Application of Element-Free Trefftz Method to Second-Order Design Sensitivity Analysis of Two-dimensional Elastic Problem*

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This paper presents a new scheme for the second-order shape design sensitivity analysis of the two-dimensional elastic problem by the element-free Trefftz method. In the Trefftz formulation, the physical quantities are approximated by the superposition of the regular T-complete functions. Therefore, direct differentiation of the approximate expressions with respect to shape parameters leads to the regular expressions of the sensitivities. The present schemes are applied to simple examples in order to confirm the validity.

Key Words: Trefftz Method, Element-Free Method, 2D Elastic Problem, Sensitivity Analysis, Second-Order Sensitivity Analysis

1. Introduction

The inverse problems such as the shape optimization, the free and the moving boundary problems are usually solved by applying the gradient search schemes employing the first-order derivative (sensitivity) of the objective functions with respect to the design variables. Some researchers, however, present the new gradient search schemes using, in addition to the first-order sensitivities, the second-order sensitivities⁽¹⁾⁻⁽⁹⁾. Mlejnek et al.⁽⁶⁾ point out that the schemes using both the first- and the second-order sensitivities have faster and more stable convergence property than the schemes using the first-order sensitivity alone.

The first-order sensitivity analysis schemes based on the finite and the boundary element methods can be classified into the adjoint variable and the direct differentiation methods. Therefore, in the ordinary

second-order sensitivity analysis schemes, the adjoint variable and the direct differentiation methods are applied twice to the basic equations of the finite and the boundary element methods. They, however, have some difficulties. In the finite element method, the physical quantities are approximated by the linear combination of the interpolation functions with nodal values. The interpolation functions are low-order polynomials of the shape parameters. When the quantities are differentiated with respect to the design variable for the sensitivity analysis, the order of the interpolation functions is reduced by the differentiation and therefore, the accuracy may become worse. On the other hands, in the boundary element method, the singularity of the integral equation is the great difficulty. Since the quantities are expressed by the singular integral equations, the sensitivities are given by the hyper-singular integral equations derived from the differentiation of the original integral equations. For regularization of the hyper-singular equation, we have to apply the special formulations⁽¹⁰⁾⁻⁽¹³⁾.

For overcoming the above-mentioned difficulties, we apply Trefftz method to the second-order sensitivity analysis. The Trefftz method is known as the boundary-type solution procedure employing the regular T-complete functions. Since the physical quantities are approximated by the superposition of the regular T-complete functions, the sensitivities are

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approximated by the regular expressions derived from direct differentiation of the original quantities. In this case, the special formulations, which are employed in the boundary element method, are not necessary. Since, moreover, the T-complete functions are independent on the shape parameter, the accuracy of the sensitivities is not reduced by the differentiation.

In this paper, the present scheme is firstly compared with the ordinary schemes. For the two-dimensional elastic problem, the Trefftz method is formulated by the means of the collocation method. Then, the collocation-type Trefftz method is applied to the second-order sensitivity analysis of the displacement and the stress components with respect to the design variable. Finally, the present method is applied to the simple numerical example whose theoretical results can be calculated easily.

2. Background

2.1 Ordinary studies

The ordinary first-order sensitivity analysis schemes are classified into the numerical differentiation, the adjoint variable and the direct differentiation methods⁽¹⁴⁾. In the numerical differentiation method, the algorithm is simple but the computational accuracy is dependent on the step size of the finite difference. Therefore, the other methods have been studied. In the adjoint variable method, by introducing the adjoint variable, the adjoint system for the sensitivities is derived from the initial system. On the other hands, the sensitivities are derived from direct differentiation of the physical quantities with respect to the design variable.

The second-order sensitivities are calculated by applying twice the adjoint variable and the direct differentiation methods to the basic equations of the finite and the boundary element methods. Haftka et al.^{(1),(2)} and Sobieszczanski-Sobieski⁽⁹⁾ present that the second-order sensitivities are derived from applying twice the direct differentiation method to the finite element method. Haftka et al.⁽²⁾ extend the present scheme to the higher-order sensitivity analysis. Haug et al.^{(3),(4)} apply twice to the adjoint variable method to the finite element method. Besides, Mlejnek et al.⁽⁶⁾ and Hou et al.⁽⁷⁾ compare four formulations based on the combinations of the adjoint variable and the direct differentiation methods. On the other hands, Zhang et al.⁽⁸⁾ and Bonnet⁽⁹⁾ present the second-order sensitivity analysis schemes based on direct differentiation of the boundary integral equations.

