plastic stress singular fields around a crack tip in a FGM are basically described by the fracture mechanics parameters(K_I and J_I) as well as in a non-FGM, using the mechanical properties of the material at the crack tip. (3) The size of the singular field decreases with an increase in the gradient of the mechanical properties. This means that the applicability of fracture mechanics, such as the small-scale-yielding condition and the validity of the J -integral, is affected by the gradient.

Key Words: Functionally Graded Material, Fracture Mechanics, Finite Element Method, Particulate-Reinforced Composite, Stress Intensity Factor, Crack-Tip Field

1. Introduction

Functionally graded materials(FGMs) in which material components or their contents varies in some direction have a large amount of potentiality as engineering materials because of their unique performance⁽¹⁾⁻⁽¹⁷⁾. A FGM plate consisting of ceramics and metals, for example, can be designed to reduce the thermal stress and to maintain a high heat resistance and ductility. The investigations about the mechanics and strength of FGMs deal with (1) thermal stress and its relaxation under the thermal loading⁽²⁾⁻⁽⁵⁾, (2) fracture behavior under thermal shock or thermal fatigue⁽⁶⁾⁻⁽¹¹⁾, and (3) stress intensity factor of a crack under thermal loading⁽¹²⁾⁻⁽¹⁴⁾. In order to apply FGMs to the engineering structure, it is necessary to estimate the strength and fracture mechanism not only under thermal loading but also under

external loading such as static and cyclic loading⁽¹⁵⁾⁻⁽¹⁷⁾. Furthermore, in the evaluation of the strength of FGMs, the application of the fracture mechanics is dispensable as in conventional materials.

In the present investigation, for the FGMs in which the ceramic particles are dispersed in the ductile metal matrix and a particle volume fraction is linearly changed, elastic and elastic-plastic analyses of a crack were carried out based on a newly developed finite element method⁽¹⁸⁾⁻⁽²⁰⁾. Influence of gradient of mechanical properties on the stress intensity factor and the crack tip singular field was discussed.

2. Fracture Mechanics in Functionally Graded Materials

Let us consider a crack emanating from ceramic side in FGM with the volume content of ceramic particles f_{p0} linearly changing from 100% to 0%, as shown in Fig. 1. Since crack initiation is easy to occur from the ceramic side under some external loading because of low fracture toughness of ceramics, the crack problem in Fig. 1 is practically important. It is critical issue whether the crack grows and leads to

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Fig. 1 Distribution of mechanical properties and crack problem in a functionally graded material

catastrophic fracture of the plate or the final failure of the plate is avoided owing to the crack arrest. When we apply the fracture mechanics to this problem, the crack extension or the crack arrest would be evaluated based on the distributions of fracture toughness and crack driving force, where the fracture toughness is determined experimentally, and the crack driving force is calculated under given external loading. However, there is no knowledge on the applicability of fracture mechanics to FGMs. The question of fracture mechanics in FGMs comes from the gradient of the mechanical properties, such as Young's modulus, Poisson's ratio, yield stress, tensile strength and fracture toughness, around a crack tip. It was already reported that the stress singular field exists around a crack tip in FGMs as well as in non-FGM^{(13),(15)}. From a viewpoint of evaluating the crack growth behavior, the next two points should be clarified.

(1) Influence of the gradient on the fracture mechanics parameter.

For cracks under constant boundary conditions, the stress intensity factor or J -integral seems to be affected by the gradient of mechanical properties. In the case of FGM, since even the stress intensity factor cannot be evaluated by the results for non-FGM, it should be analyzed for each configuration and for each material.

(2) Influence of the gradient on the stress/strain field around a crack tip.

In the concept of fracture mechanics, a crack tip field is uniquely represented by the stress intensity factor or J -integral. If the crack tip field under constant stress intensity is affected by the gradient of mechanical properties, fracture behavior from the

crack may be different between non-FGM and FGM. This means that we cannot predict the fracture behavior in FGM based on the fracture mechanics using the basic data of non-FGM consisting of the material at the crack tip in the FGM.

The present work was conducted in order to clarify the above two points.

