Elastodynamic Thin Plate Bending Analysis by Boundary Element Method with Laplace Transform*

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The boundary element method (BEM) is developed for the dynamic analysis of thin elastic plate bending problems with arbitrary boundary conditions. The formulation employs Laplace-transform technique, where the boundary integral equations are obtained on the Laplace transformed domain with the fundamental solutions derived from Kelvin's functions. The accuracy of the numerical results mainly depends on those of numerical estimation of the singular integral derived from the static term of the fundamental solutions. In the present paper, an BEM formulation based on a single boundary integral equation of the deflection, which employs a source point on both the boundary and outer region, is discussed in detail. A non-singular boundary integral equation is introduced on the transformed domain, which is obtained by superposition of the analyzed field and the referenced field with a uniform gradient of deflection. Numerical results obtained by the proposed method are compared with the analytical solutions and the other numerical solutions by means of several numerical examples. These examples also serve to illustrate the use of the proposed method.

Key Words: Boundary Element Method, Computational Mechanics, Structural Analysis, Numerical Analysis, Singular Integral, Laplace Transform, Plate Bending

1. Introduction

It is well known that the most commonly used numerical methods in the engineering field are the finite difference method (FDM), the finite-element method (FEM) and the boundary element method (BEM). The boundary element method in particular has been developed as a powerful numerical method of analyzing various boundary value problems of wave scattering, heat conduction, fluid analysis and elasticity.

In the general BEM formulation for the plate bending problems, coupled boundary integral equations for the deflection and normal rotation are employed according to the formulation of Stern⁽¹⁾, Bezine⁽²⁾, Bezine and Gamby⁽³⁾ or Hartmann^{(4),(5)}.

It is well known that the second boundary inte-

gral equation for rotation include the integration with the singular kernel of type r^{-2} , which does not converge in a general sense. In order to overcome this difficulty, some useful techniques have already been suggested to regularize the hyper-singular integrals, for example, by Matsumoto et al.⁽⁶⁾ and Arai et al.^{(7),(8)} in plate bending problems.

Instead of the above formulation based on the coupled BIE, Paiva and Venturini⁽⁹⁾ presented an alternative formulation which does not require the second boundary integral equation. In their formulation, the external source points are employed in addition to the ordinary source point with the first boundary integral equation. Strictly speaking, the former formulation, based on the coupled BIE, needs C^1 continuous elements because the tangential derivative of the deflection should be Hölder continuous at the conjunction points of the elements, but the latter Paiva and Venturini's approach enables the numerical difficulties arising from the second integral equation to be avoided and the C^0 conforming elements can be used in the discretization consistently.

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Arai et al.⁽¹⁰⁾ presented the improved formulation of Paiva and Venturini's method, and found that some difficulties will remain in this method owing to the use of the external source points. In this formulation, a nonsingular integral equation is suggested by superposition of the rigidly rotational state of the analyzed plates. This is a direct extension of the rigid body mode technique⁽¹¹⁾ and the regularization technique of the hypersingular integrals such as those of Rudolphi⁽¹²⁾, Rudolphi et al.⁽¹³⁾ and Sladek et al.⁽¹⁴⁾.

As shown in recent articles^{(15)–(17)}, the boundary element method becomes an accurate and efficient numerical method for dynamic plate bending analysis. There are basically three approaches for treating plate dynamic problems by the BEM. These are the direct BEM that employs the dynamic fundamental solution of the problem, the indirect BEM that employs the static fundamental solution of the problem and the Laplace-transformed BEM approach which employs the dynamic fundamental solution derived in the transformed domain.

The first approach is a direct method which requires not only discretization of the a boundary of the plate but also discretization of time domain, in which the complicated form of the dynamic fundamental solution increases the computational difficulties.

The second approach requires a discretization of both the boundary as well as the internal domain of the plate. The integral for the field is employed to estimate the term for inertia, which is treated by the finite difference technique or Laplace transform procedure. This enables the reduction the computational difficulties by virtue of the simple form of the static fundamental solutions. This approach was introduced by Bezzine⁽²⁾ for the forced vibrations, Providakis and Beskos⁽¹⁷⁾ developed it using the Laplace transform and Tanaka et al.⁽¹⁵⁾ extended it for free vibrations and assembled plate bending problems. Other applications of this type have been reported by Katsikadelis et al.^{(18),(19)}, Costa⁽²⁰⁾, for example.

The third approach is based on the Laplace transformation method, where the boundary integral equations are obtained on the transformed domain with the transformed fundamental solutions. This is the most powerful and useful approach, as long as the inverse Laplace transform is executed appropriately, but the complicated form of the fundamental solution increases the computational difficulties. An example of this type for the plate bending problems is given by Adachi et al.⁽²¹⁾.