2.2 Comparison of ordinary and present schemes

The ordinary schemes based on the finite and the boundary element methods have some difficulties. In the finite element method, the physical quantities are

approximated by the linear combination of the interpolation functions. Since the interpolation functions are the low-order polynomials of the shape parameters, the order of the functions is reduced when the quantities are differentiated to calculate the sensitivities. Therefore, the relatively high-order interpolation functions should be employed for original quantities. Besides, in the finite element method, the object domain is discretized by the finite elements. When the boundary profile of the domain is modified successively, the finite element mesh may be distorted by the successive shape modification and therefore, the computational accuracy may become worse. On the other hands, the boundary element methods can overcome partially the difficulties of the finite element method. In the boundary element method, however, the singularity of the integral equation is great difficulty. Since the physical quantities are expressed by the singular integral equations, the sensitivities are given by the hyper-singular integral equations derived from the differentiation of the original equations. Therefore, the special formulation techniques are necessary for the regularization of the hyper-singular equations^{(10)–(13)}.

On the other hands, in the Trefftz method, the physical quantities are expressed by the superposition of the regular T-complete functions. Then, the unknown parameters are determined so that the approximate solutions satisfy the boundary conditions. Since the T-complete functions are independent of the shape parameter, the accuracy is not reduced by the differentiation with respect to the shape parameter. Besides, the physical quantities are approximated by the regular T-complete functions and therefore, the sensitivities are also given by the regular expressions derived from the differentiation of the original quantities. Therefore, the special techniques, which are employed in the boundary element method, are not necessary.

3. Two-dimensional Elastic Problem

3.1 Basic relationship

In the two-dimensional elastic problem without body force, the governing equation and the boundary conditions are given as follows⁽¹⁴⁾:

$$\sigma_{ij,j} = 0 \quad (\text{in } \Omega) \quad (1)$$

$$u_i = \bar{u}_i \quad (\text{on } \Gamma_u), \quad t_i = \sigma_{ij}n_j = \bar{t}_i \quad (\text{on } \Gamma_t) \quad (2)$$

where Ω , Γ_u and Γ_t denote the object under consideration, its potential-specified and flux-specified boundaries, respectively. $(\cdot)_{,j}$ is partial differentiation in x_j direction and $(\bar{\cdot})$ indicates specified values on boundary. n_j is x_j directional component of the normal vector.

At the arbitrary point inside the object domain,

the displacement component u_i and the traction component t_i are approximated by the superposition of the T-complete functions:

$$u_i = a_1 u_{i1}^* + a_2 u_{i2}^* + \dots + a_N u_{iN}^* = \mathbf{a}^T \mathbf{u}_i^* \quad (3)$$

$$t_i = a_1 t_{i1}^* + a_2 t_{i2}^* + \dots + a_N t_{iN}^* = \mathbf{a}^T \mathbf{t}_i^* = \mathbf{a}^T \boldsymbol{\sigma}_{ij}^* n_j \quad (4)$$

where $\mathbf{u}_i^*$, $\mathbf{t}_i^*$ and $\boldsymbol{\sigma}_{ij}^*$ are the T-complete functions related to the displacement, the traction and the stress components. $\mathbf{u}_i^*$ and $\boldsymbol{\sigma}_{ij}^*$ are given as⁽¹⁵⁾:

$$2G\mathbf{u}_1^* = 2G\{u_{11}^*, u_{12}^*, \dots, u_{1,4\mu-1}^*, u_{1,4\mu}^*, u_{1,4\mu+1}^*, u_{1,4\mu+2}^*, \dots\}^T \\ = \{-1, 0, \dots, r^\mu[\kappa C(\mu) - \mu C(\mu-2)], -r^\mu[\kappa S(\mu) - \mu S(\mu-2)], -r^\mu C(\mu), r^\mu S(\mu), \dots\}^T \quad (5)$$

$$2G\mathbf{u}_2^* = 2G\{u_{21}^*, u_{22}^*, \dots, u_{2,4\mu-1}^*, u_{2,4\mu}^*, u_{2,4\mu+1}^*, u_{2,4\mu+2}^*, \dots\}^T \\ = \{0, 1, \dots, r^\mu[\kappa S(\mu) + \mu S(\mu-2)], r^\mu[\kappa C(\mu) + \mu C(\mu-2)], r^\mu S(\mu), r^\mu C(\mu), \dots\}^T \quad (6)$$

