Finite Expansion of an Infinitesimal Void in Elastic-Plastic Materials under Equitriaxial Stress*

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Sudden growth of an infinitesimal void to a finite size under equitriaxial tension is studied for elastic-plastic materials via a bifurcation approach. The analysis employs the Prandtl-Reuss model with finite deformation taken into account, for both strain-hardening and perfectly plastic solids. Expressions for critical stress and strain levels for finite void growth, namely, cavitation limits, are obtained in the form of integrals involving material parameters and hardening characteristics. Numerical results for the critical values and post-cavitation behavior are demonstrated for power-law hardening elastic-plastic materials, and the influence of hardening exponents as well as elastic compliance is discussed in detail.

Key Words: Plasticity, Finite Deformation Theory, Void Formation and Growth, Bifurcation, Nonlinear Problem, Prandtl-Reuss Material, Equitriaxial Stress

1. Introduction

Ductile particles and fibers are often introduced in stiff and brittle matrices to improve their mechanical properties such as fracture toughness⁽¹⁾. Soft and ductile layers are also used for bonding brittle components such as ceramics⁽²⁾. In these circumstances ductile materials tend to be subjected to highly triaxial stresses since their deformations are heavily constrained by stiffer substances surrounding them. Experimental observations have revealed that macroscopic cavities (or voids) are created in ductile materials in the presence of sufficiently high external loading^{(3),(4)}.

In a series of recent works⁽⁵⁾⁻⁽⁷⁾ growth of an infinitesimal void in infinite elastic-plastic media was analysed using elaborate finite element procedures and it was demonstrated that the void could grow without bounds when a critical loading level is

achieved. This stress level was thus referred to as the "cavitation limit", and correlated with stress values required for appearance of macroscopic voids observed experimentally.

In an alternative approach to the problem, Chung et al.⁽⁸⁾ analyzed nucleation and growth of a spherical void in elastic-plastic materials using the framework of the bifurcation problem originally introduced by Ball⁽⁹⁾ to model cavitation in nonlinear elasticity. In their analysis⁽⁸⁾ a solution containing a cavity of a finite size was assumed to exist at applied loads exceeding a certain critical point, and a closed-form expression for the critical stress was obtained in terms of strain-hardening characteristics. Compared to more direct numerical methods⁽⁵⁾⁻⁽⁷⁾, this approach is advantageous in obtaining critical values precisely in simplified situations (readers are referred to a recent review of the approach by Horgan and Polignone⁽¹⁰⁾).

Most of the preceding works for inelastic solids^{(8),(11),(12)} with respect to this topic, however, assumed for simplicity that the material was incompressible, i.e., the elastic Poisson's ratio was set at 0.5. Huang et al.⁽⁵⁾ recorded the analytical expression

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for the cavitation limit in the case of elastically compressible plastic materials with no strain hardening. In this regard it is instructive to consider the phenomenon in strain-hardening elastic-plastic materials with elastic compressibility taken into account. It is our objective in the present study to resolve this problem using the framework of the bifurcation approach, and to obtain the expression for critical stresses for sudden growth of an infinitesimal void and formation of a macroscopic one under equitriaxial (i.e., spherically symmetric) loading conditions.

In the following, the formulation of the problem to be considered will be described, along with a solution both for strain-hardening and perfectly plastic materials. An expression for critical stresses will then be obtained using a certain limiting procedure. Numerical calculation will be carried out to illustrate the dependence of critical stresses as well as post-cavitation behavior on material constitutive parameters.

2. Formulation of the Problem

Consider the problem, as schematically delineated in Fig. 1(a), where a solid sphere of elastic-plastic material with radius A is subjected to uniform radial tension on the surface. The deformation moves a generic material point at a radial position R to a point $r(R, t)$ in the deformed state, where t implies a monotonical loading parameter. Although the size of the micro-void, pre-existing at the center of the sphere, is assumed to be initially negligible, i.e., $r(0, 0) = 0$, with its possible growth to a finite size in mind a non-vanishing finite value of $r(0, t)$ will be allowed. Then if $r(0, t) > 0$, a finite spherical cavity is present at the center, otherwise the sphere remains intact (the micro-void remains infinitesimal).