This paper concerns the Laplace-transformed BEM formulation for the thin elastic plate subjected to impulsive load. In the present formulation, according to the previous formulation for the static case⁽¹⁰⁾,

only a single boundary integral equation of the deflection is used, where a nonsingular boundary integral equation^{(7),(8)} is applied for the Laplace transformed BEM. Since the non-singular boundary integral equation enables the external source point to be located near the boundary, the accuracy of the numerical results is increased by employing the arbitrary high-order elements. In the present paper, the fourth order isoparametric element is employed for the discretization in order to minimize the error derived from discontinuity of the quantities on the conjunction points or corner nodes. The results obtained from the proposed method are compared with the analytical solutions and the other numerical solutions by means of a number of numerical examples. These examples also serve to illustrate the usefulness of the proposed method.

2. Theory

2.1 BEM formulation of the Laplace-transformed domain

Consider a linear elastic, homogeneous and isotropic thin plate subjected to unsteady transverse pressure q . According to classical plate bending theory, a deflection w is governed by the following biharmonic differential equation,

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = q, \quad (1)$$

where t , ρ and h are time, density and thickness of the plate, respectively. $D = Eh^3/12(1-\nu^2)$ denotes the flexural rigidity of the plate with the elastic modulus E and Poisson's ratio ν . $\nabla^4(\cdot)$ is $\nabla^2(\nabla^2(\cdot))$, where $\nabla^2(\cdot)$ is the Laplacian $\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$.

Let Ω be a bounded region with total boundary Γ , where normal rotation T_n , bending moment M_n , twisting moment M_{ns} and equivalent shear force V_n are defined by the differential forms

$$T_n = \frac{\partial w}{\partial n} \quad (2)$$

$$M_n = -D \left(\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial s^2} \right) \quad (3)$$

$$M_{ns} = -D(1-\nu) \frac{\partial^2 w}{\partial n \partial s} \quad (4)$$

$$V_n = -D \frac{\partial \nabla^2 w}{\partial n} - \frac{\partial M_{ns}}{\partial \Gamma} \quad (5)$$

at a smooth regular point on the boundary Γ , where $\partial/\partial n$ and $\partial/\partial s$ denote normal and tangential derivative operators, respectively, and $\partial/\partial \Gamma$ is derivative operator for the coordinate along the boundary Γ .

In the Laplace-transformed domain, Eq. (1) can be rewritten as

$$D\nabla^4 \bar{w} + \rho h \kappa^2 \bar{w} = \bar{q}, \quad (6)$$

with a Laplace transformation parameter κ . Then, let us employ a fundamental solution $\bar{w}^*(p, Q)$ satisfying

$$D\nabla^4 \bar{w}^*(p, Q) + \rho h \kappa^2 \bar{w}^*(p, Q) = \delta(r), \quad r = |pQ|, \quad (7)$$

where $\delta(r)$ is a Dirac's delta function. Namely, $\bar{w}^*(p, Q)$ is a deflection on the source point p subjected to a unit concentrate impulsive load $\delta(t)$ applied on the point Q .

The concrete form of $\bar{w}^*$ is expressed by Kelvin's function^{(21),(22)}

$$\bar{w}^*(p, Q) = -\frac{1}{2\pi D \eta} Kei_0(Z), \quad (8)$$

where $\eta = \kappa \sqrt{\rho h/D}$ and $Z = \sqrt{\eta} r$.

Multiplying the fundamental solution $\bar{w}^*(p, Q)$ with Eq.(6) and expanding it according to Gauss's divergence theorem, we have the following boundary relation between physical quantities on Γ and the internal deflection $w(p)$ in Ω .

$$\begin{aligned} \bar{w}(p) &= \int_{\Gamma} \{ \bar{w}^*(p, Q) \bar{V}_n - \bar{T}_n^*(p, Q) \bar{M}_n(Q) \\ &\quad + \bar{M}_n^*(p, Q) \bar{T}_n(Q) - \bar{V}_n^*(p, Q) \bar{w}(Q) \} d\Gamma \\ &\quad + \sum_m [\Delta \bar{M}_{ns}^*(p, Q_m) \bar{w}(Q_m) - \bar{w}^*(p, Q_m) \Delta \bar{M}_{ns}(Q_m)] \\ &= \int_{\Omega} \bar{w}^*(p, Q) \bar{q}(Q) d\Omega, \quad p \in \Omega, \end{aligned} \quad (9)$$

where $\bar{T}_n^*$, $\bar{M}_n^*$, $\bar{M}_{ns}^*$ and $\bar{V}_n^*$ are the integral kernels obtained by substituting Eq.(8) into Eqs.(2), (3), (4) and (5), and Q_m denotes the corner points on Γ . $\Delta \bar{M}_{ns}^*(p, Q_m)$ and $\Delta \bar{M}_{ns}(Q_m)$ denote jump terms of the twisting moment arising on Q_m .

