Optimization of Material Composition to Minimize Thermal Stresses in Nonhomogeneous Plate Subjected to Unsteady Heat Supply*

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For a nonhomogeneous medium, both the heat conduction equation and the governing equations of an associated thermoelastic field are nonlinear in general. Therefore, theoretical treatment of these nonlinear equations is very difficult and an exact solution is almost impossible to obtain. By introducing a laminated composite model, we derived a one-dimensional temperature solution for a nonhomogeneous plate in a transient state in our previous work. In the present work, making use of this temperature solution, we describe the optimization of material composition to minimize the transient thermal stress. As a numerical example, two nonhomogeneous plates, one composed of zirconium oxide/titanium alloy and the others of alumina/aluminum alloy, are considered. Then the optimum material composition is determined by calculation. Furthermore, the temperature dependence of material properties is discussed.

Key Words: Elasticity, Thermal Stresses, Material Design, Functionally Graded Materials

1. Introduction

Nonhomogeneous materials, such as functionally graded materials (FGM), have unique thermal and mechanical characteristics due to arbitrarily distributed and continuously varied material properties. For such nonhomogeneous materials, the heat conduction equation is presented in a nonlinear form in general. Therefore, theoretical analysis with respect to temperature changes in such materials is difficult and an exact solution is almost impossible to obtain. Using an approximated analytical model for a nonhomogeneous medium, i.e., a laminated composite model of numerous homogeneous layers, we have

already analyzed several boundary value problems⁽¹⁾⁻⁽⁸⁾ of unsteady heat conduction and the associated thermoelastic field for nonhomogeneous media, and discussed the relation of nonhomogeneity of material properties with temperature changes and associated thermal stresses. However, for such nonhomogeneous materials, it may be possible to vary the material composition arbitrarily during production, and therefore the problem of optimization of nonhomogeneous material composition to minimize the thermal stresses has attracted the interest of numerous researchers. For such problems of material composition optimization, Noda and his coworkers have already analyzed one-dimensional problems for a steady state⁽⁹⁾⁻⁽¹³⁾ of a nonhomogeneous plate, a hollow circular cylinder and a hollow sphere. However, with regard to thermal stress problems, it is wellknown that maximum stress distribution occurs in the transient state which lasts from the beginning of the heating to the steady state, in general. Therefore, for such optimization problems, it is necessary to take into account the transient state in which the maximum stress distribution is predicted to occur.

In the present work, making use of the

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temperature solution⁽¹⁴⁾ for a transient state of a nonhomogeneous plate subjected to uniform heating from the surrounding medium, which was previously analyzed by us, we deal with a problem of optimization of material composition of a nonhomogeneous plate to reduce the thermal stress distributions in a transient state. Numerical calculations are carried out for two nonhomogeneous plates, one composed of zirconium oxide/titanium alloy and the others of alumina/aluminum alloy, and the optimum material compositions are determined using the nonlinear programming method. Furthermore, the temperature dependence of the material properties is also considered for the numerical calculation, and the numerical results which take into account the effect of the temperature dependence are shown in graphical form.

2. Analytical Development

As shown in Fig. 1, we now consider an infinite plate with thickness b , and assume that the plate has nonhomogeneous thermal and mechanical properties, which are varied in the thickness direction continuously and arbitrarily.

2.1 Temperature solution in a transient state

We assume that the initial temperature of the nonhomogeneous plate is zero, and that the plate is suddenly heated from the upper surface ($x=b$) by the surrounding medium so that the temperature is zero at the lower surface ($x=0$) and T_0 at the upper one. Then, the heat conduction equation for a transient state and the initial and thermal boundary conditions are given as follows:

$$\frac{\partial T}{\partial t} = \frac{1}{c(x)\rho(x)} \cdot \frac{\partial}{\partial x} \left\{ \lambda(x) \frac{\partial T}{\partial x} \right\} \quad (1)$$

$$t=0; T=0 \quad (2)$$

$$x=0; T=0, x=b; T=T_0, \quad (3)$$

where T is temperature change; t is time; x is the coordinate variable in the thickness direction; and $\lambda(x)$, $c(x)$ and $\rho(x)$ are the thermal conductivity, the

specific heat and the density, respectively.