The associated deformation gradient is

$$\mathbf{F} = r' \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \frac{r}{R} (\hat{\mathbf{i}}_\theta \otimes \hat{\mathbf{i}}_\theta + \hat{\mathbf{i}}_\phi \otimes \hat{\mathbf{i}}_\phi), \quad r' \equiv \frac{\partial r(R, t)}{\partial R}, \quad (1)$$

where $\hat{\mathbf{r}}$, $\hat{\mathbf{i}}_\theta$ and $\hat{\mathbf{i}}_\phi$ are orthonormal base vectors for polar coordinates. The Cauchy (true) stress tensor, which is presently expressed as a function of R and t ,

(a) undeformed (b) homogeneous solution (c) cavitating solution

Fig. 1 Radial deformations of an elastoplastic sphere.

reads

$$\boldsymbol{\tau}(R, t) = \tau_r(R, t) \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \tau_\theta(R, t) (\hat{\mathbf{i}}_\theta \otimes \hat{\mathbf{i}}_\theta + \hat{\mathbf{i}}_\phi \otimes \hat{\mathbf{i}}_\phi), \quad (2)$$

and, in the radial tension at hand, the Mises equivalent stress may be reduced to

$$\tau_e = \tau_\theta - \tau_r, \quad (3)$$

provided that monotonic loading is solely considered.

When the Prandtl-Reuss model, generalized to incorporate finite strains^{(13),(14)}, is adopted to represent elastoplastic behavior, the constitutive relations in rate-form reduce to

$$\dot{\tau}_r = 2\mu \frac{\partial \dot{r}/\partial R}{\partial r/\partial R} + \lambda \frac{\dot{J}}{J} + 2\mu \dot{\epsilon}_e^p, \quad (4.a)$$

$$\dot{\tau}_\theta = 2\mu \frac{\dot{r}}{r} + \lambda \frac{\dot{J}}{J} - \mu \dot{\epsilon}_e^p, \quad (4.b)$$

which are easily integrated in the present case to yield

$$\tau_r = 2\mu \ln r' + \lambda \ln J + 2\mu \epsilon_e^p, \quad (5.a)$$

$$\tau_\theta = 2\mu \ln(r/R) + \lambda \ln J - \mu \epsilon_e^p, \quad (5.b)$$

where $J = r' \cdot r^2/R^2$ is the Jacobian of the deformation gradient. In the above expressions λ and μ are Lamé's elastic constants and ϵ_e^p the Odqvist parameter of accumulated distortion (equivalent plastic strain).

The analysis is first carried out for strain-hardening materials. Then the strain hardening characteristics can be written as

$$\epsilon_e^p = \begin{cases} 0 & (\tau_e < \tau_Y) \\ \epsilon_e^p(\tau_e) & (\tau_e \geq \tau_Y) \end{cases} \quad (6)$$

giving the equivalent plastic strain as a monotonic function of the equivalent stress, where τ_Y is the yield stress. In the alternative non-hardening case the equivalent stress takes on a constant value, irrespective of the accumulated strains. Further, the stress components must satisfy the following equilibrium:

$$\frac{\partial \tau_r}{\partial R} + \frac{2r'}{r} (\tau_r - \tau_\theta) = 0, \quad (7)$$

which is the only non-trivial equation in the present spherically symmetric problem.

In posing the underlying boundary value problem a set of boundary conditions need to be stated. Thus the traction condition of the form

$$\tau_r(A, t) = p(t) > 0 \quad (8)$$

will be imposed on the surface. When $r(0, t) > 0$, a finite cavity is present at the center and assumed to be traction-free. Then the condition at the center is

$$r(0, t) = 0 \quad \text{or} \quad \tau_r(0, t) = 0. \quad (9)$$

In summary, the problem is to obtain the deformation and the stress fields satisfying the constitutive law in Eqs. (5) and (6), the equilibrium in Eq. (7) with the conditions in Eqs. (8) and (9).

A homogeneous solution with $r(0, t) = 0$, keeping the sphere intact, may be seen to exist if the stress field of the form

$$\tau_r(R, t) = \tau_\theta(R, t) = p(t) \quad (10)$$

is examined. Indeed, it is seen that this stress field satisfies the equilibrium in Eq.(7) and the condition given in Eq.(8), giving $\tau_e=0$ from Eq.(3). Furthermore, the deformation is homogeneous in the sense that

$$r'(R, t) = \frac{r(R, t)}{R} = \Lambda(t) \quad (11)$$

where $\Lambda(t)$ is given by

$$p(t) = (3\lambda + 2\mu) \ln \Lambda(t). \quad (12)$$

Thus the condition in Eq.(9) is seen to hold. That is, the sphere only expands elastically and the sphere is kept intact without creating a finite cavity (Fig. 1(b)).

