Homogenization Method for Dynamic Viscoelastic Analysis of Composite Materials*

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In this paper we propose a new homogenization method for composite materials in order to predict effective dynamic viscoelastic material properties in the frequency domain. The loss tangent, which is the ratio of storage modulus to loss modulus, is an important design variable for products made of viscoelastic materials. The proposed procedure, by which it is possible to compute the effective loss tangent of composite materials with periodic structure, is implemented in a homogenization analysis system based on the general purpose finite element analysis code, ABAQUS. The results of the numerical analysis are very close to experimental results which are obtained by using a rubber composite containing a stiff rubber and a soft rubber.

Key Words: Composite Material, Viscoelasticity, FEM, Homogenization, Loss Tangent, Rubber

1. Introduction

In the present paper a homogenization method is proposed for dynamic viscoelasticity analysis of composite materials with periodic microstructures. Moreover, in this study an effective loss tangent analysis system is developed by using a finite element technique.

It is well known that all polymers show time dependent mechanical properties, such as viscoelasticity caused by internal friction. In order to design products made of viscoelastic composite materials, global effective time dependent mechanical prop-

erties should be evaluated experimentally or by numerical methods. The loss tangent, which is defined as the ratio of the homogenized loss modulus to the homogenized storage modulus, is important when harmonic external forces are considered. Dissipated energy changes internal heat generation. The internal heat generation and rolling resistance of automobile tires made of rubber and fiber-reinforced rubber are caused by energy dissipation. High-damping materials, e.g. high-damping steel plates and high-damping alloys have been used⁽¹⁾ recently. The ability of high-damping materials to dissipate vibration energy is indicated by the loss factor. A high loss tangent is necessary in order to obtain a high loss factor. In practical applications, the damping characteristics of materials and structures are very important. Therefore, in this study, we have developed an analysis system to compute the effective loss tangent for polymer composites with periodic microstructures.

A homogenization method for viscoelastic materials in the time domain was studied by Francfort and Suquet⁽²⁾ and Suquet⁽³⁾. In these studies, the macroscopic law was derived using linear viscoelastic materials with short memory; Kelvin-Voigt's materials

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and Maxwell's materials. The homogenized macroscopic law was formulated by applying a Laplace transform. In these studies, it was demonstrated that the homogenization procedure gives rise to long range memory effects⁽⁴⁾. A homogenization procedure has been derived by Shibuya^{(5),(6)} by applying a Laplace transform to viscoelastic materials characterized by a convolution integral. Shibuya numerically computed a effective relaxation modulus⁽⁵⁾ and a effective creep compliance⁽⁶⁾ using a boundary element technique. However, a homogenization method for viscoelastic materials in the frequency domain has not yet been realized.

The purpose of the present study is:

- (i) to formulate a homogenization method to a compute effective loss tangent of composite materials with periodic microstructures;
- (ii) to develop a homogenization analysis system based on the general purpose finite element code, ABAQUS⁽⁷⁾;
- (iii) to validate the proposed homogenization method by comparing computed effective loss tangents with experimental results obtained using rubber composite materials.

2. Dynamic Viscoelastic Analysis for Frequency Domain

2.1 Complex constitutive equation

The stress response $\sigma(t)$ is expressed by the Boltzmann integral representation in terms of the strain $\epsilon(t)$ ⁽⁸⁾ as

$$\sigma(t) = \int_{-\infty}^t E(t-\bar{\tau}) \frac{d\epsilon(\bar{\tau})}{d\bar{\tau}} d\bar{\tau}. \quad (1)$$

We introduce a dimensionless relaxation function as

$$\phi(t) = \frac{E(t)}{E_{\infty}} - 1, \quad (2)$$

where E_{∞} denotes long-term modulus.

Substituting Eq. (2) into Eq. (1) yields

$$\begin{aligned} \sigma(t) &= \sigma_{\infty}(t) + \int_{-\infty}^t \phi(t-\bar{\tau}) \frac{d\sigma_{\infty}(\bar{\tau})}{d\bar{\tau}} d\bar{\tau} \\ &= \sigma_{\infty}(t) - \int_0^{\infty} \phi(\tau) \frac{d\sigma_{\infty}(t-\tau)}{d\tau} d\tau, \end{aligned} \quad (3)$$

where $\sigma_{\infty}(t)$ and τ denote $E_{\infty}\epsilon(t)$ and $t-\bar{\tau}$, respectively.