An approximated temperature solution for a transient state for the fundamental equation system (1)–(3) was derived in our previous work⁽¹⁴⁾; we now outline the process used to obtain the approximated one-dimensional temperature solution. First, we consider a composite laminated plate composed of a large number of layers with different isotropic and homogeneous material properties as shown in Fig. 2; which λ_i , κ_i , c_i and ρ_i are the thermal conductivity, the thermal diffusivity, the specific heat and the density of the i th layer, respectively. For such a multilayered laminated plate, the governing equation of heat conduction for each layer is a linear equation, the analytical solution of which can be derived easily under the conditions of Eqs. (2) and (3) and the continuity of the temperature change and the heat flux at the interfaces. For the analytical solution of temperature change of the laminated composite plate, we now consider a limit when the number of layers n tends to infinity; then the approximate temperature solution in a transient state for such a nonhomogeneous plate is presented in dimensionless form as follows:

$$\bar{T}(\bar{x}, \tau) = \frac{\int_0^{\bar{x}} \frac{d\bar{\xi}}{\bar{\lambda}(\bar{\xi})}}{\int_0^1 \frac{d\bar{\xi}}{\bar{\lambda}(\bar{\xi})}} + \sum_{m=1}^{\infty} \frac{2 \exp(-q_m^2 \tau)}{q_m \Delta'(q_m)} F(\bar{x}, q_m), \quad (4)$$

where

$$\left. \begin{aligned} F(\bar{x}, q_m) &= \sin\left(q_m \int_0^{\bar{x}} \frac{d\bar{\xi}}{\sqrt{\bar{\kappa}(\bar{\xi})}}\right) \\ &+ \int_0^{\bar{x}} \left\{ \frac{\bar{\lambda}(\bar{\xi})}{\sqrt{\bar{\kappa}(\bar{\xi})}} \frac{\partial}{\partial \bar{\xi}} \left(\frac{\sqrt{\bar{\kappa}(\bar{\xi})}}{\bar{\lambda}(\bar{\xi})} \right) \right. \\ &\times \cos\left(q_m \int_0^{\bar{\xi}} \frac{d\bar{\xi}}{\sqrt{\bar{\kappa}(\bar{\xi})}}\right) \\ &\times \sin\left(q_m \int_{\bar{\xi}}^{\bar{x}} \frac{d\bar{\xi}}{\sqrt{\bar{\kappa}(\bar{\xi})}}\right) \Big\} d\bar{\xi} \\ \Delta(q) &= \sin\left(q \int_0^1 \frac{d\bar{\xi}}{\sqrt{\bar{\kappa}(\bar{\xi})}}\right) + \int_0^1 \left\{ \frac{\bar{\lambda}(\bar{\xi})}{\sqrt{\bar{\kappa}(\bar{\xi})}} \right. \end{aligned} \right\}$$

Fig. 1 Thermal conditions and coordinate system of nonhomogeneous plate

Fig. 2 Analytical model of nonhomogeneous plate

$$\left. \begin{aligned} & \times \frac{\partial}{\partial \xi} \left(\frac{\sqrt{\kappa(\xi)}}{\lambda(\xi)} \right) \cos \left(q \int_0^{\xi} \frac{d\xi}{\sqrt{\kappa(\xi)}} \right) \\ & \times \sin \left(q \int_{\xi}^1 \frac{d\xi}{\sqrt{\kappa(\xi)}} \right) \} d\xi \\ \Delta'(q_m) &= \frac{d\Delta(q)}{dq} \Big|_{q=q_m} \end{aligned} \right\} \quad (5)$$

In Eqs. (4) and (5), q_m represents the eigenvalue, which is given by the m th positive root of the transcendental equation $\Delta(q)=0$, and the following dimensionless quantities are introduced:

$$\left. \begin{aligned} \bar{T}(\bar{x}, \tau) &= T(x, t)/T_0, \quad \bar{x}=x/b, \quad \tau=\kappa_0 t/b^2 \\ \bar{\lambda}(\bar{x}) &= \lambda(x)/\lambda_0, \quad \bar{\kappa}(\bar{x}) = \kappa(x)/\kappa_0 \end{aligned} \right\} \quad (6)$$

where $\kappa(x)$ represents the thermal diffusivity of the nonhomogeneous plate, and κ_0 and λ_0 denote typical values of the thermal diffusivity and the thermal conductivity, respectively. The temperature solution in a transient state given by Eqs. (4) and (5) is given by the integral form in terms of the thermal conductivity $\lambda(x)$ and the thermal diffusivity $\kappa(x)$; then it is concluded that the temperature solution is valid for any variation of these thermal material properties.