$$\boldsymbol{\sigma}_{11}^* = \{\sigma_{111}^*, \sigma_{112}^*, \dots, \sigma_{1,1,4\mu-1}^*, \sigma_{1,1,4\mu}^*, \sigma_{1,1,4\mu+1}^*, \sigma_{1,1,4\mu+2}^*, \dots\}^T \\ = \{0, 0, \dots, \mu r^{\mu-1}[2C(\mu-1) - (\mu-1)C(\mu-3)], -\mu r^{\mu-1}[2S(\mu-1) - (\mu-1)S(\mu-3)], \\ -\mu r^{\mu-1}C(\mu-1), \mu r^{\mu-1}S(\mu-1), \dots\}^T \quad (7)$$

$$\boldsymbol{\sigma}_{12}^* = \{\sigma_{121}^*, \sigma_{122}^*, \dots, \sigma_{1,2,4\mu-1}^*, \sigma_{1,2,4\mu}^*, \sigma_{1,2,4\mu+1}^*, \sigma_{1,2,4\mu+2}^*, \dots\}^T \\ = \{0, 0, \dots, \mu(\mu-1)r^{\mu-1}S(\mu-3), \mu(\mu-1)r^{\mu-1}C(\mu-3), \mu r^{\mu-1}S(\mu-1), \mu r^{\mu-1}C(\mu-1), \dots\}^T \quad (8)$$

$$\boldsymbol{\sigma}_{22}^* = \{\sigma_{221}^*, \sigma_{222}^*, \dots, \sigma_{2,2,4\mu-1}^*, \sigma_{2,2,4\mu}^*, \sigma_{2,2,4\mu+1}^*, \sigma_{2,2,4\mu+2}^*, \dots\}^T \\ = \{0, 0, \dots, \mu r^{\mu-1}[2C(\mu-1) + (\mu-1)C(\mu-3)], -\mu r^{\mu-1}[2S(\mu-1) + (\mu-1)S(\mu-3)], \\ \mu r^{\mu-1}C(\mu-1), -\mu r^{\mu-1}S(\mu-1), \dots\}^T \quad (9)$$

where (r, θ) denote the plane polar coordinates. $\mu = 1, 2, 3, \dots$, $C(\mu) = \cos \mu\theta$ and $S(\mu) = \sin \mu\theta$. Besides, κ is given as:

$$\kappa = \begin{cases} 3-4\nu & (\text{for plane stress state}) \\ (3-\nu)/(1+\nu) & (\text{for plane strain state}) \end{cases} \quad (10)$$

where G and ν are the shear modulus and Poisson's ratio.

3.2 Collocation formulation

Since the T-complete functions are selected to satisfy the governing equation, Eqs.(3) and (4) satisfy the governing equation. However, they do not satisfy the boundary condition (Eq.(2)), the residuals yield:

$$\begin{aligned} R_u &= \mathbf{a}^T \mathbf{u}_i^* - \bar{u}_i \neq 0 \quad \text{on } \Gamma_u \\ R_t &= \mathbf{a}^T \mathbf{t}_i^* - \bar{t}_i \neq 0 \quad \text{on } \Gamma_t \end{aligned} \quad (11)$$

where R_u and R_t denote the residuals on Γ_u and Γ_t , respectively. In the Trefftz method, $\mathbf{a}$ is determined so that the residuals R_u and R_t are minimized simultaneously.

In the collocation-type formulation, the residuals are forced to vanish on the boundary collocation point P_m . From Eq.(11), we have

$$\begin{aligned} P_m \in \Gamma_u : R_u(P_m) &= \mathbf{u}_i^*(P_m)^T \mathbf{a} - \bar{u}_i(P_m) = 0 \\ P_m \in \Gamma_t : R_t(P_m) &= \mathbf{t}_i^*(P_m)^T \mathbf{a} - \bar{t}_i(P_m) = 0 \end{aligned} \quad (12)$$

and then,

$$\mathbf{K}\mathbf{a} = \mathbf{f} \quad (13)$$

Since the coefficient matrix of Eq.(13) is non-square matrix, Eq.(13) is solved by applying the QR-decomposition method.

4. Sensitivity Analysis

We shall consider the sensitivity analysis of the displacement and the traction components with respect to the shape parameters.