Limiting $p(\in \Omega) \rightarrow P(\in \Gamma)$, Eq.(9) yields the boundary relation,

$$\begin{aligned} C(P) \bar{w}(P) &= \int_{\Gamma} \{ \bar{w}^*(P, Q) \bar{V}_n - \bar{T}_n^*(P, Q) \bar{M}_n(Q) \\ &\quad + \bar{M}_n^*(P, Q) \bar{T}_n(Q) - \bar{V}_n^*(P, Q) \bar{w}(Q) \} d\Gamma \\ &\quad + \sum_m [\Delta \bar{M}_{ns}^*(P, Q_m) \bar{w}(Q_m) - \bar{w}^*(P, Q_m) \Delta \bar{M}_{ns}(Q_m)] \\ &= \int_{\Omega} \bar{w}^*(P, Q) \bar{q}(Q) d\Omega, \quad P \in \Gamma, \end{aligned} \quad (10)$$

where $C(P)$ is a discontinuous function depending on the location of source point P , which is 1/2 on the smooth boundary point.

2.2 Nonsingular boundary integral equation

It is well known that, since the boundary integral equation (10) includes the singular kernels $\bar{M}_n^*(P, Q)$ of type $\ln r$ and $\bar{V}_n^*(P, Q)$ of type r^{-1} , even when the source point is located on the outside of the region Ω and is taken very close to the boundary, the singular behavior arises as for the source points n on Γ . Therefore, we should think about the treatment of the nearlsingular integrals with the source point the in exterior region, as well as the singular integrals with P on Γ .

In this section, a formulation which gives a non-singular boundary integral equation for static problems^{(7),(8)} is extended to the Laplace-transformed BEM. First, let us separate the integral kernel in Eq. (10) into the static fundamental solutions and the

dynamic terms;

$$\begin{aligned} &C(P) \bar{w}(P) \\ &\quad + \int_{\Gamma} \{ V_n^{*s}(P, Q) \bar{w}(Q) - M_n^{*s}(P, Q) \bar{T}_n(Q) \\ &\quad + T_n^{*s}(P, Q) \bar{M}_n(Q) - w^{*s}(P, Q) \bar{V}_n(Q) \} d\Gamma \\ &\quad + \int_{\Gamma} \{ \bar{V}_n^{*D}(P, Q) \bar{w}(Q) - \bar{M}_n^{*D}(P, Q) \bar{T}_n(Q) \\ &\quad + \bar{T}_n^{*D}(P, Q) \bar{M}_n(Q) - \bar{w}^{*D}(P, Q) \bar{V}_n(Q) \} d\Gamma \\ &\quad + \sum_m [\Delta M_{ns}^{*s}(P, Q_m) \bar{w}(Q_m) \\ &\quad - w^{*s}(P, Q_m) \Delta \bar{M}_{ns}(Q_m)] \\ &\quad + \sum_m [\Delta \bar{M}_{ns}^{*D}(P, Q_m) \bar{w}(Q_m) \\ &\quad - \bar{w}^{*D}(P, Q_m) \Delta \bar{M}_{ns}(Q_m)] \\ &= \int_{\Omega} \bar{w}^*(P, Q) q(Q) d\Omega, \end{aligned} \quad (11)$$

where w^{*s} , T_n^{*s} , M_n^{*s} and V_n^{*s} are the static terms of the integral kernels⁽¹⁾. w^{*s} is given by

$$w^{*s}(P, Q) = -\frac{r^2}{8\pi D} \ln r. \quad (12)$$

T_n^{*s} , M_n^{*s} , M_{ns}^{*s} and V_n^{*s} are obtained by substituting w^{*s} into Eqs.(2), (3), (4) and (5), respectively. Note that the kernel V_n^{*s} has a singularity of the type r^{-1} , and the kernel M_n^{*s} has a weak singularity of the type $\ln r$, $\bar{w}^{*D}$, $\dots$, $\bar{V}_n^{*D}$ denote the dynamic terms without any singularities.

Then, let us consider a reference field which has rigid rotation $\bar{w}_{,k}(p)$ and a rigid displacement $\bar{w}(p)$ at source point. In the reference field, the deflection $\bar{w}'(Q)$ is given by

$$\bar{w}'(Q) = \bar{w}(p) + \bar{w}_{,k}(p) r_k(p, Q), \quad p \in \Omega, \quad (13)$$

where

$$r_k(p, Q) = x_k(Q) - x_k(p), \quad k=1, 2. \quad (14)$$

Then, the following identity is introduced,

$$D \int_{\Omega} \bar{w}'(Q) \nabla^4 w^{*s}(p, Q) d\Omega = \bar{w}'(p), \quad (15)$$

where the definition $D \int_{\Omega} \nabla^4 \bar{w}^{*s}(p, Q) d\Omega = 1$ is used.