2.2 Thermal stress analysis

Let ε_0 and ρ be the strain component and the radius of curvature at the lower surface ($x=0$) of the nonhomogeneous plate. Assuming that the plate is isotropic in the in-plane direction ($y-z$ plane), then the thermal stress component $\sigma_{yy}(=\sigma_{zz})$ of the plate is given by the relation

$$\sigma_{yy}(x, t) = \frac{E(x)}{1-\nu(x)} \left\{ \varepsilon_0(t) + \frac{x}{\rho(t)} - \alpha(x)T(x, t) \right\}, \quad (7)$$

where $E(x)$, $\nu(x)$ and $\alpha(x)$ are Young's modulus of elasticity, Poisson's ratio and the coefficient of linear thermal expansion, respectively, and ε_0 and ρ are the unknown constants determined from the mechanical condition. Assuming that the infinitely long nonhomogeneous plate subjected to the thermal loading is traction-free, then the following equilibrium relations for bending moment and in-plane force are given to estimate the unknown constants ε_0 and ρ .

$$\int_0^b \sigma_{yy} dx = 0, \quad \int_0^b \sigma_{yy} x dx = 0 \quad (8)$$

Finally, the thermal stress component for the plate is given in dimensionless form by

$$\left. \begin{aligned} \bar{\sigma}_{yy}(\bar{x}, \tau) &= \frac{\bar{E}(\bar{x})}{1-\nu(\bar{x})} \\ & \times \left\{ \frac{(C_2 D_0 - C_1 D_1) + (-C_1 D_0 + C_0 D_1) \bar{x}}{C_0 C_2 - C_1^2} \right. \\ & \left. - \bar{\alpha}(\bar{x}) \bar{T}(\bar{x}, \tau) \right\}, \end{aligned} \right\} \quad (9)$$

where

$$\left. \begin{aligned} C_i &= \int_0^1 \frac{\bar{E}(\xi)}{1-\nu(\xi)} \xi^i d\xi; \quad i=0, 1, 2 \\ D_i &= \int_0^1 \frac{\bar{\alpha}(\xi) \bar{E}(\xi) \bar{T}(\xi, \tau)}{1-\nu(\xi)} \xi^i d\xi; \quad i=0, 1 \end{aligned} \right\} \quad (10)$$

In Eqs. (9) and (10), the following dimensionless quantities are introduced:

$$\left. \begin{aligned} \bar{\sigma}_{yy}(\bar{x}, \tau) &= \sigma_{yy}(x, t)/(\alpha_0 E_0 T_0) \\ \bar{E}(\bar{x}) &= E(x)/E_0, \quad \bar{\alpha}(\bar{x}) = \alpha(x)/\alpha_0 \end{aligned} \right\} \quad (11)$$

where α_0 and E_0 are typical values of the coefficient of linear thermal expansion and Young's modulus of elasticity, respectively.

2.3 Analytical method taking into consideration the temperaturedependence of material properties

Making use of the temperature solution given by Eqs. (4) and (5), temperature change taking into consideration the temperaturedependence of the thermal material properties can be estimated by using the iteration method for numerical calculations, which is carried out in the following manner. Let T_a [K] and T_b [K] be the actual temperatures of the temperature change on the basis ($T=0$) and the constant temperature change T_0 of the surrounding medium denoted in section 2.1. Then, the following relation is obtained.

$$T_0 = T_b - T_a \quad (12)$$

Therefore, the actual temperature distribution T_R [K] for the nonhomogeneous plate is given by the following relation.

$$T_R(\bar{x}, \tau) = T_a + (T_b - T_a) \bar{T}(\bar{x}, \tau) \quad (13)$$

The temperature solution for a transient state taking into consideration the temperaturedependence of the material properties can be obtained using Eqs. (4) and (5) by putting the thermal material properties in the following forms.

$$\bar{\lambda}(\xi) \rightarrow \bar{\lambda}\{\xi(T_R)\}, \quad \bar{\kappa}(\xi) \rightarrow \bar{\kappa}\{\xi(T_R)\} \quad (14)$$