3. Constitutive Relations

Here, the constitutive relations for particulate-reinforced composites used in the following finite element analyses are briefly explained^{(18),(19)}. The incremental constitutive relation ($d\epsilon_{ij} - d\sigma_{ij}$ relation) is described by the form decomposed into the hydrostatic part ($d\epsilon_{kk} - d\sigma_{kk}$ relation) and the deviatoric part ($d\epsilon'_{ij} - d\sigma'_{ij}$ relation). $d\sigma_{ij}$ and $d\epsilon_{ij}$ are incremental total stress and strain, $(1/3)d\sigma_{kk}$, $(1/3)d\epsilon_{kk}$ and $d\sigma'_{ij}$, $d\epsilon'_{ij}$ are their hydrostatic and deviatoric parts, respectively. The incremental total stress and strain are given by

$$d\sigma_{ij} = d\sigma'_{ij} + \frac{1}{3}d\sigma_{kk}\delta_{ij}, \quad d\epsilon_{ij} = d\epsilon'_{ij} + \frac{1}{3}d\epsilon_{kk}\delta_{ij} \quad (1)$$

In the composite system the elastic spherical particles are homogeneously dispersed in the elastic-plastic matrix.

3.1 Properties of constituent materials

The elastic incremental stress-strain relation of the isotropic particle follows as

$$d\epsilon_{kk} = \frac{1}{3\kappa_1}d\sigma_{kk}, \quad d\epsilon'_{ij} = \frac{1}{2\mu_1}d\sigma'_{ij} \quad (2)$$

where κ_1 and μ_1 are the bulk modulus and shear modulus of the particle. The elastic behavior of the matrix is also taken to be isotropic with the moduli κ_0 and μ_0 . κ_0 , κ_1 , and μ_0 , μ_1 are related to Young's moduli E_0 , E_1 and Poisson's ratios ν_0 , ν_1 by

$$\kappa_i = \frac{E_i}{3(1-2\nu_i)}, \quad \mu_i = \frac{E_i}{2(1+\nu_i)}, \quad i=0 \text{ or } 1. \quad (3)$$

The elastic-plastic deformation of the matrix is described by the Prandtl-Ruess equation (the J_2 -flow theory), which is approximated by the following isotropic relation⁽¹⁸⁾:

$$d\epsilon_{kk} = \frac{1}{3\kappa_0}d\sigma_{kk}, \quad d\epsilon'_{ij} = \frac{1}{2\mu_0'}d\sigma'_{ij} \quad (4)$$

where

$$\mu_0' = \frac{\mu_0}{1 + \frac{3\mu_0}{H'}}, \quad \nu_0' = \frac{\nu_0 + \frac{\mu_0}{H'}(1 + \nu_0)}{1 + 2\frac{\mu_0}{H'}(1 + \nu_0)}, \quad (5)$$

and μ_0' and ν_0' are the tangent shear modulus and tangent Poisson's ratio of the matrix under elastic-plastic deformation. H' is the work-hardening ratio of the matrix:

$$H' = \frac{d\sigma_e}{d\epsilon_e^{pl}} \quad (6)$$

where

$$\sigma_e = \left(\frac{3}{2} \sigma'_{ij} \sigma'_{ij} \right)^{1/2}, \quad d\epsilon_e^{pl} = \left(\frac{2}{3} d\epsilon_{ij}^{pl} d\epsilon_{ij}^{pl} \right)^{1/2}. \quad (7)$$

Here σ_e and $d\epsilon_e^{pl}$ are the von Mises' equivalent stress and incremental equivalent plastic strain, respectively, and $d\epsilon_{ij}^{pl}$ is the incremental plastic strain. Equation (4) are strictly valid in the case of monotonic proportional loading.

In the composite system the stress and strain of the particle and matrix are represented with the superscripts "p" and "m", respectively, and those of the composite are shown by symbols without superscript.