3. Cavitating Solution for Strain-Hardening Materials

3.1 Variable transformation

Next a solution with a cavity of a finite size as shown in Fig. 1(c), i.e., cavitating solution, is examined. The description of the solution presented here makes use of a theory developed by Durban and Baruch⁽¹³⁾, who obtained analytical solutions to the elastoplastic hollow sphere problem employing a non-linear kinematics. As shown below, it is convenient in the case of strain-hardening materials to regard the equivalent stress as an independent variable in place of the radial coordinate R .

From Eq.(5), the equivalent stress in Eq.(3) is given by

$$\tau_e = 2\mu \ln \left(\frac{r}{r'R} \right) - 3\mu \varepsilon_e^p, \quad (13)$$

which after rearrangement yields

$$\frac{r}{r'R} = \exp \left(\frac{\tau_e}{2\mu} + \frac{3\varepsilon_e^p}{2} \right). \quad (14)$$

In circumstances where the equivalent plastic strain is a monotonically increasing function of the equivalent stress, the right-hand side of Eq.(14) is monotonically increasing with respect to τ_e and is greater than unity, so that $0 < r' < r/R$. If it is assumed a priori that the left-hand side of Eq.(14) is monotonically decreasing with R , the equivalent stress τ_e will be monotonically decreasing with respect to R . This in turn enables the variable transformation $(R, t) \rightarrow (\tau_e, t)$ to be justified.

On the above assumption, Eq.(14) is substituted into the equilibrium equation, Eq.(7), to obtain

$$\frac{\partial \tau_r}{\partial R} = \frac{2\tau_e}{Re(\tau_e)}, \quad (15)$$

where an auxiliary function $e(\tau_e)$ is defined by

$$e(\tau_e) = \exp \left(\frac{\tau_e}{2\mu} + \frac{3\varepsilon_e^p(\tau_e)}{2} \right). \quad (16)$$

Since the right-hand side of Eq.(15) is positive, the radial stress τ_r is monotonically increasing with respect to R . In this setting the following differential relations between primary variables can be obtained

from Eqs.(5), and (13)–(16) via lengthy but straightforward manipulation (the derivation can be found elsewhere⁽¹³⁾);

$$\frac{\partial R}{\partial \tau_e} = - \frac{Re(\tau_e) \{ \lambda + 2\mu + \mu(3\lambda + 2\mu) \cdot d\varepsilon_e^p/d\tau_e \}}{2\mu(3\lambda + 2\mu) \{ e(\tau_e) - 1 \} + 4\mu\tau_e}, \quad (17.a)$$

$$\frac{\partial r}{\partial \tau_e} = - \frac{r \{ \lambda + 2\mu + \mu(3\lambda + 2\mu) \cdot d\varepsilon_e^p/d\tau_e \}}{2\mu(3\lambda + 2\mu) \{ e(\tau_e) - 1 \} + 4\mu\tau_e}, \quad (17.b)$$

$$\frac{\partial \tau_r}{\partial \tau_e} = - \frac{\tau_e \{ \lambda + 2\mu + \mu(3\lambda + 2\mu) \cdot d\varepsilon_e^p/d\tau_e \}}{\mu(3\lambda + 2\mu) \{ e(\tau_e) - 1 \} + 2\mu\tau_e}. \quad (18)$$

It is seen in Eq.(17.a) that the relation between R and τ_e is monotonic as has been assumed, for hardening solids. If the value of the equivalent stress at $R=A$ is denoted by τ_{eA} , integration of Eq.(17.a) yields the desired relation between τ_e and R for $0 < R \leq A$. It is seen from Eq.(14) that $R \rightarrow 0$ corresponds to $\tau_e \rightarrow \infty$ for strain-hardening materials. Further, if Eq.(18) is integrated with the end condition (8),

$$\tau_r = p - \int_{\tau_{eA}}^{\tau_e} \frac{\tau_e \{ \lambda + 2\mu + \mu(3\lambda + 2\mu) \cdot d\varepsilon_e^p/d\tau_e \}}{\mu(3\lambda + 2\mu) \{ e(\tau_e) - 1 \} + 2\mu\tau_e} d\tau_e \quad (19)$$

is obtained for the expression of the radial stress with τ_e as a parameter. Thus the boundary condition (9)₂ at the cavity reduces to

$$p = \int_{\tau_{eA}}^{\infty} \frac{\tau_e \{ \lambda + 2\mu + \mu(3\lambda + 2\mu) \cdot d\varepsilon_e^p/d\tau_e \}}{\mu(3\lambda + 2\mu) \{ e(\tau_e) - 1 \} + 2\mu\tau_e} d\tau_e. \quad (20)$$

The above equation specifies the value of τ_{eA} for the prescribed value of p . If such τ_{eA} exists, a solution with a finite cavity is also seen to exist. Once τ_{eA} is determined, the deformation field is expressed in a parametric representation by τ_e through integration of Eqs.(17.a) and (17.b) and use of Eq.(14) as shown below.