We consider a steady-state harmonic oscillating strain

$$\epsilon(t) = \epsilon^* e^{i\omega t}. \quad (4)$$

Substituting Eq. (4) into Eq. (3), the long-term stress relaxation equation yields:

$$\begin{aligned} \sigma(t) &= E_{\infty}\epsilon(t) - \int_0^{\infty} \phi(\tau) \frac{d}{d\tau} (E_{\infty}\epsilon^* e^{i\omega(t-\tau)}) d\tau \\ &= E_{\infty} \left(1 + i\omega \int_0^{\infty} \phi(\tau) e^{-i\omega\tau} d\tau \right) \epsilon(t). \end{aligned} \quad (5)$$

Since $\phi(\tau)=0$ as $\tau < 0$,

$$\phi^*(\omega) = \int_0^{\infty} \phi(\tau) e^{-i\omega\tau} d\tau \quad (6)$$

is a Fourier transform of $\phi(t)$. Therefore, Eq. (5) can be rewritten as

$$\sigma(t) = E^*(\omega) \epsilon(t), \quad (7)$$

where

$$E^*(\omega) = E_{\infty}(1 + i\omega\phi^*(\omega)). \quad (8)$$

Moreover, separating the complex function $\phi^*(\omega)$ into a real part ϕ' and an imaginary part ϕ'' , Eq. (8) can be rewritten as

$$\begin{aligned} E^*(\omega) &= E_{\infty}(1 - \omega\phi'') + iE_{\infty}\omega\phi' \\ &= E' + iE'', \end{aligned} \quad (9)$$

where E' and E'' denote the storage modulus and the loss modulus, respectively. The loss tangent is defined as

$$\tan \delta = \frac{E''}{E'}. \quad (10)$$

Substituting Eq. (4) and Eq. (9) into Eq. (7) yields the complex constitutive equation

$$\begin{Bmatrix} \sigma' \\ \sigma'' \end{Bmatrix} = \begin{bmatrix} E' & -E'' \\ E'' & E' \end{bmatrix} \begin{Bmatrix} \epsilon' \\ \epsilon'' \end{Bmatrix}, \quad (11)$$

where we separate the complex function into a real and imaginary part.

Extending the complex constitutive equation to three dimensions yields

$$\begin{aligned} \begin{Bmatrix} \sigma' \\ \sigma'' \end{Bmatrix} &= \begin{bmatrix} D' & -D'' \\ D'' & D' \end{bmatrix} \begin{Bmatrix} \epsilon' \\ \epsilon'' \end{Bmatrix} \\ &= \begin{bmatrix} D'_d + D'_v & -D''_d - D''_v \\ D''_d + D''_v & D'_d + D'_v \end{bmatrix} \begin{Bmatrix} \epsilon' \\ \epsilon'' \end{Bmatrix}, \end{aligned} \quad (12)$$

where D_d and D_v denote the deviatoric part and volumetric part of the elasticity matrix, respectively, and they are expressed as

$$D_d = G \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} & & \\ & \frac{4}{3} & -\frac{2}{3} & & \\ & & \frac{4}{3} & & \\ & & & 1 & \\ \text{sym.} & & & & 1 \end{bmatrix} \quad (13)$$

and

$$D_v = K \begin{bmatrix} 1 & 1 & 1 & & \\ & 1 & 1 & & \\ & & 1 & & \\ & & & 0 & \\ \text{sym.} & & & & 0 \end{bmatrix}, \quad (14)$$

where G and K are the shear modulus and bulk modulus, respectively, and hence viscoelastic properties are considered independently.

2.2 Principle of virtual work

When the d'Alembert force is used for the inertia effect the principle of virtual work is

$$\int_{\Omega} \rho \mathbf{v}^T \ddot{\mathbf{u}} d\Omega + \int_{\Omega} \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \right)^T \boldsymbol{\sigma} d\Omega = \int_{\Omega} \mathbf{v}^T \mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{v}^T \mathbf{t} d\Gamma, \quad (15)$$

where ρ , $\dot{\mathbf{u}}$, $\mathbf{v}$, $\mathbf{b}$ and $\mathbf{t}$ denote mass density, velocity, virtual displacement, body force and traction, respectively. The superscript T denotes transpose.