3.2 Incremental constitutive relation for composite

Tohgo and Chou⁽¹⁸⁾ developed the incremental damage theory, which describes the plasticity of the matrix and debonding damage of the particles from the matrix, based on Eshelby's equivalent inclusion method⁽²¹⁾ and Mori and Tanaka's mean field concept⁽²²⁾. This theory is applicable not only to the deformation and damage of particulate-reinforced composites but also to hybrid composite containing particles and voids and porous material. Here, the constitutive relation for the hybrid composites without the debonding damage will be shown. If the volume fractions of the particles and voids are denoted by f_{p0} and f_{v0} , respectively, the constitutive relation is given as follows^{(18),(19)}:

$$d\epsilon_{kk} = \frac{1}{3\kappa_t} d\sigma_{kk}, \quad (8)$$

$$d\epsilon'_{ij} = \frac{1}{2\mu_t} d\sigma'_{ij}, \quad (9)$$

where

$$\kappa_t = \frac{\kappa_0(1-\alpha)A_h}{\{(1-\alpha)(1-f_{p0}-f_{v0})+f_{v0}\}\{\kappa_0+(\kappa_1-\kappa_0)\alpha\} + \kappa_0(1-\alpha)f_{p0}} \quad (10)$$

$$\mu_t = \frac{\mu'_0(1-\beta)A_d}{\{(1-\beta)(1-f_{p0}-f_{v0})+f_{v0}\}\{\mu'_0+(\mu_1-\mu'_0)\beta\} + \mu'_0(1-\beta)f_{p0}} \quad (11)$$

and

$$A_h = (1-f_{p0}-f_{v0})\{\kappa_0+(\kappa_1-\kappa_0)\alpha\} + f_{p0}\kappa_1 \quad (12)$$

$$A_d = (1-f_{p0}-f_{v0})\{\mu'_0+(\mu_1-\mu'_0)\beta\} + f_{p0}\mu_1. \quad (13)$$

Here $(d\sigma_{kk}, d\sigma'_{ij})$ and $(d\epsilon_{kk}, d\epsilon'_{ij})$ are the incremental stress and strain of the composite. With spherical particles and voids the constants α and β follow from Eshelby's relations:

$$\alpha = \frac{1}{3} \frac{1+\nu'_0}{1-\nu'_0}, \quad \beta = \frac{2}{15} \frac{4-5\nu'_0}{1-\nu'_0} \quad (14)$$

The incremental average stresses of the matrix and particles are given by

$$d\sigma_{kk}^m = \frac{\kappa_0+(\kappa_1-\kappa_0)\alpha}{A_h} d\sigma_{kk}, \quad (15)$$

$$d\sigma_{ij}^m = \frac{\mu'_0+(\mu_1-\mu'_0)\beta}{A_d} d\sigma'_{ij}, \quad (16)$$

$$d\sigma_{kk}^p = \frac{\kappa_1}{A_h} d\sigma_{kk}, \quad (17)$$

$$d\sigma_{ij}^p = \frac{\mu_1}{A_d} d\sigma'_{ij}. \quad (18)$$

3.3 Equivalent stress of the matrix in the composite

The equivalent stress of the matrix in the composite are estimated by Tohgo and Weng's theory⁽¹⁹⁾. According to the theory, after the incremental deformation the equivalent stress of the matrix is estimated by $\sigma_e^m + d\sigma_e^m$, where σ_e^m and $d\sigma_e^m$ denote the current equivalent stress before the incremental deformation and its increment, respectively. In the numerical analysis σ_e^m is known and $d\sigma_e^m$ is given by

$$d\sigma_e^m = \frac{3\mu'_0}{\sigma_e^m(1-f_{p0}-f_{v0})} dU - \frac{3\mu'_0}{\sigma_e^m \kappa_0} \sigma_m^m d\sigma_m^m - \frac{3\mu'_0 f_p}{\sigma_e^m(1-f_{p0}-f_{v0})} \sigma_{ij}^p d\epsilon_{ij}^p, \quad (19)$$

where $\sigma_m^m (= (1/3)\sigma_{kk}^m)$ and $d\sigma_m^m (= (1/3)d\sigma_{kk}^m)$ are the hydrostatic stress of the matrix and its increment, respectively. dU is the incremental energy of the composite:

$$dU = \sigma_{ij} d\epsilon_{ij}. \quad (20)$$

If the matrix is in the elastic state, μ'_0 and ν'_0 in all the equations in Sections 3.2 and 3.3 reduce to their elastic counterparts μ_0 and ν_0 , respectively.