In this case, the procedure of numerical calculation is as follows. First, we assume that the first approximation of the actual temperature distribution $T_R(\bar{x}, \tau)$ is given by the actual temperature T_a , namely,

$$T_{R1}(\bar{x}, \tau) = T_a. \quad (15)$$

Then, the material properties $\bar{\lambda}\{\xi(T_{R1})\}$ and $\bar{\kappa}\{\xi(T_{R1})\}$ can be calculated from the expressions with the temperaturedependence as is mentioned below. Making use of these material properties, the eigenvalue q_m is determined using Eq. (5). Thereafter, the first approximation $\bar{T}_1(\bar{x}, \tau)$ of the temperature solution $\bar{T}(\bar{x}, \tau)$ taking into consideration the temperaturedependence of the material properties is calculated using Eq. (4). Secondly, substituting the first approximated solution $\bar{T}_1(\bar{x}, \tau)$ into Eq. (13), the second approximation $T_{R2}(\bar{x}, \tau)$ of the actual temperature distribution $T_R(\bar{x}, \tau)$ is obtained. Then the material properties $\bar{\lambda}\{\xi(T_{R2})\}$ and $\bar{\kappa}\{\xi(T_{R2})\}$ can be

calculated. Furthermore, the second approximated solution $\bar{T}_2(\bar{x}, \tau)$ of the temperature solution $\bar{T}(\bar{x}, \tau)$ is obtained in the same manner as the first approximated solution $\bar{T}_1(\bar{x}, \tau)$. We may continue this iterative calculation until the temperature solution can be regarded to have converged with sufficient accuracy. Finally, the temperature distribution considered the temperature dependence of the material properties is calculated numerically.

The associated thermal stress distribution taking into consideration the temperature dependence of the material properties can be obtained using Eqs. (9) and (10) by putting the mechanical properties of the material in the following forms.

$$\begin{aligned} \bar{E}(\xi) &\rightarrow \bar{E}\{\xi(T_R)\}, \quad \nu(\xi) \rightarrow \nu\{\xi(T_R)\} \\ \bar{\alpha}(\xi) &\rightarrow \bar{\alpha}\{\xi(T_R)\} \end{aligned} \quad (16)$$

The procedure for numerical calculation for a transient thermal stress distribution is as follows. Substituting the temperature solution $\bar{T}(\bar{x}, \tau)$ into Eq. (13), the actual temperature distribution $T_R(\bar{x}, \tau)$ can be obtained. Then the material properties $\bar{E}\{\xi(T_R)\}$, $\nu\{\xi(T_R)\}$ and $\bar{\alpha}\{\xi(T_R)\}$ are calculated using the expressions with the temperature dependence as is stated below. Thereafter, we may proceed with the numerical calculations using Eqs. (9) and (10) making use of these values of material constants, and finally the transient thermal stress distribution considered the temperature dependence of the material properties is calculated.

3. Method of Numerical Calculation

Making use of the analytical solution for unsteady temperature distribution and the associated thermal stress distribution described in the previous section, numerical calculations for optimization of material composition to minimize the thermal stresses are carried out. The practical procedure for numerical calculation is as follows.

3.1 Nonhomogeneous materials adapted to the numerical calculations

We consider two nonhomogeneous plates composed of different materials for the numerical calculations. One of these is of an alumina (Al_2O_3)/aluminum alloy (A7075) functionally graded material, in which alumina and aluminum alloy are placed to the heated surface ($\bar{x}=1$) and the cooled one ($\bar{x}=0$), respectively. These material properties are shown in Table 1, in which σ_{Bt} and σ_{Bc} denote the tensile strength and the compressive strength. The numerical calculations for this FGM plate are carried out under the condition of disregarding the temperature dependence of the material properties.

The other plate is of a zirconium oxide (ZrO_2)/titanium alloy (Ti-6Al-4V) FGM, the material

Table 1 Material properties of alumina and aluminum alloy

	Al_2O_3	A7075
λ [W/(mK)]	36.0	154.0
κ [m^2/s]	11.9×10^{-6}	57.1×10^{-6}
α [1/K]	8.0×10^{-6}	23.0×10^{-6}
E [GPa]	343.0	73.0
ν	0.22	0.33
σ_{Bt} [MPa]	259.7	485.0
σ_{Bc} [MPa]	2930.2	1940.0

properties⁽¹⁵⁾ of which taking into consideration their temperature dependence for the temperature range of $300 \text{ K} \leq T \leq 1300 \text{ K}$ are as follows.