For later purposes it is convenient to use Young's modulus E and Poisson's ratio ν in place of Lamé's constants used above and also to normalize stresses by

$$T = \frac{\tau_e}{\tau_Y}, \quad T_A = \frac{\tau_{eA}}{\tau_Y} \quad (21)$$

and the auxiliary function in Eq.(16) as $e(\tau_e) = \tilde{e}(T)$. Then the results obtained above may be summarized in the following manner. First the deformation field is

$$R = A \exp \left[- \int_{T_A}^T \frac{\tilde{e}(T) \{ 2(1-\nu)\varepsilon_Y + d\varepsilon_e^p/dT \}}{2\{ \tilde{e}(T) - 1 \} + 4(1-2\nu)\varepsilon_Y T} dT \right], \quad (22.a)$$

$$r = \Lambda A \exp \left[- \int_{T_A}^T \frac{2(1-\nu)\varepsilon_Y + d\varepsilon_e^p/dT}{2\{ \tilde{e}(T) - 1 \} + 4(1-2\nu)\varepsilon_Y T} dT \right], \quad (22.b)$$

where $\varepsilon_Y = \tau_Y/E$ is the yield strain. The stress field is

$$\frac{\tau_r}{\tau_Y} = \frac{p}{\tau_Y} - \int_{T_A}^T \frac{T \{ 2(1-\nu)\varepsilon_Y + d\varepsilon_e^p/dT \}}{\tilde{e}(T) - 1 + 2(1-2\nu)\varepsilon_Y T} dT, \quad (23.a)$$

$$\frac{\tau_\theta}{\tau_Y} = T + \frac{p}{\tau_Y} - \int_{T_A}^T \frac{T \{ 2(1-\nu)\varepsilon_Y + d\varepsilon_e^p/dT \}}{\tilde{e}(T) - 1 + 2(1-2\nu)\varepsilon_Y T} dT. \quad (23.b)$$

In Eq.(22.b), Λ represents the value of r/R on the surface of the sphere and is given by

$$\Lambda = \exp[(1-2\nu)p/E + (1-\nu)\varepsilon_Y T_A + \varepsilon_e^p(\tau_{eA})/2] \quad (24)$$

from Eq.(5) and Eq.(14) in terms of the surface traction p . It gives the ratio between the undeformed and deformed radii of the sphere. The radius of the initiated cavity r_c is written, corresponding to $T \rightarrow \infty$ in Eq.(22.b), as

$$r_c = \Lambda A \exp\left[-\int_{T_A}^{\infty} \frac{2(1-\nu)\varepsilon_Y + d\varepsilon_e^p/dT}{2\{\tilde{\varepsilon}(T)-1\} + 4(1-2\nu)\varepsilon_Y T} dT\right]. \quad (25)$$

3.2 Bifurcation point

At bifurcation of the cavitating solution from the homogeneous one, the parameter T_A approaches zero since it corresponds to $r_c \rightarrow 0$ in Eq.(25). Then the critical stress corresponding to the bifurcation point is given by

$$\frac{p_{cr}}{\tau_Y} = \int_0^{\infty} \frac{T\{2(1-\nu)\varepsilon_Y + d\varepsilon_e^p/dT\}}{\tilde{\varepsilon}(T)-1+2(1-2\nu)\varepsilon_Y T} dT. \quad (26)$$

Thus calculation of the critical stress only requires integration of a known function when the material hardening data $\varepsilon_e^p(\tau_e)$ are given.

From a practical point of view, macroscopic void formation in highly constrained deformation processes is of relevance to what is ordinarily observed. Thus critical stretches, i.e., critical values of the expansion ratio Λ , are important quantities to be examined. Here, the critical stretch is readily obtained from Eq.(24) as

$$\Lambda_{cr} = \exp\left\{(1-2\nu)\frac{p_{cr}}{E}\right\}, \quad (27)$$

which gives the ratio between the deformed and undeformed radii of the sphere at the instant of cavitation.