4. Numerical Procedure

Elastic and elastic-plastic analyses of a crack in functionally graded materials were carried out using a newly developed finite element method (FEM) based on the incremental constitutive relation for the hybrid composites containing particles and voids. This FEM is easily applied to the analysis of functionally graded materials (FGMs), by setting the distribution of the initial particle volume fraction and void volume fraction. On the ceramic-metal system, the ceramic particles are dispersed in the metal matrix in the material with low content of ceramics, while the metal is dispersed in the ceramic matrix in the material with high content of ceramics. Therefore, in the actual FGM the microstructure of the ceramic particles and metal matrix on the metal rich side is inversely changed to that of the metal particles and ceramic matrix on the ceramic rich side. However, the present analyses were carried out under the assumption that the ceramic particles are always dispersed in the metal matrix.

4.1 Elastic analysis

Elastic analysis was conducted to investigate the influence of the gradient of the mechanical properties on the stress intensity factor K_I . Figure 2 shows the

Fig. 2 Crack models used in elastic analysis

Fig. 3 Distribution of Young's modulus and Poisson's ratio in the FGMs used in elastic analysis

Fig. 4 Crack models used in elastic-plastic analysis

FGM specimens under uniform displacement tension (UDT), uniform stress tension (UST) and three point bending (3 PB), in which the particle volume fraction

Table 1 Material properties used in elastic analysis

Specimen	Material properties		Loading condition		
	Matrix	Particle	UDT	UST	3PB
Non-FGM	E_0	E_1			
FGM-1	ν_0	ν_1			
FGM-2					
FGM-3					
FGM-4					
FGM-5					

σ_0 : Yield stress of matrix material

O : Analyzed loading condition

Table 2 Material properties used in elastic-plastic analysis

Specimen	Particle volume fraction	Material properties	
		Matrix (Elastic-plastic)	Particle (Elastic)
EP-50	50%	$E_0=500\sigma_0$	$E_1=5000\sigma_0$
EP-FGM	Graded 100% to 0%	$\nu_0=0.3$	$\nu_1=0.17$
EP-FGM2	Graded 100% to 0% in interphase	$\sigma^m = \sigma_0 \left(1 + \frac{\epsilon^m p}{\epsilon_0} \right)^{0.1}$	

σ_0 : Yield stress of matrix material

is linearly graded from 100% to 0% and a edge crack emanates from the ceramic side. The material properties of the matrix and particle are shown in Table 1. One non-FGM and five FGMs are designed by the combinations of the material properties. By changing the material properties of the particle under the constant matrix material, the FGMs with different gradient of the mechanical properties are created. Figure 3 shows the distributions of Young's modulus and Poisson's ratio along the crack direction for each material.

4.2 Elastic-plastic analysis

Elastic-plastic analysis was conducted to study the plastic zone and stress field around a crack-tip in FGM. Figure 4 shows the single edge cracked specimens under uniform displacement tension for a non-FGM (EP-50) and two FGMs (EP-FGM and EP-FGM 2). In EP-FGM, the particle volume fraction is linearly graded from 100% to 0% and a edge crack emanates from the ceramic side. EP-FGM 2 simulates a crack in ceramic-metal joining with interphase where the particle volume fraction varies within the very narrow region. EP-50 is a particulate reinforced composite with particle volume fraction of $f_{p0}=50\%$ which corresponds to the material at the crack tip in EP-FGM and EP-FGM 2. Young's modulus and Poisson's ratio of the matrix and particle, and the equivalent stress-strain relation of the matrix are shown in Table 2. σ_0 in Table 2 is the yield stress of the matrix. Figure 5 shows distributions of Young's

Fig. 5 Distribution of Young's modulus and Poisson's ratio in the FGMs used in elastic-plastic analysis

Fig. 6 Stress-strain curves at ceramic side and metal side surfaces and at a crack tip in the FGMs

Fig. 7 Finite element mesh used in analysis

modulus and Poisson's ratio along the crack direction in three materials, and Fig. 6 exhibits the stress-strain relations under uniaxial tension for the materials at the ceramic side, crack tip and metal side in FGMs. Three kinds of materials are the same in the mechanical properties at the crack tip, but are different in their gradient; for example, the gradient of Young's modulus $d(E/\sigma_0)/dx$ is 0, 480 and 2 400 (1/mm) for EP-50, EP-FGM and EP-FGM 2, respectively.