4.1 First-order sensitivity analysis

As shown in Eqs.(3) and (4), the displacement

and the traction components are approximated by the superposition of the related T-complete functions. While $\mathbf{a}$ is dependent on the boundary profile of the object under consideration and the specified values on the boundary, the T-complete functions are on the coordinates of the estimation points. Therefore, direct differentiation of Eqs.(3) and (4) with respect to the design variable φ_l leads to

$$\frac{\partial u_i}{\partial \varphi_l} = \left(\frac{\partial \mathbf{a}}{\partial \varphi_l} \right)^T \mathbf{u}_i^*, \quad \frac{\partial t_i}{\partial \varphi_l} = \left(\frac{\partial \mathbf{a}}{\partial \varphi_l} \right)^T \mathbf{t}_i^* \quad (14)$$

By differentiating Eq.(13), $\partial \mathbf{a} / \partial \varphi_l$ is calculated as follows.

$$\frac{\partial \mathbf{K}}{\partial \varphi_l} \mathbf{a} + \mathbf{K} \frac{\partial \mathbf{a}}{\partial \varphi_l} = \frac{\partial \mathbf{f}}{\partial \varphi_l} \quad (15)$$

$$\mathbf{K} \frac{\partial \mathbf{a}}{\partial \varphi_l} = \frac{\partial \mathbf{f}}{\partial \varphi_l} - \frac{\partial \mathbf{K}}{\partial \varphi_l} \mathbf{a} \quad (16)$$

If the specified values on the boundary are independent of the design variable, $\partial \mathbf{f} / \partial \varphi_l = 0$ and therefore,

$$\mathbf{K} \frac{\partial \mathbf{a}}{\partial \varphi_l} = -\frac{\partial \mathbf{K}}{\partial \varphi_l} \mathbf{a} \quad (17)$$

From Eq.(16) or (17), $\partial \mathbf{a} / \partial \varphi_l$ is calculated and then, substituted into Eq.(14).

4.2 Second-order sensitivity analysis

The second-order sensitivities can be calculated in the same manner as the first-order sensitivities. By differentiating again Eq.(14) with respect to the another design variable φ_k , we have

$$\frac{\partial^2 u_i}{\partial \varphi_k \partial \varphi_l} = \left(\frac{\partial^2 \mathbf{a}}{\partial \varphi_k \partial \varphi_l} \right)^T \mathbf{u}_i^*, \quad \frac{\partial^2 t_i}{\partial \varphi_k \partial \varphi_l} = \left(\frac{\partial^2 \mathbf{a}}{\partial \varphi_k \partial \varphi_l} \right)^T \mathbf{t}_i^* \quad (18)$$

Besides, by differentiating Eq.(16) with respect to φ_k , $\partial^2 \mathbf{a} / \partial \varphi_k \partial \varphi_l$ can be calculated as

$$\begin{aligned} \frac{\partial^2 \mathbf{K}}{\partial \varphi_k \partial \varphi_l} \mathbf{a} + \frac{\partial \mathbf{K}}{\partial \varphi_l} \frac{\partial \mathbf{a}}{\partial \varphi_k} + \frac{\partial \mathbf{K}}{\partial \varphi_k} \frac{\partial \mathbf{a}}{\partial \varphi_l} + \mathbf{K} \frac{\partial^2 \mathbf{a}}{\partial \varphi_k \partial \varphi_l} \\ = \frac{\partial^2 \mathbf{f}}{\partial \varphi_k \partial \varphi_l} \end{aligned} \quad (19)$$

and then,

$$\mathbf{K} \frac{\partial^2 \mathbf{a}}{\partial \varphi_k \partial \varphi_l} = \frac{\partial^2 \mathbf{f}}{\partial \varphi_k \partial \varphi_l} - \left(\frac{\partial^2 \mathbf{K}}{\partial \varphi_k \partial \varphi_l} \mathbf{a} + \frac{\partial \mathbf{K}}{\partial \varphi_l} \frac{\partial \mathbf{a}}{\partial \varphi_k} + \frac{\partial \mathbf{K}}{\partial \varphi_k} \frac{\partial \mathbf{a}}{\partial \varphi_l} \right) \quad (20)$$

If the specified values on the boundary are independent of the design variable, $\partial^2 \mathbf{f} / \partial \varphi_k \partial \varphi_l = 0$ and therefore,

$$\mathbf{K} \frac{\partial^2 \mathbf{a}}{\partial \varphi_k \partial \varphi_l} = - \left(\frac{\partial^2 \mathbf{K}}{\partial \varphi_k \partial \varphi_l} \mathbf{a} + \frac{\partial \mathbf{K}}{\partial \varphi_l} \frac{\partial \mathbf{a}}{\partial \varphi_k} + \frac{\partial \mathbf{K}}{\partial \varphi_k} \frac{\partial \mathbf{a}}{\partial \varphi_l} \right) \quad (21)$$

From Eq.(20) or (21), $\partial^2 \mathbf{a} / \partial \varphi_k \partial \varphi_l$ is calculated and then, substituted into Eq.(18).