[ZrO₂]

$$\lambda = 1.71 + 0.21 \times 10^{-3} T + 0.116 \times 10^{-6} T^2$$

$$[\text{W}/(\text{mK})]$$

$$c = 2.74 \times 10^2 + 7.95 \times 10^{-1} T - 6.19 \times 10^{-4} T^2 + 1.71 \times 10^{-7} T^3 \quad [\text{J}/(\text{kgK})]$$

$$\rho = 3657.0 / \{1.0 + \alpha(T - 300.0)\}^3 \quad [\text{kg}/\text{m}^3]$$

$$\kappa = \lambda / \rho c \quad [\text{m}^2/\text{s}]$$

$$\alpha = 13.31 \times 10^{-6} - 18.9 \times 10^{-9} T + 12.7 \times 10^{-12} T^2$$

$$[1/\text{K}]$$

$$E = 132.2 - 50.3 \times 10^{-3} T - 31.4 \times 10^{-6} T^2 \quad [\text{GPa}]$$

$$\nu = 0.333$$

$$\sigma_{Bt} = 148.1 + 1.184 \times 10^{-3} T - 31.4 \times 10^{-6} T^2$$

$$[\text{MPa}]$$

$$\sigma_{Bc} = 3136.0 / 146.0 \times \sigma_{Bt} \quad [\text{MPa}]$$

(17)

[Ti-6Al-4V]

$$\lambda = 1.1 + 0.017 T \quad [\text{W}/(\text{mK})]$$

$$c = 3.5 \times 10^2 + 8.78 \times 10^{-1} T - 9.74 \times 10^{-4} T^2 + 4.43 \times 10^{-7} T^3 \quad [\text{J}/(\text{kgK})]$$

$$\rho = 4420.0 / \{1.0 + \alpha(T - 300.0)\}^3 \quad [\text{kg}/\text{m}^3]$$

$$\kappa = \lambda / (\rho c) \quad [\text{m}^2/\text{s}]$$

$$\alpha = 7.43 \times 10^{-6} + 5.56 \times 10^{-9} T - 2.69 \times 10^{-12} T^2$$

$$[1/\text{K}] : 300 \text{ K} \leq T \leq 1100 \text{ K}$$

$$\alpha = 10.291 \times 10^{-6} \quad [1/\text{K}] : 1100 \text{ K} \leq T \leq 1300 \text{ K}$$

$$E = 122.7 - 0.0565 T \quad [\text{GPa}]$$

$$\nu = 0.289 + 32.0 \times 10^{-6} T$$

$$\sigma_{Bt} = \sigma_{Bc} = 1252.0 - 0.8486 T \quad [\text{MPa}]$$

(18)

3.2 Volume fractions of the material composition and law of mixture for functionally graded materials

For expression of the material composition of the nonhomogeneous plate, volume fractions of light metal V_m and ceramics V_c are defined by the relations

$$V_m(\bar{x}) = \begin{cases} 1 - \bar{x}^M & : M \geq 1 \\ (1 - \bar{x})^{1/M} & : M < 1 \end{cases}, \quad V_c(\bar{x}) = 1 - V_m(\bar{x}),$$

(19)

in which the numerical parameter M is a design

Fig. 3 Variation of volume fraction V_m

variable for the problem of optimization to minimize the thermal stress distribution. The variation of the volume fraction V_m of the light metal with the parameter M is shown in Fig. 3.

To estimate the material properties of the nonhomogeneous materials, we applied mainly the law of mixture proposed by Kerner^{(16),(17)}, which was derived based on the assumption that the granular phase is in a matrix phase. However, the details of expression for the law are omitted here for brevity.

3.3 Optimization problem

For nonhomogeneous materials such as FGMs, the tensile strength $\sigma_{Bt}(\bar{x})$ and the compressive strength $\sigma_{Bc}(\bar{x})$ vary in accordance with the variation of the volume fractions of light metal and ceramics. In order to deal with the problem of optimization of material composition to minimize the thermal stress distribution, we now introduce stress ratio f defined by the following relation.

$$\left. \begin{aligned} f(\bar{x}) &= \sigma_{yy}(\bar{x}) / \sigma_{Bt}(\bar{x}); \sigma_{yy}(\bar{x}) \geq 0 \\ f(\bar{x}) &= \sigma_{yy}(\bar{x}) / \sigma_{Bc}(\bar{x}); \sigma_{yy}(\bar{x}) \leq 0 \end{aligned} \right\} \quad (20)$$

Stress ratio f is the ratio of the stress distribution and the tensile or compressive strength at an arbitrary point of the nonhomogeneous plate. Therefore, the optimization problem in this case is defined as a minimization problem of maximum tensile/compressive stress ratio $f_{\max}$ to determine the design variable M for the time interval from the initial state of heating to the steady state. The numerical calculations are carried out under the following conditions.