Figure 7 is an example of finite element mesh used

Fig. 8 K_I - r relation in Non-FGM and FGM-4 under uniform displacement tension

in the analyses. A very thin notch is regarded as a crack. Both elastic and elastic-plastic analyses were carried out under the assumption of plane strain deformation.

5. Stress Intensity Factor

Stress intensity factors were determined by a stress method and a J -integral method which are conventionally used for homogeneous material. In the stress method which utilizes the elastic singular stress field around a crack tip, the stress intensity factor is obtained by extrapolating $K_I (= \sigma_y \sqrt{2\pi r})$ to the crack tip, where K_I is calculated from the stress σ_y at the position of the distance r ahead of the crack tip. Figure 8 shows the K_I - r relations for the Non-FGM and FGM-4 under uniform displacement tension. It is found that the gradient of the elastic moduli affects the gradient of K_I - r relation, namely non singular stress field around a crack tip.

In homogeneous materials, J -integral is path-independent and the stress intensity factor is easily obtained from the J -integral. In the FGMs, however, as shown in Appendix, the J -integral is path-dependent and a value for a path adjacent to a crack tip is identical to that for homogeneous materials. In the present analyses, the J -integral is calculated for the many circular paths with different radius around a crack tip. Figure 9 shows the path-dependency of the J -integral obtained circular paths surrounding a crack-tip. Although the J -integral for the Non-FGM is path-independent, FGMs exhibit the path-dependency depending on the gradient of mechanical properties. Using the value of J -integral extrapolated to

Fig. 9 Path-dependency of J -integral in the FGMs under uniform displacement tension

the crack tip and the mechanical properties (E and ν) at the crack tip in FGM, the stress intensity factor is obtained by

$$K_I = \sqrt{\frac{EJ_I}{(1-\nu^2)}} \quad (21)$$

From K_I obtained by both the stress method and the J -integral method, the correction factor F_I is calculated by

$$F_I = \frac{K_I}{\sigma \sqrt{\pi a}}, \quad (22)$$

where, σ is average tensile stress for the UST and UDT specimens, and $\sigma = 3PS/(2W^2B)$ for the 3PB specimen. P is load, and a , W , B and S are crack length, width, thickness and span length of the 3PB specimen. Table 3 shows the correction factors of K_I . F_I and F_{II} are the results obtained by the stress method and the J -integral method, respectively. In all cases F_I and F_{II} give the same value, and also in the non-FGM they are well consistent with the results of conventional equations for fracture mechanics specimens. It is concluded that the stress intensity factor of a crack in FGM can be determined by the stress method and the J -integral method. The difference due to the materials in the correction factors indicates the influence of the gradient of mechanical properties. For example, in the UDT specimen, the correction factor is larger when the gradient of the mechanical properties is steeper. This means that the stress intensity factor for FGMs cannot be evaluated by the results for non-FGM collected in handbooks, and it has to be analyzed for given specimens and given materials; this is significant problem on the application of fracture mechanics to FGMs.

Table 3 Stress intensity factor for a crack in the FGMs

Specimen	UDT		UST			3PB		
	F_I	F_{II}	F_I	F_{II}	F_{IS}	F_I	F_{II}	F_{IS}
Non-FGM	1.783	1.763	2.843	2.827	2.827	1.420	1.402	1.416
FGM-1	1.751	1.735	-	-	-	-	-	-
FGM-2	2.084	2.074	-	-	-	-	-	-
FGM-3	2.472	2.472	-	-	-	-	-	-
FGM-4	2.695	2.691	3.384	3.358	-	1.863	1.846	-
FGM-5	2.950	2.927	-	-	-	-	-	-

F_{IS} : by Brown and Srawley

F_{IS} : by Srawley

Fig. 10 Plastic zone around a crack tip

6. Crack Tip Singular Field

Here, influence of the gradient of mechanical properties on the plastic zone and stress field around a crack tip will be discussed based on the results of the elastic-plastic analyses for EP-50, EP-FGM and EP-FGM 2 on the basis of the stress intensity factor. The stress intensity factor is calculated from the applied stress and the correction factors obtained by elastic analysis of the specimens in Fig. 4, and it is represented in the form of $K_I = \sigma_0 \sqrt{\pi a}$ using the yield stress of the matrix σ_0 measured in MPa.