The coefficient matrices $\mathbf{K}$ of Eqs.(16), (17), (20) and (21) is identical to Eq.(13). Therefore, if the matrix decomposed in the stress analysis is re-used in the sensitivity analysis, the computational cost should be reduced fairly.

4.3 Sensitivity analysis in normal direction on boundary

As the typical numerical examples of the sensitiv-

ity analysis, we consider that the coordinates of the boundary collocation points are taken as the design variables, which are moved in the normal direction on the boundary. In this case, the above-mentioned design variables φ_l and φ_k are taken as the normal vectors n_l and n_k on the boundary collocation points P_l and P_k , respectively.

4.3.1 First-order derivative of coefficient matrix The first-order derivative of the coefficient matrix is given as

$$\frac{\partial \mathbf{K}}{\partial \varphi_l} = \frac{\partial \mathbf{K}}{\partial n_l} = \begin{bmatrix} 0 & & \\ \frac{\partial K_{l1}}{\partial n_l} & \cdots & \frac{\partial K_{lN}}{\partial n_l} \\ 0 & & \end{bmatrix} \quad (l=1, \dots, 2M) \quad (22)$$

where M is the total number of the collocation points. Besides, $\partial K_{ln} / \partial n_l (n=1, \dots, N)$ is given as the normal first-order derivative of the T-complete functions related to the displacement or the traction component ;

$$\begin{aligned} 2G \frac{\partial \mathbf{u}_1^*}{\partial n_l} &= 2G \frac{\partial}{\partial n_l} \{u_{11}^*, u_{12}^*, \dots, u_{1,4\mu-1}^*, u_{1,4\mu}^*, u_{1,4\mu+1}^*, u_{1,4\mu+2}^*, \dots\}^T \\ &= \{0, 0, \dots, \mu r^{\mu-1} [\{\kappa C(\mu) - \mu C(\mu-2)\} n_r - r \{\kappa S(\mu) - (\mu-2) S(\mu-2)\} n_\theta], \\ &\quad - \mu r^{\mu-1} [\{\kappa S(\mu) - \mu S(\mu-2)\} n_r + r \{\kappa C(\mu) - (\mu-2) C(\mu-2)\} n_\theta], \\ &\quad - \mu r^{\mu-1} [C(\mu) n_r - r S(\mu) n_\theta], \mu r^{\mu-1} [S(\mu) n_r + r C(\mu) n_\theta], \dots\}^T \end{aligned} \quad (23)$$

$$\begin{aligned} 2G \frac{\partial \mathbf{u}_2^*}{\partial n_l} &= 2G \frac{\partial}{\partial n_l} \{u_{21}^*, u_{22}^*, \dots, u_{2,4\mu-1}^*, u_{2,4\mu}^*, u_{2,4\mu+1}^*, u_{2,4\mu+2}^*, \dots\}^T \\ &= \{0, 0, \dots, \mu r^{\mu-1} [\{\kappa S(\mu) - \mu S(\mu-2)\} n_r + r \{\kappa C(\mu) + (\mu-2) C(\mu-2)\} n_\theta], \\ &\quad \mu r^{\mu-1} [\{\kappa C(\mu) + \mu C(\mu-2)\} n_r - r \{\kappa S(\mu) + (\mu-2) S(\mu-2)\} n_\theta], \\ &\quad \mu r^{\mu-1} [S(\mu) n_r + r C(\mu) n_\theta], \mu r^{\mu-1} [C(\mu) n_r - r S(\mu) n_\theta], \dots\}^T \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{\partial \sigma_{11}^*}{\partial n_l} &= \frac{\partial}{\partial n_l} \{\sigma_{111}^*, \sigma_{112}^*, \dots, \sigma_{1,1,4\mu-1}^*, \sigma_{1,1,4\mu}^*, \sigma_{1,1,4\mu+1}^*, \sigma_{1,1,4\mu+2}^*, \dots\}^T \\ &= \{0, 0, \dots, \mu(\mu-1) r^{\mu-2} [2C(\mu-1) - (\mu-1)C(\mu-3)] n_r - r \{2S(\mu-1) - (\mu-3)S(\mu-3)\} n_\theta], \\ &\quad - \mu(\mu-1) r^{\mu-2} [2S(\mu-1) - (\mu-1)S(\mu-3)] n_r + r \{2C(\mu-1) - (\mu-3)C(\mu-3)\} n_\theta], \\ &\quad - \mu(\mu-1) r^{\mu-2} [C(\mu-1) n_r - r S(\mu-1) n_\theta], \mu(\mu-1) r^{\mu-2} [S(\mu-1) n_r + r C(\mu-1) n_\theta], \dots\}^T \end{aligned} \quad (25)$$