We assume that the responses of displacements $\mathbf{u}$ and external forces $\mathbf{f}$ ($=\mathbf{b}, \mathbf{t}$) are steady-state harmonic vibration, so that

$$\mathbf{u}(t) = (\mathbf{u}' + i\mathbf{u}'')e^{i\omega t} \quad (16)$$

and

$$\mathbf{f}(t) = (\mathbf{f}' + i\mathbf{f}'')e^{i\omega t}. \quad (17)$$

Substituting Eq. (16), Eq. (17) and Eq. (12) into Eq. (15) yields

$$\begin{aligned} & -\omega^2 \int_{\Omega} \rho [\mathbf{v}' \quad \mathbf{v}''] \left\{ \begin{matrix} \mathbf{u}' \\ \mathbf{u}'' \end{matrix} \right\} d\Omega \\ & + \int_{\Omega} \left[\frac{\partial \mathbf{v}'}{\partial \mathbf{x}} \quad \frac{\partial \mathbf{v}''}{\partial \mathbf{x}} \right] \left[\begin{matrix} \mathbf{D}' & -\mathbf{D}'' \\ \mathbf{D}'' & \mathbf{D}' \end{matrix} \right] \left\{ \begin{matrix} \frac{\partial \mathbf{u}'}{\partial \mathbf{x}} \\ \frac{\partial \mathbf{u}''}{\partial \mathbf{x}} \end{matrix} \right\} d\Omega \\ & = \int_{\Omega} [\mathbf{v}' \quad \mathbf{v}''] \left\{ \begin{matrix} \mathbf{b}' \\ \mathbf{b}'' \end{matrix} \right\} d\Omega + \int_{\Gamma_t} [\mathbf{v}' \quad \mathbf{v}''] \left\{ \begin{matrix} \mathbf{t}' \\ \mathbf{t}'' \end{matrix} \right\} d\Gamma. \end{aligned} \quad (18)$$

3. Homogenization Method for Dynamic Viscoelastic Materials

3.1 Variational formulation

The homogenization method applies to composite materials with periodic microstructure. One period of this periodicity is called a unit cell. Let $\mathbf{x}$ be a macroscopic description of a structural body and let $\mathbf{y}$ be a microscopic description of a unit cell. $\mathbf{y}$ is expressed as

$$\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}, \quad (19)$$

where ε is a ratio between the macroscopic level and the microscopic level.

It is reasonable to assume that complex displacements can be expressed as an asymptotic expansion with respect to the parameter ε ; namely,

$$\mathbf{u}^{\varepsilon} = \mathbf{u}^{0*}(\mathbf{x}, \mathbf{y}) + \varepsilon \mathbf{u}^{1*}(\mathbf{x}, \mathbf{y}) + \dots, \quad \mathbf{y} = \frac{\mathbf{x}}{\varepsilon}. \quad (20)$$

By substituting the above equation into Eq. (2) and by using Eq. (3), we can obtain the following two uncoupled weak forms of the equilibrium equations: one is on the microscopic level and the other is on the macroscopic level.

$$\begin{aligned} & \int_Y \left[\frac{\partial \mathbf{v}'}{\partial \mathbf{y}} \quad \frac{\partial \mathbf{v}''}{\partial \mathbf{y}} \right] \left[\begin{matrix} \mathbf{D}' & -\mathbf{D}'' \\ \mathbf{D}'' & \mathbf{D}' \end{matrix} \right] \left\{ \begin{matrix} \frac{\partial \boldsymbol{\chi}'}{\partial \mathbf{y}} \\ \frac{\partial \boldsymbol{\chi}''}{\partial \mathbf{y}} \end{matrix} \right\} dY \\ & = \int_Y \left[\frac{\partial \mathbf{v}'}{\partial \mathbf{y}} \quad \frac{\partial \mathbf{v}''}{\partial \mathbf{y}} \right] \left[\begin{matrix} \mathbf{D}' \\ -\mathbf{D}'' \end{matrix} \right] dY \\ & - \omega^2 \int_{\Omega} \rho^H [\mathbf{v}' \quad \mathbf{v}''] \left\{ \begin{matrix} \mathbf{u}^{0'} \\ \mathbf{u}^{0''} \end{matrix} \right\} d\Omega \end{aligned} \quad (21)$$