6.1 Plastic zone around a crack-tip

Development of the plastic zone from a crack-tip in the EP-50, EP-FGM and EP-FGM 2 are shown in Fig. 10, and the maximum plastic zone size as a function of the stress intensity factor standardized by the yield stress of the matrix is shown in Fig. 11. It is found from these figures that the size and shape of the plastic zone are almost the same in all materials under low stress intensity, while under the high stress intensity the plastic zone size is affected by the gradient of the mechanical properties and it becomes smaller in

Fig. 11 Plastic zone size as a function of K_I/σ_0

order of the EP-50, EP-FGM and EP-FGM 2.

6.2 Stress field around a crack-tip

Stress distributions along the line ahead of a crack tip under the four K_I levels are shown in Figs. 12(a), (b) and (c). In these figures, the normalized stresses (σ_y/σ_0 , σ_y^m/σ_0 , σ_y^p/σ_0) are presented as a function of normalized distance ($r/(J_I/\sigma_0)$) from a crack tip in the log-log diagram. Here, J_I is calculated from the stress intensity factor K_I under the assumption of small scale yielding and σ_y , σ_y^m , and σ_y^p are macroscopic stress and average microscopic stresses in the matrix and particles, respectively. In these figures the distribution under lowest K_I level represents the result of the first elastic analysis. In the EP-50 and EP-FGM, the elastic singular K_I field described by the line of gradient 1/2 and the plastic singular HRR field described by the line of gradient $Nc/(Nc+1)$ are created around a crack tip. The EP-FGM 2 exhibits almost the same feature except for the HRR field depending on the K_I level. Figures 13(a) and (b) show the comparison of the stress distributions amongst the EP-50, EP-FGM and EP-FGM 2. Figure 13(a) is the stress distributions along the line ahead of a crack tip and Fig. 13(b) is the angular distributions of the stresses (σ_{rr} , $\sigma_{\theta\theta}$, $\sigma_{r\theta}$, σ_e) at the distance of $r/(J_I/\sigma_0)=10$ from a crack tip for the third step of K_I level. The K_I field is the same in all materials, but the HRR field is affected by the gradient of mechanical properties. On both distributions, the HRR field in the EP-FGM is very close to that in the EP-50, while the characteristic in the EP-FGM 2 is shifted to lower side of stress with increasing stress intensity. However, the influence of the gradient of mechanical properties on the HRR field is not strong and almost the same HRR field is created in both FGM and non-FGM. Furthermore, Nc which characterizes the HRR field corresponds to work-hardening exponent of the composite material containing particles of 50% in

(a) EP-50

(b) EP-FGM

(c) EP-FGM 2

Fig. 12 Stress distribution along the line ahead of a crack tip

volume content. It is concluded that the HRR field in FGMs is basically described by the J -integral and the mechanical properties of the material at the crack tip.

(a) Distribution along the line ahead of a crack tip

(b) Angular distribution

Fig. 13 Comparison of stress distributions around a crack tip in FGMs

From the comparison of the stress distribution under each K_I level in Fig. 13(a), it is noted that in the region apart from the crack tip the material with higher gradient exhibits lower stress. This result well corresponds to that the plastic zone in the EP-FGM and EP-FGM 2 is smaller than that in the EP-50 under high stress intensity. This means that the size of the elastic singular field and HRR singular field decreases with an increase in the gradient of mechanical properties. As a result, in the highly graded FGM, the singular field is limited in the vicinity of a crack tip or vanishes, and it is expected that the applicability of fracture mechanics such as the small scale yielding condition and the validity of J -integral is affected by the gradient of the mechanical properties.

7. Conclusions

Elastic and elastic-plastic analyses of a crack in

Fig. 14 J -integral path

FGMs have been carried out by the finite element method based on the constitutive relation for particulate reinforced composite. The obtained results are summarized as follows.