In order for a cavitating solution to exist, the right-hand side of Eq.(26), which is actually improper, must be finite. Since the integrand may be regarded as a continuous function of T , it suffices to examine its behavior at both ends of the interval of integration. When $T \rightarrow 0$, l'Hôpital's rule gives

$$\lim_{T \rightarrow 0} \frac{T\{2(1-\nu)\varepsilon_Y + d\varepsilon_e^p/dT\}}{\tilde{\varepsilon}(T)-1+2(1-2\nu)\varepsilon_Y T} = \frac{2}{3}. \quad (28)$$

When $T \rightarrow \infty$, on the other hand, it may be expected for ordinary elastic-plastic power-law hardening solids that $\varepsilon_e^p(T) = O(T^m)$, $m > 1$, where the constant m characterizes strain hardening. Then it is clear that the integral in question will indeed converge to give a finite value of p_{cr}/τ_Y (note the exponential growth of $\tilde{\varepsilon}(T)$ as $T \rightarrow \infty$).

In the present spherically symmetric problem, the elastoplastic constitutive law adopted is essentially a nonlinear elastic one obeying the stress-strain relation given by Eqs.(5) and (6). It can be noted, however,

that the present material model is rather a simplifying one, and such an explicit argument concerning critical stresses as derived in this section is in general not to be expected in the case of nonlinear elastic materials possessing material compressibility^{(15),(16)}.

3.3 Cavitation stress for incompressible materials

It is noteworthy that the results obtained so far may be reduced to a special case in which the material becomes incompressible. The critical stress in this limiting case can be readily obtained by setting $\nu \rightarrow 1/2$ in Eq.(26). Further, to derive the desired expression, the monotonic stress-strain relation

$$\varepsilon_e = \varepsilon_e(\tau_e) = \tau_e/E + \varepsilon_e^p(\tau_e) \quad (29)$$

is regarded as a one-to-one correspondence between the equivalent stress τ_e and the equivalent (total) strain ε_e , hence yielding $\tau_e = \tau_e(\varepsilon_e)$. This enables the further transformation $T \rightarrow \varepsilon_e$ and gives a simple formula;

$$p_{cr} = \int_0^{\infty} \frac{\tau_e(\varepsilon_e)}{\exp(3\varepsilon_e/2)-1} d\varepsilon_e. \quad (30)$$

This simple expression for the critical stress for incompressible elastic-plastic solids conforms with the formula derived separately⁽⁸⁾.

Using the finite element method Huang et al.⁽⁶⁾ recently analysed a problem in which a spherical cavity in an incompressible elastic-plastic infinite medium is subjected to remote equitriaxial tension. They referred to the critical stress, at which the cavity radius grew without bounds, as the "cavitation limit". The cavitation limit thus obtained by these investigators for incompressible materials is identical to Eq.(30), although the general case with elastic compressibility, given presently in Eq.(26), was not discussed in their investigation.

4. Cavitating Solution for Non-Hardening Materials

4.1 Variable transformation

The above discussion has been confined to strain-hardening materials for which there is a one-to-one correspondence between the equivalent stress and the plastic strain. In the case of perfectly plastic materials where no strain hardening occurs, the procedure requires a slight modification as the equivalent stress cannot be employed as an independent variable for the whole domain. The equivalent strain, however, can be regarded as a primary variable in the plastically deformed region, and the following transformation of variables may instead be conducted;

$$(i) \text{ for } R \in (0, R_Y], R \rightarrow \varepsilon_e^p, \varepsilon_e^p \in [0, \infty),$$

$$(ii) \text{ for } R \in (R_Y, A], R \rightarrow \tau_e, \tau_e \in [\tau_{eA}, \tau_Y],$$

where the undeformed coordinate of the elastic-plastic boundary is denoted by R_Y to be given explicitly

below. In the plastic region (i), the equivalent stress remains constant and the relation in Eq. (14) can be used to transform variables employing ε_e^p , in essentially the same manner as described in the previous chapter. In the elastic region (ii) the equivalent stress can still be employed as has been done and Eqs. (22) and (23) are valid.

Since $R=R_Y$ (elastic-plastic boundary) corresponds to $T=1$ (initiation of plastic yielding), the radius of the boundary R_Y is given from Eq. (22) as

$$R_Y = A \exp \left[- \int_{T_A}^1 \frac{(1-\nu)\varepsilon_Y \exp\{(1-\nu)\varepsilon_Y T\}}{\exp\{(1+\nu)\varepsilon_Y T\} - 1 + 2(1-2\nu)\varepsilon_Y T} dT \right] \quad (31)$$

The elastic-plastic boundary given above initiates at $R=0$ and proceeds to advance in the radial direction as T_A increases from zero.

When details of the intermediary derivation are suppressed, the outcome may be summarized as follows. Two cases must be distinguished, i.e., a case where a certain (outer) part of the sphere deforms elastically and the rest plastically, and a case where the whole sphere is in a plastic state.