$$\left. \begin{aligned} 0.001 \leq \tau \leq 100 (\div \infty) \\ 0 \leq \bar{x} \leq 1.0 \\ 1/40 \leq M \leq 40 \end{aligned} \right\} \quad (21)$$

4. Numerical Results and Discussion

For the two different functionally graded materials described in section 3.1, we calculated stress ratio $f(M)$ for the alumina/aluminum alloy FGM plate in accordance with the variations of numerical

parameter M , position $\bar{x}$ and the time τ given by Eq. (21). From the numerical results obtained, it is found that the maximum value of the compressive stress ratio is always small compared with the maximum value of the tensile stress ratio because of the large value of the compressive strength σ_{Bc} compared with the tensile strength σ_{Bt} .

For the zirconium oxide/titanium alloy FGM plate, we could not find the compressive strength of titanium alloy in literatures, so we assigned the same value with the tensile strength as for the compressive strength was stated in Eq.(18), and performed the numerical calculations for stress ratio $f(M)$ with variations of M , $\bar{x}$ and τ in a manner similar to that in the case of the alumina/aluminum alloy FGM plate. From the numerical results obtained, it is found that in this case the maximum value of the compressive stress ratio is large compared with the maximum value of the tensile stress ratio for a certain value of M . We suspect the reason of this phenomenon to be that the estimation of the compressive strength of the titanium alloy is inappropriate. From the above-mentioned consideration, numerical calculations for the optimization in this case are carried out based on the tensile stress ratio.

4.1 Alumina/aluminum alloy FGM plate

Numerical calculations for the alumina/aluminum alloy FGM plate are carried out disregarding the temperature dependence of the material properties. From the numerical results obtained, the optimal value of M for the material composition and the corresponding maximum stress ratio $f_{\max}(M)$ are shown as

$$M = 2.1453, f_{\max}(M) = 0.74273. \quad (22)$$

The volume fraction V_m for aluminum alloy, the temperature change and the associated thermal stress distribution corresponding to the optimal value of M are shown in Figs. 4(a)-4(c), and the variation of the stress ratio f with time τ ($0.001 \leq \tau \leq 100$) and the position $\bar{x}$ ($0 \leq \bar{x} \leq 1$) is shown in Fig. 4(d). Moreover, the variation of the maximum value of stress ratio $f_{\max}$ with M is shown in Fig. 4(e), and the time τ and the position $\bar{x}$ at which the maximum value of stress ratio $f_{\max}$ occurs for each value of M are shown in Fig. 4(f). It can be found from Fig. 4(e) that the range of M where $f_{\max}$ is less than unity (namely, where the nonhomogeneous plate is not broken down) is extremely narrow, and restricted to the region of $1.0 < M < 2.3$. Moreover, it is shown in Fig. 4(f) that the position $\bar{x}$ and the time τ for the maximum value of stress ratio $f_{\max}$ corresponding to the optimal value $M = 2.1453$ exist two different cases, one of which corresponds to $\bar{x} = 1.0$, $\tau = \infty$, and the another of which to $\bar{x} \cong 0.8$, $\tau \cong 0.01$. From these results, the

optimization problem in this case is a minimization problem of a bimodal function of tensile stresses near the heated end at the beginning of heat supply and on the heated surface in the steady state.

(a) Volume fraction V_m

(b) Temperature change $\bar{T}$

(c) Thermal stress distribution $\bar{\sigma}_{yy}$

4.2 Zirconium oxide/titanium alloy FGM plate

Taking into consideration the temperature dependence of the material properties given by Eqs. (17) and (18), numerical calculations for the zirconium oxide/titanium alloy FGM plate are carried out. As illustrated in section 2.3, letting T_a and $T_b (=T_0 + T_a)$ be the

(d) Variation of stress ratio f

(e) Variation of $f_{\max}$ with M

(f) Variation of $f_{\max}$ with position $\bar{x}$ and time τ

Fig. 4 Numerical results for optimal value $M=2.1453$ (alumina/aluminum alloy FGM plate)

actual temperatures of the cooled surface ($\bar{x}=0$) and the heated one ($\bar{x}=1$), the following four cases are considered.

Case 1: $T_a=300$ K, $T_b=700$ K, temperature dependences are disregarded.

Case 2: $T_a=300$ K, $T_b=700$ K, temperature de-

pendences are considered.

Case 3: $T_a=300$ K, $T_b=1\,000$ K, temperature dependences are considered.