$$\begin{aligned} \frac{\partial \sigma_{12}^*}{\partial n_l} &= \frac{\partial}{\partial n_l} \{\sigma_{121}^*, \sigma_{122}^*, \dots, \sigma_{1,2,4\mu-1}^*, \sigma_{1,2,4\mu}^*, \sigma_{1,2,4\mu+1}^*, \sigma_{1,2,4\mu+2}^*, \dots\}^T \\ &= \{0, 0, \dots, \mu(\mu-1) r^{\mu-2} [(\mu-1)S(\mu-3) n_r + (\mu-3) r C(\mu-3) n_\theta], \\ &\quad \mu(\mu-1) r^{\mu-2} [(\mu-1)C(\mu-3) n_r - (\mu-3) r S(\mu-3) n_\theta], \\ &\quad \mu(\mu-1) r^{\mu-2} [S(\mu-1) n_r + r C(\mu-1) n_\theta], \mu(\mu-1) r^{\mu-2} [C(\mu-1) n_r + r S(\mu-1) n_\theta], \dots\}^T \end{aligned} \quad (26)$$

$$\begin{aligned} \frac{\partial \sigma_{22}^*}{\partial n_l} &= \frac{\partial}{\partial n_l} \{\sigma_{221}^*, \sigma_{222}^*, \dots, \sigma_{2,2,4\mu-1}^*, \sigma_{2,2,4\mu}^*, \sigma_{2,2,4\mu+1}^*, \sigma_{2,2,4\mu+2}^*, \dots\}^T \\ &= \{0, 0, \dots, \mu(\mu-1) r^{\mu-2} [2C(\mu-1) + (\mu-1)C(\mu-3)] n_r - r \{2S(\mu-1) + (\mu-3)S(\mu-3)\} n_\theta], \\ &\quad - \mu(\mu-1) r^{\mu-2} [2S(\mu-1) + (\mu-1)S(\mu-3)] n_r + r \{2C(\mu-1) + (\mu-3)C(\mu-3)\} n_\theta], \\ &\quad \mu(\mu-1) r^{\mu-2} [C(\mu-1) n_r - r S(\mu-1) n_\theta], - \mu(\mu-1) r^{\mu-2} [S(\mu-1) n_r + r C(\mu-1) n_\theta], \dots\}^T \end{aligned} \quad (27)$$

where n_r and n_θ denote the r - and θ -components of the normal vector n_l on the boundary collocation point P_l .

4.3.2 Second-order derivative of coefficient matrix The second-order derivative of the coefficient matrix $\mathbf{K}$ is calculated in the same manner as the first-order derivative ;

$$\frac{\partial^2 \mathbf{K}}{\partial \varphi_k \partial \varphi_l} = \frac{\partial^2 \mathbf{K}}{\partial n_k \partial n_l} = \frac{\partial^2 \mathbf{K}}{\partial n_l^2} = \begin{bmatrix} 0 & & \\ \frac{\partial^2 K_{l1}}{\partial n_l^2} & \cdots & \frac{\partial^2 K_{lN}}{\partial n_l^2} \\ 0 & & \end{bmatrix} \quad (l=1, \dots, 2M) \quad (28)$$

$\partial^2 K_{ln} / \partial n_l^2$ is given as the second-order derivative of the T-complete functions. The second-order derivatives of the T-complete functions are as follows.

$$\begin{aligned}
2G \frac{\partial^2 \mathbf{u}_i^*}{\partial n_i^2} &= 2G \frac{\partial^2}{\partial n_i^2} \{u_{i1}^*, u_{i2}^*, \dots, u_{i,4\mu-1}^*, u_{i,4\mu}^*, u_{i,4\mu+1}^*, u_{i,4\mu+2}^*, \dots\}^T \\
&= \{-1, 0, \dots, \mu(\mu-1)r^{\mu-2}[\chi C(\mu) - \mu C(\mu-2)]n_r^2 - 2\mu^2 r^{\mu-1}[\chi S(\mu) - (\mu-2)S(\mu-2)]n_r n_\theta \\
&\quad - \mu r^\mu [\mu \chi C(\mu) - (\mu-2)^2 C(\mu-2)]n_\theta^2, -\mu(\mu-1)r^{\mu-2}[\chi S(\mu) - \mu S(\mu-2)]n_r^2 \\
&\quad - 2\mu^2 r^{\mu-1}[\chi C(\mu) - (\mu-2)C(\mu-2)]n_r n_\theta + \mu r^\mu [\mu \chi S(\mu) - (\mu-2)^2 S(\mu-2)]n_\theta^2, \\
&\quad -\mu r^{\mu-2}[(\mu-1)C(\mu)n_r^2 - 2\mu r S(\mu)n_r n_\theta - \mu r^2 C(\mu)n_\theta^2], \\
&\quad \mu r^{\mu-2}[(\mu-1)S(\mu)n_r^2 + 2\mu r C(\mu)n_r n_\theta - \mu r^2 S(\mu)n_\theta^2], \dots\}^T
\end{aligned} \quad (29)$$