Expressions for the deformation and stress fields will be first presented in the case where elastic and plastic regions coexist. In this case, Eqs. (22) and (23) apply for the outer elastic region $R_Y < R \leq A$, i.e., $T_A \leq T < 1$. For the inner plastic region, i.e., $0 < R < R_Y$, resulting expressions are:

$$R = A \exp \left[- \int_{T_A}^1 \frac{(1-\nu)\varepsilon_Y \exp\{(1+\nu)\varepsilon_Y T\}}{\exp\{(1+\nu)\varepsilon_Y T\} - 1 + 2(1-2\nu)\varepsilon_Y T} dT - \int_0^{\varepsilon_e^p} \frac{h(\varepsilon_e^p)}{2\{h(\varepsilon_e^p) - 1\} + 4(1-2\nu)\varepsilon_Y} d\varepsilon_e^p \right], \quad (32.a)$$

$$r = \Lambda A \exp \left[- \int_{T_A}^1 \frac{(1-\nu)\varepsilon_Y}{\exp\{(1+\nu)\varepsilon_Y T\} - 1 + 2(1-2\nu)\varepsilon_Y T} dT - \int_0^{\varepsilon_e^p} \frac{d\varepsilon_e^p}{2\{h(\varepsilon_e^p) - 1\} + 4(1-2\nu)\varepsilon_Y} \right], \quad (32.b)$$

$$\frac{\tau_r}{\tau_Y} = \frac{p}{\tau_Y} - \int_{T_A}^1 \frac{2(1-\nu)\varepsilon_Y T}{\exp\{(1+\nu)\varepsilon_Y T\} - 1 + 2(1-2\nu)\varepsilon_Y T} dT - \int_0^{\varepsilon_e^p} \frac{d\varepsilon_e^p}{h(\varepsilon_e^p) - 1 + 2(1-2\nu)\varepsilon_Y}, \quad (33.a)$$

$$\frac{\tau_\theta}{\tau_Y} = \frac{\tau_r}{\tau_Y} + 1, \quad (33.b)$$

with $\varepsilon_e^p \in [0, \infty)$ as a parameter. In the above expressions an auxiliary function was defined by

$$h(\varepsilon_e^p) = \exp \left(\frac{3}{2} \varepsilon_e^p + \frac{\tau_Y}{2\mu} \right) \quad (34)$$

and the stretch of the sphere Λ in Eq. (32.b) is given by

$$\Lambda = \exp[(1-2\nu)p/E + (1-\nu)\varepsilon_Y T_A]. \quad (35)$$

From Eq. (32.a) it is seen that $R \rightarrow 0$ corresponds to $\varepsilon_e^p \rightarrow \infty$. Namely, the plastic strain will increase to infinity as the initiated cavity is approached. Further, in this case the surface value T_A is related to the applied stress p by

$$\frac{p}{\tau_Y} = \int_{T_A}^1 \frac{2(1-\nu)\varepsilon_Y T}{\exp\{(1+\nu)\varepsilon_Y T\} - 1 + 2(1-2\nu)\varepsilon_Y T} dT + \int_0^{\varepsilon_e^p} \frac{d\varepsilon_e^p}{h(\varepsilon_e^p) - 1 + 2(1-2\nu)\varepsilon_Y}. \quad (36)$$

After the elastic-plastic boundary reaches the outer surface, $R_Y=A$, the whole sphere is covered by the plastic zone (after R_Y in Eq. (31) exceeds the radius A , it no longer implies the elastic-plastic boundary). In this fully plastic case, the results are expressed as

$$R = A \exp \left[- \int_{\varepsilon_{eA}^p}^{\varepsilon_e^p} \frac{h(\varepsilon_e^p)}{2\{h(\varepsilon_e^p) - 1\} + 4(1-2\nu)\varepsilon_Y} d\varepsilon_e^p \right], \quad (37.a)$$

$$r = \Lambda A \exp \left[- \int_{\varepsilon_{eA}^p}^{\varepsilon_e^p} \frac{d\varepsilon_e^p}{2\{h(\varepsilon_e^p) - 1\} + 4(1-2\nu)\varepsilon_Y} \right], \quad (37.b)$$

$$\frac{\tau_r}{\tau_Y} = \frac{p}{\tau_Y} - \int_{\varepsilon_{eA}^p}^{\varepsilon_e^p} \frac{d\varepsilon_e^p}{h(\varepsilon_e^p) - 1 + 2(1-2\nu)\varepsilon_Y}, \quad (38)$$

and Eq. (33.b), where the surface value of the equivalent plastic strain ε_{eA}^p is given by the condition

$$\frac{p}{\tau_Y} = \int_{\varepsilon_{eA}^p}^{\varepsilon_e^p} \frac{d\varepsilon_e^p}{h(\varepsilon_e^p) - 1 + 2(1-2\nu)\varepsilon_Y}, \quad (39)$$

and the overall stretch of the sphere becomes

$$\Lambda = \exp[(1-2\nu)p/E + (1-\nu)\varepsilon_Y + \varepsilon_{eA}^p/2]. \quad (40)$$