Case 4: $T_a=300$ K, $T_b=1\,300$ K, temperature dependences are considered.

From the numerical results obtained, the optimal value of M for the material composition and the

(a) Volume fraction V_m

(b) Temperature distribution T

(c) Thermal stress distribution σ_{yy}/E_0

(d) Variation of stress ratio f

(e) Variation of $f_{\max}$ with M

(f) Variation of $f_{\max}$ with position $\bar{x}$ and time τ

Fig. 5 Numerical results for optimal value $M=40.0$ (Case 1 for zirconium oxide/titanium alloy FGM plate)

corresponding maximum value of the stress ratio $f_{\max}(M)$ are given as follows.

$$\left. \begin{array}{l} \text{Case 1: } M=40.0, f_{\max}(M)=0.093\,701 \\ \text{Case 2: } M=1.416\,7, f_{\max}(M)=0.151\,45 \\ \text{Case 3: } M=2.886\,8, f_{\max}(M)=0.190\,55 \\ \text{Case 4: } M=2.878\,3, f_{\max}(M)=0.295\,29 \end{array} \right\} \quad (23)$$

Numerical results for Case 1 and Case 2 are shown in

Figs. 5 and 6, respectively. In detail for these two cases, volume fractions V_m of titanium alloy are shown in Figs. 5(a) and 6(a), the temperature distributions are shown in Figs. 5(b) and 6(b), and the associated thermal stress distributions are shown in Figs. 5(c) and 6(c), respectively. Moreover, for Case 1 and Case 2, variations of stress ratio f with the

(a) Volume fraction V_m

(b) Temperature distribution T

(c) Thermal stress distribution σ_{yy}/E_0

(d) Variation of stress ratio f

(e) Variation of $f_{\max}$ with M

(f) Variation of $f_{\max}$ with position $\bar{x}$ and time τ

Fig. 6 Numerical results for optimal value $M=1.416\,7$ (Case 2 for zirconium oxide/titanium alloy FGM plate)

(a) Case 3

(b) Case 4

Fig. 7 Variations of $f_{\max}$ with M

time τ and the position $\bar{x}$ are shown in Figs. 5(d) and 6(d), variations of the maximum value of stress ratio $f_{\max}$ with the M are shown in Figs. 5(e) and 6(e), and the time τ and the position $\bar{x}$ where the maximum value of the stress ratio $f_{\max}$ occurs for each value of M are shown in Figs. 5(f) and 6(f). Upon comparison of the numerical results shown in Figs. 5 and 6, the effect of the temperature dependence of the material properties is found to be remarkable; in other words, the temperature distribution, the associated thermal stress distribution and the optimal value of the material composition change characteristically when the temperature dependence is taken into consideration. For example, as shown in Fig. 5(a), optimal material composition $M=40.0$, when the temperature dependence of the material properties is disregarded, means that the FGM plate is composed of a thin layer of the ceramics on the heated surface, and it is shown from Fig. 5(e) that $f_{\max}$ is less than unity for each value of $M(1/40 \leq M \leq 40)$, which means that the FGM plate is free from the fracture for any material composition.

On the other hand, as shown in Fig. 6(a), the optimal material composition $M=1.4167$ when the temperature dependence of the material properties is considered shows a linearly changing material composition of light metal and ceramics, practically. From Fig. 6(e), the value of M being free from fracture is restricted to within the range of $M < 6$. Moreover, it is found from Figs. 5(f) and 6(f) that optimization problem when the temperature dependence is disregarded is a minimization problem of a unimodal function, whereas when the temperature dependence is considered, the optimization problem is a minimization problem of a bimodal function of tensile stresses near the heated end at the beginning of the heat supply and on the heated surface in the steady state. As is stated in Eq. (23), when the temperature dependence of the material properties is considered, the optimal material composition is markedly changed due to the temperature rise $T_0 (= T_b - T_a)$. However, from the numerical results obtained, it is found that the optimal material composition is restricted to within the range of $1.4 < M < 2.9$. The variations of the maximum value of stress ratio $f_{\max}$ with M for Case 3 and Case 4 are shown in Figs. 7(a) and 7(b). From these figures, it is found that the ranges of the parameter M to be free from fracture, in other words, the ranges of $f_{\max}$ less than unity, are $M < 8$ for Case 3, and $M > 0.4$ for Case 4, and yet, for these two cases, it is shown that the optimization problems are minimization problems of a bimodal function for the beginning of the heat supply and the steady state, which are omitted to show in graphical forms.