$$\begin{aligned}
2G \frac{\partial^2 \mathbf{u}_i^*}{\partial n_i^2} &= 2G \frac{\partial^2}{\partial n_i^2} \{u_{i1}^*, u_{i2}^*, \dots, u_{i,4\mu-1}^*, u_{i,4\mu}^*, u_{i,4\mu+1}^*, u_{i,4\mu+2}^*, \dots\}^T \\
&= \{0, 1, \dots, \mu(\mu-1)r^{\mu-2}[\chi S(\mu) + \mu S(\mu-2)]n_r^2 + 2\mu^2 r^{\mu-1}[\chi C(\mu) - (\mu-2)C(\mu-2)]n_r n_\theta \\
&\quad - \mu r^\mu [\mu \chi S(\mu) + (\mu-2)^2 S(\mu-2)]n_\theta^2, \mu(\mu-1)r^{\mu-2}[\chi C(\mu) + \mu C(\mu-2)]n_r^2 \\
&\quad - 2\mu^2 r^{\mu-1}[\chi S(\mu) + (\mu-2)S(\mu-2)]n_r n_\theta - \mu r^\mu [\mu \chi C(\mu) + (\mu-2)^2 C(\mu-2)]n_\theta^2, \\
&\quad \mu r^{\mu-2}[(\mu-1)S(\mu)n_r^2 + 2\mu r C(\mu)n_r n_\theta - \mu r^2 S(\mu)n_\theta^2], \\
&\quad \mu r^{\mu-2}[(\mu-1)C(\mu)n_r^2 - 2\mu r S(\mu)n_r n_\theta - \mu r^2 C(\mu)n_\theta^2], \dots\}^T
\end{aligned} \quad (30)$$

$$\begin{aligned}
\frac{\partial^2 \sigma_{i1}^*}{\partial n_i^2} &= \frac{\partial^2}{\partial n_i^2} \{\sigma_{i11}^*, \sigma_{i12}^*, \dots, \sigma_{i,1,4\mu-1}^*, \sigma_{i,1,4\mu}^*, \sigma_{i,1,4\mu+1}^*, \sigma_{i,1,4\mu+2}^*, \dots\}^T \\
&= \{0, 0, \dots, \mu(\mu-1)r^{\mu-3}[(\mu-2)\{2C(\mu-1) - (\mu-1)C(\mu-3)\}n_r^2 - 2(\mu-1)r\{2S(\mu-1) \\
&\quad - (\mu-3)S(\mu-3)\}n_r n_\theta + r^2\{2(\mu-1)C(\mu-1) - (\mu-3)^2 C(\mu-3)\}n_\theta^2], \\
&\quad -\mu(\mu-1)r^{\mu-3}[(\mu-2)\{2S(\mu-1) - (\mu-1)S(\mu-3)\}n_r^2 - 2(\mu-1)r\{2C(\mu-1) \\
&\quad - (\mu-3)C(\mu-3)\}n_r n_\theta + r^2\{2(\mu-1)S(\mu-1) - (\mu-3)^2 S(\mu-3)\}n_\theta^2], \\
&\quad -\mu(\mu-1)r^{\mu-3}[(\mu-2)C(\mu-1)n_r^2 - 2(\mu-1)r \times r S(\mu-1)n_r n_\theta - (\mu-1)r^2 C(\mu-1)n_\theta^2], \\
&\quad \mu(\mu-1)r^{\mu-3}[(\mu-2)S(\mu-1)n_r^2 + 2(\mu-1)r \times r C(\mu-1)n_r n_\theta - (\mu-1)r^2 S(\mu-1)n_\theta^2], \dots\}^T
\end{aligned} \quad (31)$$