$$\begin{aligned} & + \int_{\Omega} \left[\frac{\partial \mathbf{v}'}{\partial \mathbf{x}} \quad \frac{\partial \mathbf{v}''}{\partial \mathbf{x}} \right] \left[\begin{matrix} \mathbf{D}^{H'} & -\mathbf{D}^{H''} \\ \mathbf{D}^{H''} & \mathbf{D}^{H'} \end{matrix} \right] \left\{ \begin{matrix} \frac{\partial \mathbf{u}^{0'}}{\partial \mathbf{x}} \\ \frac{\partial \mathbf{u}^{0''}}{\partial \mathbf{x}} \end{matrix} \right\} d\Omega \\ & = \int_{\Omega} [\mathbf{v}' \quad \mathbf{v}''] \left\{ \begin{matrix} \mathbf{b}^{H'} \\ \mathbf{b}^{H''} \end{matrix} \right\} d\Omega + \int_{\Gamma_t} [\mathbf{v}' \quad \mathbf{v}''] \left\{ \begin{matrix} \mathbf{t}' \\ \mathbf{t}'' \end{matrix} \right\} d\Gamma, \end{aligned} \quad (22)$$

where $\boldsymbol{\chi}$ is a characteristic function of a microstructure. In physical meaning, the characteristic function denotes displacements subjected to internal stresses caused by unit strain at each material point under periodic boundary conditions. Furthermore, the following parameters

$$\rho^H = \frac{1}{|Y|} \int_Y \rho dY \quad (23)$$

$$\mathbf{D}^{H'} = \frac{1}{|Y|} \int_Y \left[\mathbf{D}' - \left(\mathbf{D}' \frac{\partial \boldsymbol{\chi}'}{\partial \mathbf{y}} - \mathbf{D}'' \frac{\partial \boldsymbol{\chi}''}{\partial \mathbf{y}} \right) \right] dY \quad (24)$$

$$\mathbf{D}^{H''} = \frac{1}{|Y|} \int_Y \left[\mathbf{D}'' - \left(\mathbf{D}'' \frac{\partial \boldsymbol{\chi}'}{\partial \mathbf{y}} + \mathbf{D}' \frac{\partial \boldsymbol{\chi}''}{\partial \mathbf{y}} \right) \right] dY \quad (25)$$

$$\mathbf{b}^{H'} = \frac{1}{|Y|} \int_Y \mathbf{b}' dY \quad (26)$$

and

$$\mathbf{b}^{H''} = \frac{1}{|Y|} \int_Y \mathbf{b}'' dY \quad (27)$$

are homogenized mass density, homogenized storage elastic matrix, homogenized loss elastic matrix, the real part of the homogenized body force and the imaginary part of the homogenized body force, respectively. $|Y|$ indicates the volume of a unit cell.

A homogenized storage elasticity matrix and a homogenized loss elasticity matrix are computed by substituting the solution of Eq. (21) into Eq. (24) and Eq. (25). For the case of $\mathbf{D}'' = \mathbf{0}$, Eq. (21) and Eq. (22) agree with the homogenized weak form of the equilibrium equations for an elastic body⁽⁹⁾.

The homogenized effective loss tangent can be derived as follows.