Applying the divergence theorem to Eq.(15) limiting $p \rightarrow P$, the above identity yields

$$\begin{aligned} &C(P) \bar{w}(P) + \int_{\Gamma} V_n^{*s}(P, Q) \{ \bar{w}(P) \\ &\quad + \bar{w}_{,k}(P) r_k(P, Q) \} d\Gamma \\ &\quad + \sum_m \Delta M_{ns}^{*s}(P, Q_m) \{ \bar{w}(P) + \bar{w}_{,k}(P) r_k(P, Q_m) \} \\ &= \int_{\Gamma} M_n^{*s}(P, Q) \bar{w}_{,k}(P) n_k(Q) d\Gamma, \end{aligned} \quad (16)$$

where $n_k(Q)$ denotes the component of the unit normal vector.

Subtracting Eq.(16) from Eq.(10) gives

$$\begin{aligned} &\int_{\Gamma} [V_n^{*s}(P, Q) \{ \bar{w}(Q) - \bar{w}(P) - \bar{w}_{,k}(P) r_k(P, Q) \} \\ &\quad - M_n^{*s}(P, Q) \{ \bar{T}_n(Q) - \bar{w}_{,k}(P) n_k(Q) \} \\ &\quad + T_n^{*s}(P, Q) \bar{M}_n(Q) - w^{*s}(P, Q) \bar{V}_n(Q)] d\Gamma \\ &\quad + \int_{\Gamma} \{ \bar{V}_n^{*D}(P, Q) \bar{w}(Q) - \bar{M}_n^{*D}(P, Q) \bar{T}_n(Q) \\ &\quad + \bar{T}_n^{*D}(P, Q) \bar{M}_n(Q) - \bar{w}^{*D}(P, Q) \bar{V}_n(Q) \} d\Gamma \\ &\quad + \sum_m [\Delta M_{ns}^{*s}(P, Q_m) \{ \bar{w}(Q_m) - \bar{w}(P) \} \end{aligned}$$

$$\begin{aligned}
& -\bar{w}_{,k}(P)r_k(P, Q_m)\} - w^{*s}(P, Q_m)\Delta\bar{M}_{ns}(Q_m)] \\
& + \sum [\Delta M_{ns}^{*D}(P, Q_m)\bar{w}(Q_m) \\
& - \bar{w}^{*D}(P, Q_m)\Delta\bar{M}_{ns}(Q_m)] \\
& = \int_{\Omega} \bar{w}^*(P, Q)\bar{q}(Q)d\Omega. \quad (17)
\end{aligned}$$

When $r=|PQ|\rightarrow 0$, it can be easily confirmed that

$$\begin{aligned}
& \bar{w}(Q) - \bar{w}(P) - \bar{w}_{,k}(P, Q)r_k(P, Q) \\
& = \bar{w}(P) + \frac{\partial \bar{w}}{\partial r}\bigg|_P dr + \frac{1}{2!} \frac{\partial^2 \bar{w}}{\partial r^2}\bigg|_P dr^2 + \dots, \\
& - \frac{\partial \bar{w}}{\partial r}\bigg|_P dr - \bar{w}(P) \\
& = \frac{1}{2} \frac{\partial^2 \bar{w}}{\partial r^2}\bigg|_P dr^2 \cong O(r^2), \quad (18)
\end{aligned}$$

$$\begin{aligned}
& \bar{T}_n(Q) - \bar{w}_{,k}(P, Q)n_k(Q) \\
& = \bar{T}_n(P) + \frac{\partial \bar{T}_n(Q)}{\partial r}\bigg|_P dr, \dots, -\bar{T}_n(P) \\
& = \frac{\partial \bar{T}_n(P)}{\partial r}\bigg|_P dr \cong O(r). \quad (19)
\end{aligned}$$

Note that above relations Eqs.(18), (19) require the Hölder continuity of $\bar{w}(Q)$, $\bar{T}_n(Q)$ and $\bar{w}_{,k}(Q)$ on the source point P . If the tangential derivative $\partial \bar{w}/\partial s$ is discontinuous at the conjunction of the elements, namely, when the deflection $\bar{w}(Q)$ is discretized with general conforming elements (e.g., quadratic elements), Eq.(18) should be regarded as

$$\bar{w}(Q) - \bar{w}(P) - \bar{w}_{,k}(P, Q)r_k(P, Q) \cong O(r) \quad (20)$$

when the source point P is located on the conjunction of the elements.

The characteristic of Eq.(18) is necessary for a general formulation using coupled BIE^{(1),(6)}, because the second BIE for rotation has a hypersingularity. Strictly speaking, this formulation requires the C^1 element for the approximation of $\bar{w}(Q)$ in order to retain the continuity of derivatives $\partial \bar{w}/\partial s$ ⁽²³⁾.

On the other hand, the present formulation uses only the first BIE for deflection. Hence, the general C^0 -conforming elements (quadratic or higher-order elements) can be employed because the behavior shown in Eq.(20) is sufficient to cancel the singularity of the kernel V_n^{*s} .

Owing to these characteristics shown in Eqs.(19), (20), the singularities of V_n^{*s} (type r^{-1}) and of M_n^{*s} (type $\ln r$) in Eq.(17) are canceled, respectively. Therefore, all integrands in Eq.(17) can be evaluated by a conventional numerical quadrature formula.