(1) The stress intensity factor of a crack in FGM can be determined by the stress method or the J -integral method.

(2) The stress intensity factor of a crack in FGM is affected by the gradient of mechanical properties, and therefore it cannot be evaluated by the results for non-FGM in handbooks.

(3) The plastic zone around a crack tip is affected by the gradient of the mechanical properties. In the present specimen configuration, it becomes smaller in the material with higher gradient.

(4) The elastic and plastic stress singular fields around a crack tip in FGM are basically described by the fracture mechanics parameters (K_I and J_I) as well as in non-FGM, using the mechanical properties of the material at the crack tip.

(5) In the present specimen configuration, the size of singular field decreases with an increase in the gradient of mechanical properties. As a result, in the highly graded FGM, the singular field is limited in the vicinity of a crack tip or vanishes. This means that the small scale yielding condition and the applicability of J -integral are affected by the gradient.

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Appendix

Rice's J -integral for the path Γ surrounding a crack tip from lower surface to upper surface as shown in Fig. 14 is given by

$$J = \int_{\Gamma} \left(W n_1 - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} \right) ds \quad (23)$$

where σ_{ij} , ϵ_{ij} , u_j and n_i are stress, strain, displacement and normal unit vector of the path Γ , respectively.

Furthermore, for the linear elastic body, the equilibrium equation of stresses, the strain-displacement relation, the constitutive relation and the strain energy density W are expressed as follows:

$$\sigma_{ij,j} = 0 \quad (24)$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (25)$$

$$\sigma_{ij} = D_{ijkl}\varepsilon_{kl} \quad (26)$$

$$W = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} = \int_0^{\varepsilon_{ij}} D_{ijkl}\varepsilon_{kl} d\varepsilon_{ij} \quad (27)$$

Using the Gauss' divergence theorem, the J -integral for the closed path Γ_0 in Fig. 14 becomes

$$\begin{aligned} & \int_{\Gamma_0} \left(W n_1 - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} \right) ds_0 \\ &= \iint_{A_0} \left(\frac{\partial W}{\partial x_1} - \sigma_{ij,j} \frac{\partial u_i}{\partial x_1} - \sigma_{ij} \frac{\partial u_{i,j}}{\partial x_1} \right) dA_0 \end{aligned} \quad (28)$$

where

$$\begin{aligned} \frac{\partial W}{\partial x_1} &= \frac{\partial}{\partial x_1} \int_0^{\varepsilon_{ij}} D_{ijkl}\varepsilon_{kl} d\varepsilon_{ij} \\ &= \int_0^{\varepsilon_{ij}} \frac{D_{ijkl}}{\partial x_1} \varepsilon_{kl} d\varepsilon_{ij} + \sigma_{ij} \frac{\partial u_{i,j}}{\partial x_1} \end{aligned} \quad (29)$$

In the conventional non-FGM, since the first term of right hand side in the above equation vanishes, the J -integral for Γ_0 equals to zero and the J -integral for surrounding crack tip is path-independent. In the FGM, however, since the first term in the above equation does not vanish, Eq.(28) is written as

$$\int_{\Gamma_0} \left(W n_1 - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} \right) ds_0 = \iint_{A_0} \int_0^{\varepsilon_{ij}} \frac{D_{ijkl}}{\partial x_1} \varepsilon_{kl} d\varepsilon_{ij} dA_0 \quad (30)$$

Therefore, the J -integral for a crack in FGM is path-dependent, and the path-independent integral is given by the terms of Rice's J -integral and area integral as follows:

$$\begin{aligned} J_{FGM} &= \int_{\Gamma} \left(W n_1 - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} \right) ds \\ &- \iint_A \int_0^{\varepsilon_{ij}} \frac{D_{ijkl}}{\partial x_1} \varepsilon_{kl} d\varepsilon_{ij} dA \end{aligned} \quad (31)$$

This corresponds to the expression for FGM of T^* -integral proposed by Atluri et al.^{(23),(24)}. Since the second term in the above equation is the area integral of a function with singularity of order r^{-1} , when the path Γ infinitely approaches the crack tip, the second term vanishes. This means that Rice's J -integral for the path adjacent a crack tip in FGM gives consistency with the J -integral in non-FGM.

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