First, we define the homogenized compliance matrix as

$$\mathbf{C}^{H*} = (\mathbf{D}^{H*})^{-1} = \begin{bmatrix} \mathbf{D}^{H'} & -\mathbf{D}^{H''} \\ \mathbf{D}^{H''} & \mathbf{D}^{H'} \end{bmatrix}^{-1}. \quad (28)$$

Next, the effective storage modulus and the effective loss modulus can be defined as

$$E_{ij}^{H'} = \Re \left(\frac{1}{C_{ijij}^{H*}} \right) = \frac{C_{ijij}^{H'}}{C_{ijij}^{H'} r^2 + C_{ijij}^{H''} n^2} \quad (29)$$

and

$$E_{ij}^{H''} = \Im \left(\frac{1}{C_{ijij}^{H*}} \right) = \frac{C_{ijij}^{H''}}{C_{ijij}^{H'} r^2 + C_{ijij}^{H''} n^2}, \quad (30)$$

where $i, j = 1 \sim 3$, and $\Re(\cdot)$ and $\Im(\cdot)$ denote the real and imaginary part, respectively.

Let $\tan \delta^H$ be a homogenized loss tangent, which is defined as the ratio of the homogenized loss modulus to the homogenized storage modulus.

$$\tan \delta_{ij}^{H'} = \frac{E_{ij}^{H''}}{E_{ij}^{H'}} = \frac{C_{ijij}^{H''}}{C_{ijij}^{H'}} \quad (31)$$

Namely, the effective loss tangent is defined as

the ratio of the homogenized loss compliance to the homogenized storage compliance.

3.2 Analysis system

We develop a homogenization analysis system for frequency domain viscoelastic bodies based on the general purpose finite element code, ABAQUS.

Applying finite element discretization in the weak form of microscopic equilibrium equations for viscoelastic body yields

$$\begin{bmatrix} \mathbf{K}_y' & -\mathbf{K}_y'' \\ \mathbf{K}_y'' & \mathbf{K}_y' \end{bmatrix} \begin{Bmatrix} \boldsymbol{\chi}' \\ \boldsymbol{\chi}'' \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_y' \\ -\mathbf{F}_y'' \end{Bmatrix}. \quad (32)$$

In the above equations, $\mathbf{K}_y'$, $\mathbf{K}_y''$, $\mathbf{F}_y'$ and $\mathbf{F}_y''$ are denoted as

$$\mathbf{K}_y' = \sum_{e=1}^{n_{el}} \int_{Y^e} \mathbf{B}_y^T \mathbf{D}' \mathbf{B}_y dY \quad (33)$$

$$\mathbf{K}_y'' = \sum_{e=1}^{n_{el}} \int_{Y^e} \mathbf{B}_y^T \mathbf{D}'' \mathbf{B}_y dY \quad (34)$$

$$\mathbf{F}_y' = \sum_{e=1}^{n_{el}} \int_{Y^e} \mathbf{B}_y^T \mathbf{D}' dY \quad (35)$$

and

$$\mathbf{F}_y'' = \sum_{e=1}^{n_{el}} \int_{Y^e} \mathbf{B}_y^T \mathbf{D}'' dY, \quad (36)$$

where $\mathbf{B}_y (= \partial \mathbf{N} / \partial \mathbf{y})$ is the discrete strain operator and n_{el} is the number of elements.

In order to solve Eq.(32), a finite element code with an unsymmetric matrix solver can be used, but, a steady-state linear dynamic analysis procedure in ABAQUS is used in the present study.

The developed analysis system, as shown in Table 1, consists of three modules. The first is a pre-processor which computes the load vector of the microscopic equilibrium equation, Eq.(32), and complete input data for ABAQUS are obtained automatically in this