Adding back the static terms and dynamic terms again in Eq.(17), we have

$$\begin{aligned}
& \int_{\Gamma} \{ \bar{V}_n^*(P, Q)\bar{w}(Q) - \bar{M}_n^*(P, Q)\bar{T}_n(Q) \\
& + \bar{T}_n^*(P, Q)\bar{M}_n(Q) - \bar{w}^*(P, Q)\bar{V}_n(Q) \} d\Gamma \\
& - \int_{\Gamma} [V_n^{*s}(P, Q)\{ \bar{w}(P) + \bar{w}_{,k}(P)r_k(P, Q) \} \\
& - M_n^{*s}(P, Q)\bar{w}_{,k}(P)n_k(Q)] d\Gamma \\
& + \sum [\Delta \bar{M}_{ns}^*(P, Q_m)\bar{w}(Q_m) - \bar{w}^*(P, Q_m)\Delta \bar{M}_{ns}(Q_m)]
\end{aligned}$$

$$\begin{aligned}
& - \sum \Delta M_{ns}^{*s}(P, Q_m)\{ \bar{w}(P) + \bar{w}_{,k}(P)r_k(P, Q_m) \} \\
& = \int_{\Omega} \bar{w}^*(P, Q)\bar{q}(Q)d\Omega. \quad (21)
\end{aligned}$$

This is a nonsingular boundary integral representation.

Even when the source point is located on the outside or inside of the region Ω , the nonsingular integral representation can be obtained according to the same procedure. Here, we rewrite Eq.(9) as,

$$\begin{aligned}
K(p)\bar{w}(p) & = \int_{\Gamma} \{ \bar{w}^*(p, Q)\bar{V}_n - \bar{T}_n^*(p, Q)\bar{M}_n(Q) \\
& + \bar{M}_n^*(p, Q)\bar{T}_n(Q) - \bar{V}_n^*(p, Q)\bar{w}(Q) \} d\Gamma \\
& + \sum [\Delta \bar{M}_{ns}^*(p, Q_m)\bar{w}(Q_m) - \bar{w}^*(p, Q_m)\Delta \bar{M}_{ns}(Q_m)] \\
& = \int_{\Omega} \bar{w}^*(p, Q)\bar{q}(Q)d\Omega, \quad p \in \Omega \text{ or } \bar{\Omega}, \quad (22)
\end{aligned}$$

where $K(p)$ is 1 for the interior source point $p \in \Omega$, and 0 for the exterior source point $p \in \bar{\Omega}$.

Consider a reference field which has rigid rotation $\bar{w}_{,k}(P_0)$ and a rigid displacement $\bar{w}(P_0)$ at the projection point P_0 , where the point P_0 should be selected as the nearest boundary point from the source point p shown in Fig. 1. According to the previous discussion, the integral representation of the reference field is given as

$$\begin{aligned}
& K(p)\{ \bar{w}(P_0) + \bar{w}_{,k}(P_0)r_k(P_0, p) \} \\
& + \int_{\Gamma} V_n^{*s}(p, Q)\{ \bar{w}(P) + \bar{w}_{,k}(P_0)r_k(P_0, Q) \} d\Gamma \\
& + \sum \Delta M_{ns}^{*s}(p, Q_m)\{ \bar{w}(P_0) + \bar{w}_{,k}(P_0)r_k(P_0, Q_m) \} \\
& = \int_{\Gamma} M_n^{*s}(p, Q)\bar{w}_{,k}(P_0)n_k(Q)d\Gamma, \\
& p \in \Omega \text{ or } \bar{\Omega}, P_0 \in \Gamma. \quad (23)
\end{aligned}$$

Subtracting Eq.(23) from Eq.(22), we have non-singular integral representation for the deflection which cancels the nearly-singular behavior of the kernels.

$$\begin{aligned}
& K(p)\{ \bar{w}(p) - \bar{w}(P_0) - \bar{w}_{,k}(P_0)r_k(P_0, p) \} \\
& + \int_{\Gamma} \{ \bar{V}_n^*(p, Q)\bar{w}(Q) - \bar{M}_n^*(p, Q)\bar{T}_n(Q) \\
& + \bar{T}_n^*(p, Q)\bar{M}_n(Q) - \bar{w}^*(p, Q)\bar{V}_n(Q) \} d\Gamma \\
& - \int_{\Gamma} [V_n^{*s}(p, Q)\{ \bar{w}(P) + \bar{w}_{,k}(P_0)r_k(P_0, Q) \} \\
& - M_n^{*s}(p, Q)\bar{w}_{,k}(P_0)n_k(Q)] d\Gamma \\
& + \sum [\Delta \bar{M}_{ns}^*(p, Q_m)\bar{w}(Q_m) - \bar{w}^*(p, Q_m)\Delta \bar{M}_{ns}(Q_m)] \\
& - \sum \Delta M_{ns}^{*s}(p, Q_m)\{ \bar{w}(P_0) + \bar{w}_{,k}(P_0)r_k(P_0, Q_m) \} \\
& = \int_{\Omega} \bar{w}^*(p, Q)\bar{q}(Q)d\Omega, \quad p \in \Omega \text{ or } \bar{\Omega}, P_0 \in \Gamma \quad (24)
\end{aligned}$$