4.2 Bifurcation point

At cavitation, Eqs. (32)-(36) apply. Then the critical stress p_{cr} for perfectly plastic materials is given by $T_A \rightarrow \infty$ in Eq. (36). In this case the integration in the second term can be carried out analytically to yield

$$\frac{p_{cr}}{\tau_Y} = \int_0^1 \frac{2(1-\nu)\varepsilon_Y T}{\exp\{(1+\nu)\varepsilon_Y T\} - 1 + 2(1-2\nu)\varepsilon_Y T} dT + \frac{2}{3\{1-2(1-2\nu)\varepsilon_Y\}} \times \ln \left\{ \frac{\exp\{(1+\nu)\varepsilon_Y\}}{\exp\{(1+\nu)\varepsilon_Y\} + 2(1-2\nu)\varepsilon_Y - 1} \right\}. \quad (41)$$

It is of interest to examine the limiting case of incompressible elastic-perfectly plastic materials. In this case with $\nu \rightarrow 1/2$, the above expression reduces to

$$\frac{p_{cr}}{\tau_Y} = - \int_0^1 \ln \left\{ 1 - \exp \left(- \frac{3}{2} \varepsilon_Y T \right) \right\} dT. \quad (42)$$

Equations (41) and (42) coincide with the cavitation limits for perfectly plastic solids earlier obtained by Huang et al.⁽⁵⁾, but can be regarded as special examples in the present framework.

5. Numerical Results for Power-Law Hardening Materials

5.1 Bifurcation point

Attention is now turned to the following simple power-law stress-strain relation

$$\begin{cases} \tau_e = E\varepsilon_e, & (\tau_e < \tau_Y), \\ \tau_e/\tau_Y = (\varepsilon_e/\varepsilon_Y)^{1/m}, & (\tau_e \geq \tau_Y), \end{cases} \quad (43)$$

where the parameter m represents strain hardening. It then becomes clear that the critical stress p_{cr} , when normalized by the yield stress τ_Y , depends on the non-

dimensional parameters which are the yield strain ε_Y , hardening exponent m and Poisson's ratio ν . This dependence is delineated in Fig. 2 by calculating Eqs. (26) and (30) by a numerical quadrature. The results for $m \rightarrow \infty$ (no strain hardening) were calculated by Eqs. (41) and (42). It is found that the critical stress increases when the hardening exponent m becomes small, i.e., when there occurs significant hardening.

Also, the variation of the critical stretch Λ_{cr} with the hardening exponent m for $\nu=0.3$, which shows monotonical dependence of Λ_{cr} on m , is presented in Fig. 3. It is shown that the critical stretch increases to a certain extent when the magnitude of yield strain becomes appreciable.

Since the present analysis is capable of taking the volumetric elastic compliance into account, it is of definite interest here to examine the influence of Poisson's ratio. In Fig. 2, two values of Poisson's ratio were used, i.e., $\nu=0.3$ and $\nu=0.5$, the latter being for the incompressible limit. For the critical stress p_{cr} , Fig. 2 indicates that the critical stress is reduced slightly when $\nu=0.3$. As to the critical stretch Λ_{cr} , on the other hand, the case of $\nu=0.5$ gives $\Lambda_{cr}=1$ identically and thus is not shown explicitly in Fig. 3. Namely, the volumetric elastic compliance reduces the critical stress, but increases the stretch. This finding could be understood intuitively, but has not been presented in an explicit form in the previous literature.

From a practical point of view, attention should be paid to the magnitude of the cavitation stress p_{cr} in comparison with the yield stress of the material. Ashby et al.⁽³⁾ carried out tensile testing of lead wires with highly constrained lateral deformation, and reported that a macroscopic void was formed with rapid load drop when the tensile stress reached around four

to seven times the uniaxial flow stress. Although this tensile load cannot be directly regarded to give the hydrostatic stress occurring in the wire, due to the highly constrained lateral deformation it may be inferred that triaxial stress of similar magnitude was present. Further, if the material parameters are set as $m=4$, $\varepsilon_Y=0.004$ and $\nu=0.3$ to approximate the characteristics of the material used by Ashby et al.⁽³⁾, Eq. (26) gives $p_{cr}/\tau_Y \approx 7.70$ which is in fair accordance with the result obtained experimentally, as was similarly described by Huang et al.⁽⁵⁾ in their paper.