Table 1 Procedure for computing effective loss tangent

Pre-processor	
• Compute load vector and obtain input data for ABAQUS ;	
$\mathbf{F}_y' = \sum_{e=1}^{n_{el}} \int_{Y^e} \mathbf{B}_y^T \mathbf{D}' dY, \quad \mathbf{F}_y'' = \sum_{e=1}^{n_{el}} \int_{Y^e} \mathbf{B}_y^T \mathbf{D}'' dY$	
Solver	
• Solve microscopic boundary value problem using ABAQUS ;	
$\begin{bmatrix} \mathbf{K}_y' & -\mathbf{K}_y'' \\ \mathbf{K}_y'' & \mathbf{K}_y' \end{bmatrix} \begin{Bmatrix} \boldsymbol{\chi}' \\ \boldsymbol{\chi}'' \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_y' \\ -\mathbf{F}_y'' \end{Bmatrix}$	
Post-processor	
• Compute homogenized complex D-matrix and complex compliance matrix ;	
$\mathbf{D}^{H'} = \frac{1}{ Y } \int_Y [\mathbf{D}' - \mathbf{D}'(\mathbf{B}_y \boldsymbol{\chi}') + \mathbf{D}''(\mathbf{B}_y \boldsymbol{\chi}'')] dY$	
$\mathbf{D}^{H''} = \frac{1}{ Y } \int_Y [\mathbf{D}'' - \mathbf{D}''(\mathbf{B}_y \boldsymbol{\chi}') - \mathbf{D}'(\mathbf{B}_y \boldsymbol{\chi}'')] dY$	
$\mathbf{C}^{H'} = \Re((\mathbf{D}^{H*})^{-1}), \quad \mathbf{C}^{H''} = \Im((\mathbf{D}^{H*})^{-1})$	
• Compute effective loss tangent ;	
$\tan \delta_{ij}^H = \mathbf{C}_{ij,ij}^{H''} / \mathbf{C}_{ij,ij}^{H'}$	

phase. The second is ABAQUS to solve Eq.(32). The last is a post-processor to compute the effective loss tangent.

4. Numerical Analysis and Discussion

In order to demonstrate the efficiency and accuracy of the proposed method, numerical analysis about two phase rubber composite is conducted, and computed effective loss tangents are compared with experimentally obtained ones.

4.1 Laminate-type

Figure 1 shows a laminate-type rubber composite. A molded rubber composite block, composed of a stack of alternate stiff sheet-rubber and soft sheet-rubber layers, was cut to composite sheets. They were finally cut with arbitrary fiber orientation to dimensions of 40 mm×5 mm×1 mm. Experiments were conducted on composite specimens with fiber orientations of 0°, 20°, 45°, 70° and 90°. The composite consists of a 45% volume fraction of stiff rubber.

The storage modulus E' and phase angle δ can be determined from the viscoelastic response of the dynamic tension test. Each measurement was carried out five times at 20°C. The strain amplitude is 2% and the static pre-strain is 5%. The frequency is 20 Hz.

Experiments were also conducted on the stiff rubber and the soft rubber, respectively. Table 2 shows the loss tangents and storage moduli of stiff rubber and soft rubber that are obtained experimentally under the same conditions. Experimental results shown in Table 2 are mean values of five measurements.

The effective loss tangent of the laminate-type

Table 2 Dynamic viscoelastic properties of soft rubber and stiff rubber (Frequency: 20 Hz, Temperature: 20°C)

	$\tan \delta$	E' (MPa)
soft rubber	0.087	2.77
stiff rubber	0.253	15.2

Fig. 1 Laminate-type rubber composite

rubber composite is computed by the developed analysis system using experimental results for the soft and stiff rubber, as given in Table 2. We assume that the long-term modulus is equivalent to the storage modulus, and Poisson's ratio is equal to 0.49. Figure 2 shows a finite element model of a unit cell with fiber orientation of 20° . The unit cell is modelled by 1600 3-node plane stress elements. Figure 3 shows the computed effective loss tangent and experimental results. Experimental results shown in Fig. 3 are mean values of five measurements. The effective loss tangent computed by the law of mixture and compliance matrix transformation proposed by Abe et al.⁽¹⁰⁾ is shown in the same figure. In this figure, the solid line represents the results obtained by the proposed homogenization method with the compliance matrix transformation.

We describe the compliance matrix transformation to compute the effective loss tangent with arbitrary fiber orientation below. We can simultaneously compute both the effective loss tangent with fiber orientation of 0° and that of 90° by using one unit cell model with fiber orientation of 0° , which we call a basic unit cell.

We consider the LT plane shown in Fig. 1.