Fig. 1 Definition of a projection point P_0

In the nonsingular boundary integral equation (21) and (24), the distributions of the density functions $\bar{w}(Q)$, $\bar{T}_n(Q)$, $\bar{M}_n(Q)$ and $\bar{V}_n(Q)$ are approximated by a polynomial function. Therefore we must discuss how to treat the deflection $\bar{w}(P)$, the derivatives $\bar{w}_{,k}(P)$ and the twisting moment $\bar{M}_{ns}$.

The deflection $\bar{w}(P)$ can be expressed generally by the nodal values and polynomials,

$$\bar{w}(P) = \sum_{i=1}^{g+1} \phi_i(\xi) \bar{w}^i(P) |_{\xi=\xi_P}, \quad (25)$$

where ξ is a natural coordinates, ξ_P is the location of the point P , $\phi_i(\xi)$ is the interpolate function of order g and $\bar{w}^i(P)$ is the nodal deflection of the current elements.

Then the following relations are employed for the derivative $\bar{w}_{,k}(P)$,

$$\begin{pmatrix} \partial \bar{w} / \partial x_1 \\ \partial \bar{w} / \partial x_2 \end{pmatrix} = \begin{pmatrix} n_1 & n_2 \\ -n_2 & n_1 \end{pmatrix} \begin{pmatrix} \partial \bar{w} / \partial n \\ \partial \bar{w} / \partial s \end{pmatrix} \quad (26)$$

Similarly to Eq. (25), the rotation $\bar{T}_n(P)$ is expressed as

$$\bar{T}_n(P) = \sum_{i=1}^{g+1} \phi_i(\xi) \bar{T}_n^i(P) |_{\xi=\xi_P} \quad (27)$$

with the nodal values of $\bar{T}_n^i(P)$ of the current element.

The tangential derivative in Eq. (26) can be expressed by differentiating the deflection $\bar{w}$ with respect to the local coordinate ξ ,

$$\begin{aligned} \frac{\partial \bar{w}}{\partial s} &= \frac{\partial \bar{w}}{\partial \xi} \frac{\partial \xi}{\partial s} = \frac{1}{J(\xi)} \frac{\partial}{\partial \xi} \sum_{i=1}^{g+1} \phi_i(\xi) \bar{w}^i \\ &= \frac{1}{J(\xi)} \sum_{i=1}^{g+1} \phi'_i(\xi) \bar{w}^i, \end{aligned} \quad (28)$$

where $J(\xi)$ denotes the Jacobian and $\phi'_i(\xi)$ are the derivatives of $\phi_i(\xi)$.

In addition, the twisting moment $\bar{M}_{ns}$ can be interpolated as

$$\begin{aligned} \bar{M}_{ns} &= D(1-\nu) \frac{\partial}{\partial \Gamma} \left(\frac{\partial \bar{w}}{\partial n} \right) - \frac{1}{R} \frac{\partial \bar{w}}{\partial \Gamma} \\ &= \frac{D(1-\nu)}{J(\xi)} \sum_{i=1}^{g+1} \phi'_i(\xi) \left\{ \bar{T}^i - \frac{1}{R} \bar{w}^i \right\}, \end{aligned} \quad (29)$$

with the nodal values $\bar{T}^i$, $\bar{w}^i$ and the curvature $(1/R)$ defined at the regular boundary point.

If we specify nodal points Q_j ($j=1, \dots, N$) on the boundary, four nodal values, namely, $\bar{w}(Q_j)$, $\bar{T}_n(Q_j)$, $\bar{M}_n(Q_j)$ and $\bar{V}_n(Q_j)$, are defined on the each element. Two of them are prescribed by the boundary condition. Therefore, the total number of unknowns is $2N$. In the present paper, the single integral equation method presented by Paiva and Venturini⁽⁹⁾ is used, which is based on the boundary integral equation of the deflection only. Hence, the source points are provided not only on the boundary but also outside of the region Ω shown in Fig. 2. The external source points are located perpendicular to the boundary at a constant distance from it. At the corner nodes, where the quantities other than deflections become discontinuous, the double nodal point scheme⁽²⁴⁾ is employed.