5. Concluding Remarks

In the present research, we considered a non-homogeneous plate with variation of material properties in the thickness direction as an analytical model. Assuming that the plate is heated from one end surface, we have dealt with a problem of optimization of material composition to minimize the thermal stress distribution from the beginning of the heat supply to the steady state. From the numerical results obtained, it is found that the optimal material composition and the corresponding temperature and thermal stress distributions are affected remarkably by the temperature dependence of the material properties and the magnitude of the temperature rise.

References

- (1) Ootao, Y., Tanigawa, Y. and Fukuda, T., Axisymmetric Transient Thermal Stress Analysis of a Multilayered Composite Hollow Cylinder, *J. Thermal Stresses*, Vol. 14(1991), p. 201.
- (2) Tanigawa, Y., Ootao, Y. and Kawamura, R.,

- Thermal Bending of Laminated Composite Rectangular Plates and Nonhomogeneous Plates due to Partial Heating, *J. Thermal Stresses*, Vol. 14(1991), p. 285.
- (3) Ootao, Y., Tanigawa, Y. and Kasai, T., Inverse Problem Analysis of Thermal Deformation for a Nonhomogeneous Solid Cylinder under Axisymmetric Unsteady Heating, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 57, No. 537, A(1991), p. 1161.
 - (4) Ootao, Y., Tanigawa, Y. and Nakanishi, N., Transient Thermal Stress Analysis of a Nonhomogeneous Hollow Sphere due to Axisymmetric Heat Supply, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 57, No. 539, A(1991), p. 1581.
 - (5) Tanigawa, Y. and Kawamura, R., Nonlinear Behavior of Nonhomogeneous Beam due to Unsteady Heat Supply, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 57, No. 540, A(1991), p. 1847.
 - (6) Ootao, Y., Tanigawa, Y., Kasai, T. and Nakanishi, N., Inverse Problem Analysis of Transient Thermal Stress Analysis in a Nonhomogeneous Solid Circular Cylinder due to Axisymmetrical Heating, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 58, No. 548, A(1992), p. 567.
 - (7) Ootao, Y., Akai, T. and Tanigawa, Y., Three-Dimensional Transient Thermal Stress Analysis of Nonhomogeneous Hollow Circular Cylinder, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 59, No. 563, A(1993), p. 1684.
 - (8) Ootao, Y., Akai, T. and Tanigawa, Y., Three-Dimensional Transient Thermal Stress Analysis of a Nonhomogeneous Hollow Circular Cylinder with Respect to Rotating Heat Source, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 59, No. 568, A(1993), p. 2953.
 - (9) Noda, N. and Tsuji, T., Steady Thermal Stresses in a Plate of Functionally Gradient Material, *Proc. First International Symposium on Functionally Gradient Materials*, (1990), p. 339.
 - (10) Noda, N. and Tsuji, T., Steady Thermal Stresses in a Plate of Functionally Gradient Material, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 57, No. 533, A(1991), p. 98.
 - (11) Noda, N. and Tsuji, T., Steady Thermal Stresses in a Plate of Functionally Gradient Material with Temperature-Dependent Properties, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 57, No. 535, A(1991), p. 625.
 - (12) Obata, Y., Noda, N. and Tsuji, T., Steady Thermal Stresses in a Functionally Gradient Material Plate (Influence of Mechanical Boundary Conditions), *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 58, No. 553, A(1992), p. 1689.
 - (13) Obata, Y. and Noda, N., Steady Thermal Stresses in a Hollow Circular Cylinder and a Hollow Sphere of a Functionally Gradient Material, *J. Thermal Stresses*, (in Japanese), Vol. 17(1994), p. 471.
 - (14) Tanigawa, Y., Akai, T. and Kasai, T., One-Dimensional Transient Heat Conduction and Associated Thermal Stress Problems of a Plate with Nonhomogeneous Material Properties, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 61, No. 584, A(1995), p. 607.
 - (15) Touloukian, Y. S., *Thermophysical Properties of Matter*, 1(1973), p. 1272, IFI/Plenum.
 - (16) Kerner, E. H., The Electrical Conductivity of Composite Media, *Proc. Phys. Soc. Lond., B*, Vol. 69(1956), p. 802.
 - (17) Kerner, E. H., The Elastic and Thermoelastic Properties of Composite Media, *Proc. Phys. Soc. Lond., B*, Vol. 69(1956), P. 808.