Fig. 2 Finite element model for laminate-type rubber composite with fiber orientation of 20°

Fig. 3 Comparison between predicted effective loss tangent using homogenization and experimental results

Denoting the L axis and T axis as 1 and 2, respectively, the effective compliance of the unit cell is expressed as

$$\mathbf{C}^{H*} = \begin{bmatrix} C_{1111}^{H*} & C_{1122}^{H*} & 0 \\ C_{2211}^{H*} & C_{2222}^{H*} & 0 \\ 0 & 0 & C_{1212}^{H*} \end{bmatrix}. \quad (37)$$

Introducing the transformation matrix, $\mathbf{R}$, the effective compliance with arbitrary fiber orientation is $\overline{\mathbf{C}}^{H*} = \mathbf{R}^T \mathbf{C}^{H*} \mathbf{R}$. (38)

Namely, the transformed effective complex compliance referring to the X axis is

$$\begin{aligned} C_X^{H*} &= \overline{C}_{1111}^{H*} \\ &= C_{1111}^{H*} \cos^4 \theta + (C_{1122}^{H*} + C_{2211}^{H*}) \sin^2 \theta \cos^2 \theta \\ &\quad + C_{1212}^{H*} \sin^2 \theta \cos^2 \theta + C_{2222}^{H*} \sin^4 \theta \\ &= C_X^{H'} + i C_X^{H''}. \end{aligned} \quad (39)$$

The transformed effective loss tangent referring to the X axis is defined as the ratio of the transformed loss compliance to the transformed storage compliance.

$$\tan \delta_X^{\text{def}} = \frac{C_X^{H''}}{C_X^{H'}} \quad (40)$$

Figure 3 shows that computed effective loss tangents using a skewed unit cell with off-axis angles of 20° , 45° and 70° are in quite good agreement with the predicted effective loss tangent using Eq. (40). Therefore, the effective loss tangent with arbitrary fiber orientation can be predicted by the compliance matrix transformation based on the computed compliance of the basic unit cell with 0° fiber orientation. As shown in Fig. 3, effective loss tangents computed using the proposed method agree well with the experimental results, however, those computed by the law of mixture are lower than those obtained experimentally for angles over 50° .

4.2 Cylindrical inclusion-type

The unidirectional composite considered in this example contains cylindrical stiff rubber inclusions, whose diameter is 1.5 mm, embedded in a soft rubber matrix as shown in Fig. 4. Specimens have the following dimensions: 10 mm $\times$ 40 mm $\times$ 2 mm, and consist of a 45% volume fraction of stiff rubber. Table 3

Fig. 4 Cylindrical inclusion-type rubber composite

Table 3 Dynamic viscoelastic properties of soft rubber and stiff rubber (Frequency: 20 Hz, Temperature: 20°C)

	$\tan \delta$	E' (MPa)
soft rubber	0.026	1.93
stiff rubber	0.237	11.7

Fig. 5 Finite element model of unit cell for cylindrical inclusion-type rubber composite

shows the dynamic viscoelastic properties of both stiff and soft rubber. Measurements were carried out under the same conditions as described in the previous section.

Figure 5 shows a finite element model of a unit cell with fiber orientation of 0°. The unit cell is modelled by 780 8-node three dimensional solid elements with selective reduced integration. The computed effective loss tangent and experimental results are shown in Fig. 6. In this figure, the results obtained by the homogenization method with matrix transformation are also shown by the solid line, while the effective loss tangents computed by the law of mixture with compliance matrix transformation are shown by the broken line. As shown in this figure, numerical results using the proposed method agree well with experimental results. The results of this numerical analysis indicate that the proposed method is useful from the viewpoints of efficiency and accuracy. With increasing fiber orientation angle, the difference between the results obtained using the law of mixture and experimental results increases, because the predicted loss tangent with fiber orientation of 90° does not agree with the experimental

Fig. 6 Comparison between predicted effective loss tangent using homogenization and experimental results

result. Therefore, we conclude that the proposed method based on a homogenization method is an effective numerical tool.

5. Conclusion

In this study, we have proposed a new homogenization method in order to predict effective dynamic viscoelastic material properties in the frequency domain.

An effective loss tangent analysis system has been developed based on the general purpose finite element analysis code, ABAQUS.

The results of numerical experiments indicate that the proposed method is useful from the viewpoints of efficiency and accuracy.

Furthermore, we have shown that an effective loss tangent with arbitrary fiber orientation can be computed by the compliance matrix transformation based on the computed compliance of a basic unit cell with 0° fiber orientation.

The effective loss tangent of a practical rubber-based composite can be determined using the proposed method, so that the developed system is useful for material design processes.

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