2.3 Numerical implementation

In this section, the numerical treatment of the boundary integral equation (21) is discussed. The following discretized equation is derived from Eq. (21) with N nodal points:

$$\begin{aligned} &\sum_{i=1}^N \{a_i \bar{w}(Q_i) + b_i \bar{T}_n(Q_i) + c_i \bar{M}_n(Q_i) + d_i \bar{V}_n(Q_i)\} \\ &= \sum_{i=1}^N A_i \{ \bar{w}(P) + r_k(P, Q_i) \bar{w}_{,k}(P) \} \\ &+ \sum_{i=1}^N B_i \{ n_k(Q_i) \bar{w}_{,k}(P) \}, \end{aligned} \quad (30)$$

where Q_i denotes nodal points defined on the boundary and $\bar{w}(Q_i)$, $\bar{T}_n(Q_i)$, $\bar{M}_n(Q_i)$ and $\bar{V}_n(Q_i)$ are density functions defined on each nodal points. The coefficients a_i , b_i , c_i and d_i are obtained by numerical integration of the kernels $\bar{w}^{*D}$, $\bar{T}_n^{*D}$, $\bar{M}_n^{*D}$ and $\bar{V}_n^{*D}$, respectively. Similarly, the coefficients A_i and B_i are obtained from the kernels $\bar{M}_n^{*S}$ and $\bar{V}_n^{*S}$. Note that the coefficient for the twisting moment $\bar{M}_{ns}$ is already included in a_i and b_i in Eq. (30). The regional integrand for transverse pressure q is omitted here.

By virtue of Eq. (26), the right-hand side (RHS) of Eq. (30) can be rewritten using the tangential derivative $\partial \bar{w} / \partial s|_P$ and normal derivative $\bar{T}_n(P)$,

RHS of Eq. (30)

$$\begin{aligned} &= \bar{w}(P) \sum_{i=1}^N A_i + \bar{T}_n(P) \sum_{i=1}^N \{A_i n_k(P) r_k(P, Q_i) \\ &+ B_i n_k(P) n_k(Q_i)\} \\ &+ \frac{\partial \bar{w}}{\partial s} \Big|_P \sum_{i=1}^N \{B_i s_k(P) r_k(P, Q_i) + B_i s_k(P) n_k(Q_i)\}, \end{aligned} \quad (31)$$

where $s_k(P)$ denotes the components of the unit tangential vector on P . Substituting Eqs. (25), (27) and (28) into Eq. (31), the RHS of Eq. (30) becomes

$$\sum_{j=1}^{g+1} \{F_j \bar{w}^j(P) + G_j \bar{T}_n^j(P)\}, \quad (32)$$

where

$$\left. \begin{aligned} F_j &= \phi_j(\xi_P) \sum_{i=1}^N A_i + \frac{1}{J(\xi_P)} \phi'_j(\xi_P) \\ &\times \sum_{i=1}^N \{B_i s_k(P) r_k(P, Q_i) + B_i s_k(P) n_k(Q_i)\} \\ G_j &= \phi_j(\xi_P) \sum_{i=1}^N \{A_i n_k(P) r_k(P, Q_i) \\ &+ B_i n_k(P) n_k(Q_i)\} \quad (j=1, \dots, g+1) \end{aligned} \right\} \quad (33)$$

Adding F_j and G_j to a_i and b_i in Eq. (30), respectively,

$$\sum_{i=1}^N \{a'_i \bar{w}(Q_i) + b'_i \bar{T}_n(Q_i) + c_i \bar{M}_n(Q_i) + d_i \bar{V}_n(Q_i)\} = 0. \quad (34)$$

From a system of simultaneous equations (34) specifying the number of source points equal to that of unknowns, the unknown nodal values are determined. Then the internal deflection can be calculated from Eq. (24).

The quantities obtained in the Laplace-transformed domain are numerically transformed into the time domain by the FFT algorithm according to Krings

and Waller's method⁽²⁵⁾.

$$f(t_k) = \frac{\exp(\gamma k \Delta t)}{T} \sum_{n=0}^{N_s-1} \bar{f}(\kappa_n) \exp \frac{j 2 \pi n k}{N_s}, \quad (35)$$

$$\Delta t = T/N_s, \Delta \omega = 2\pi/T, k=0, 1, 2, \dots, N_s-1$$

$$\gamma = \alpha/T, t_k = k\Delta t, \kappa_n = \gamma + jn\Delta\omega, j = \sqrt{-1}$$

Here T and N_s are the analyzed time range and the number of sample points, and α is specified as 6 in the present paper.

3. Numerical Results

In this section, some numerical calculations are demonstrated under the following conditions.

- All boundary integrands are estimated using the Gaussian numerical quadrature formula of the 10-th order.
- The fourth-order conforming isoparametric elements are employed.
- The double nodal point scheme⁽²⁴⁾ is applied at the corners when normal slope T_n , bending moment M_n and equivalent shear force V_n are discretized.
- An 'analytical solution' is obtained as a reference of numerical results by numerical inverse transform of the theoretical solution in Laplace-transformed domain.

3.1 Example 1

Yuki et al.⁽²⁶⁾ discussed the direct regular method (DRM) for the elastostatic problems which employ an external source point instead of a boundary source

Fig. 2 Setting of the external source points

Fig. 3 Model (A): A rectangular plate clamped with one side subjected to uniform step shear force

point, and found that it gives an accurate solution if the external source points are arranged very close to the boundary.