5.2 Post-cavitation behavior

By solving Eq. (20) numerically and using Eqs. (24) and (25), the relation between the radial stress p on the surface and the radius of the cavity r_c subsequent to cavity appearance is shown in Fig. 4. It can be seen that for the cavitating solution the radial traction decreases as the cavity radius grows.

Figure 5 shows the evolution of the cavity radius with the increasing stretch Λ (expansion ratio of the sphere) for the cavitating solution (the one for the homogeneous solution corresponds trivially to the abscissa). As may be seen, the radius of the cavity grows rapidly when the stretch Λ exceeds the critical value Λ_{cr} . Major features of this growth are not influenced by the hardening exponent m to a great extent. As a matter of fact, when $\nu=0.5$ the stretch and the cavity radius obey the relation for incompressibility constraint, $\Lambda^3 - (r_c/A)^3 = 1$ irrespective of the yield strain ε_Y and the exponent m . The corresponding relation for $\nu=0.3$ approaches that for $\nu=0.5$ as ε_Y decreases, which may be understood in the sense that the decrease of the yield strain makes insignificant the relative effect of elastic volumetric strain compared to the dominating isochoric plastic deformation.

Fig. 2 Variation of the critical stress with the hardening exponent m for different values of ε_Y (solid lines: $\nu=0.3$, dashed lines: $\nu=0.5$)

Fig. 3 Variation of the critical stretch with the hardening exponent m for different values of ε_Y when $\nu=0.3$

Fig. 4 Relation between the radial stress p and the cavity radius r_c for different values of m (solid lines: $\nu=0.3$, dashed lines: $\nu=0.5$)

Fig. 5 Relation between the stretch Λ and the cavity radius r_c for different values of m (solid lines: $\nu=0.3$, dashed lines: $\nu=0.5$)

Figure 6 shows how the radial stress p varies when the stretch Λ of the sphere is increased from unity. The figure indicates that the cavity growth causes change in the stress-strain relation of the sphere, and there occurs decrease in the stress as the cavitating deformation proceeds. Such softening characteristics of materials due to formation and growth of voids have been demonstrated in more general three-dimensional finite element simulations⁽¹⁷⁾.

When a cavity is present at the center, it is

Fig. 6 Relation between the stretch Λ and the radial stress p for different values of m (solid lines: $\nu=0.3$, dashed lines: $\nu=0.5$)

Fig. 7 Distribution of equivalent stresses in the sphere (a) $\tau_{eA}/\tau_Y=0.005$, (b) $\tau_{eA}/\tau_Y=0.01$ (c) $\tau_{eA}/\tau_Y=0.02$, (d) $\tau_{eA}/\tau_Y=0.04$

expected that the deformation and the stress are in a highly localized state in the vicinity of the cavity. Figure 7 shows that the Mises equivalent stress becomes unbounded as the cavity surface is approached. The singularity of the stress near the center depends on the hardening parameter m , and it is higher for a smaller value of m , i.e., when the hardening is more significant. In this circumstance, bifurcation into further asymmetric deformation modes or other types of failure⁽¹⁸⁾ may be expected in the neighborhood of the cavity. The discontinuity in the slope of the stress distribution is found to be due to the presence of the elastic-plastic boundary, at which $\tau_e = \tau_Y$.

6. Concluding Remarks

Sudden finite expansion of an infinitesimal void in elastic-plastic materials under conditions of equitriaxial tension was studied employing the framework of a bifurcation problem. Expressions for critical radial stresses and critical expansion ratios of the

elastoplastic sphere, at which a finite spherical cavity may appear in the interior, were derived in the form of integrals involving material constants and hardening data. By certain limiting procedures, the results obtained were shown to match those for incompressible materials earlier reported in the literature. Numerical calculations were also carried out to delineate the influence of the material parameters and the hardening characteristics on critical stresses and stretches, as well as the post-cavitation behavior.

Compared to the incompressible case, the volumetric elastic compliance was found to reduce the critical level of triaxial stress for macroscopic void formation slightly, while it gave non-trivial critical values for the expansion ratio depending on material parameters. For highly constrained deformation modes where great equitriaxial stress could be present, the present analysis was seen to give a fair estimate of stress levels required for formation of a macroscopic cavity in elastic-plastic materials.

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