There are some differences between the present formulation and DRM because the present formulation employs both the boundary source point and the external source point. However, the location of the external source points should be selected carefully as the same to the DRM scheme.

First, the appropriate distance d from the boundary to an external source point is considered in Model (A) shown in Fig. 3. Model (A) is a cantilever-type rectangular plate with one clamped edge. The opposite edge is subjected to uniform shear force $Q_0 H(t)$, where $H(t)$ is a Heviside's step function. The entire boundary is divided into 8 elements and 32 nodes, by fourth-order isoparametric elements. The number of sample points N_s in inverse Laplace-transform is 256 and the analyzed time range $\tau_{\max} = T\sqrt{D/\rho h/a^2}$ is 6.0, respectively. Poisson's ratio is 0 for comparison with the analytical solution obtained by beam theory.

Figures 4 and 5 show the results of the deflection

Fig. 4 History of deflection w at point A of Model (A) under several values of d/l

history at point A and the bending moment history at point B with several values of d . Although the error increases when d/l is less than 0.01 or larger than 1.0, it seems that the ratio d/l in the range from 0.01 to 1.0 gives accurate, stable results. Since it is confirmed that the same range of d/l gives accurate results in several other models, the ratio d/l is set at 0.1 in the following calculations.

3.2 Example 2

Let us consider a simply supported rectangular plate subjected to concentrated step load $f_0 H(t)$ at its center as shown in Fig. 6. The boundary is divided into 8 elements (32 nodes) with the fourth-order isoparametric elements. N_s , $\tau_{\max}$ and Poisson's ratio are specified to 256, 1.5 and 0.3, respectively. Figure 7 shows the histories of the deflection on the internal points O, A and B in Model (B). Figure 8 shows the histories of the normal slope T_n on the nodal points C, D and E. From these results, it can be confirmed that the present method has high accuracy compared with analytical solutions.

Fig. 5 History of bending moment M_n at point B of Model (A) under several values of d/l

3.3 Example 3

We then analyzed a clamped circular plate subjected to impulsive concentrated load $f_0 H(\tau - \tau_0)$, as shown in Fig. 9. The boundary is divided into 8 elements (32 nodes), and N_s , $\tau_{\max}$, τ_0 and Poisson's ratio are 64, 1.5, 0.1 and 0.3 respectively. The numerical results for the history of the deflection at the center of the plate are shown in Fig. 10 with the analytical solution and the BEM results obtained by Bezine and Gamby⁽³⁾. The present results agree well with the

Fig. 6 Model (B): A simply supported rectangular plate subjected to concentrated step load at the center

Fig. 7 History of deflection w on the points O, A and B of Model (B)

Fig. 8 History of normal slope T_n on the points C, D and E of Model (B)

Fig. 9 Model (C): A simply supported circular plate subjected to concentrated impulsive load at the center

Fig. 10 History of deflection w at the loading point of Model (C)

analytical solution⁽²⁷⁾ over a long time range.

3.4 Example 4

The fourth model is a rectangular plate simply supported at the two opposite edges, as shown as Fig. 11. This plate is subjected to uniform distributed pressure $q_0 H(t)$, the boundary is divided into 8 elements with 32 nodes, and N_s , $\tau_{\max}$ and Poisson's ratio are 128, 3.0 and 0.3, respectively. The analyzed region is a quarter ($x_1 \geq 0$, $x_2 \geq 0$) of the hole plate because of symmetry. A two dimensional linear element is employed as the integral of the transverse pressure term, where the region is divided equally into 4 square parts. These regional integrals are evaluated with $8 \times 8 = 64$ two dimensional Gaussian quadrature formula.

Figure 12 shows the histories of the bending moment M_2 of the three points O, A and B along the x_1 -axis, and Fig. 13 shows the histories of the deflection of points C, D and E along the free edge. In these figures, the analytical solution⁽²⁸⁾ for the static case is indicated by the broken line. It is confirmed that the present BEM gives reasonable results, because the maximum value of BEM results is approximately twice that of the static case.

Fig. 11 Model (D): A rectangular plate supported simply with two opposite sides subjected to uniformly distributed step pressure

Fig. 12 History of bending moment M_2 along the x_1 axis of Model (D)

Fig. 13 History of deflection w along the free edge of Model (D)

4. Conclusion

In the present paper, the direct BEM employing the Paiva and Venturini's approach was developed for the elasto-dynamic plate bending analysis with Laplace transform. The singularity included in the static terms of the integral kernels was canceled by employing the proposed nonsingular boundary integral

equation. The non-singular integral equation has been derived by superposing the uniform rotated field. Since the external source points can be located very close to the boundary with the non-singular boundary integral equation, the accuracy of the numerical results is increased employing the high-order elements. Through some typical examples, it is shown that the present method gives reasonable and accurate results.

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