$$\begin{aligned}
\frac{\partial^2 \sigma_{i2}^*}{\partial n_i^2} &= \frac{\partial^2}{\partial n_i^2} \{\sigma_{i21}^*, \sigma_{i22}^*, \dots, \sigma_{i,2,4\mu-1}^*, \sigma_{i,2,4\mu}^*, \sigma_{i,2,4\mu+1}^*, \sigma_{i,2,4\mu+2}^*, \dots\}^T \\
&= \{0, 0, \dots, \mu(\mu-1)r^{\mu-3}[(\mu-1)(\mu-2)S(\mu-3)n_r^2 + 2(\mu-1)(\mu-3)rC(\mu-3)n_r n_\theta \\
&\quad - (\mu-3)^2 r^2 S(\mu-3)n_\theta^2], \mu(\mu-1)r^{\mu-3}[(\mu-1)(\mu-2)C(\mu-3)n_r^2 \\
&\quad - 2(\mu-1)(\mu-3)rS(\mu-3)n_r n_\theta - (\mu-3)^2 r^2 C(\mu-3)n_\theta^2], \\
&\quad \mu(\mu-1)r^{\mu-3}[(\mu-2)S(\mu-1)n_r^2 + 2(\mu-1)r \times r C(\mu-1)n_r n_\theta - (\mu-1)r^2 S(\mu-1)n_\theta^2], \\
&\quad \mu(\mu-1)r^{\mu-3}[(\mu-2)C(\mu-1)n_r^2 - 2(\mu-1)r \times r S(\mu-1)n_r n_\theta - (\mu-1)r^2 C(\mu-1)n_\theta^2], \dots\}^T
\end{aligned} \quad (32)$$

$$\begin{aligned}
\frac{\partial^2 \sigma_{i3}^*}{\partial n_i^2} &= \frac{\partial^2}{\partial n_i^2} \{\sigma_{i31}^*, \sigma_{i32}^*, \dots, \sigma_{i,3,4\mu-1}^*, \sigma_{i,3,4\mu}^*, \sigma_{i,3,4\mu+1}^*, \sigma_{i,3,4\mu+2}^*, \dots\}^T \\
&= \{0, 0, \dots, \mu(\mu-1)r^{\mu-3}[(\mu-2)\{2C(\mu-1) + (\mu-1)C(\mu-3)\}n_r^2 - 2(\mu-1)r\{2S(\mu-1) \\
&\quad + (\mu-3)S(\mu-3)\}n_r n_\theta - r^2\{2(\mu-1)C(\mu-1) + (\mu-3)^2 C(\mu-3)\}n_\theta^2], \\
&\quad -\mu(\mu-1)r^{\mu-3}[(\mu-2)\{2S(\mu-1) + (\mu-1)S(\mu-3)\}n_r^2 - 2(\mu-1)r\{2C(\mu-1) \\
&\quad + (\mu-3)C(\mu-3)\}n_r n_\theta + r^2\{2(\mu-1)S(\mu-1) + (\mu-3)^2 S(\mu-3)\}n_\theta^2], \\
&\quad \mu(\mu-1)r^{\mu-3}[(\mu-2)C(\mu-1)n_r^2 - 2(\mu-1)r \times r S(\mu-1)n_r n_\theta - (\mu-1)r^2 C(\mu-1)n_\theta^2], \\
&\quad -\mu(\mu-1)r^{\mu-3}[(\mu-2)S(\mu-1)n_r^2 + 2(\mu-1)r \times r C(\mu-1)n_r n_\theta - (\mu-1)r^2 S(\mu-1)n_\theta^2], \dots\}^T
\end{aligned} \quad (33)$$

5. Numerical Examples

5.1 Object under consideration

A circular cylinder under internal pressure is considered as a numerical example (Fig. 1). A quarter of the cross-section is solved as a plane strain state. Physical parameters are taken as; Young's modulus $E=1(Pa)$ and Poisson's ratio $\nu=0.3$. The boundary conditions are as follows; $t_r=p_1=1$ and $t_\theta=0$ on the internal wall, $t_r=t_\theta=0$ on the external wall and $t_r=0$ and $u_\theta=0$ on the symmetric axes, respectively. u_r , u_θ , t_r and u_θ denote r - and θ -components of the displacement and the traction on the boundary, respectively.

We use two placements of the collocation points. In one case, 34 points are taken on the boundary; 11 points on internal and external walls and 6 points on the symmetric axes. In another case, 64 points are taken on the boundary; 21 points on internal and

external walls and 11 points on the symmetric axes. Numerical analysis is carried out by Hewlett

Fig. 1 Circular cylinder under internal pressure

Fig. 2 Error convergence $\partial^2 u_r / \partial^2 r_1$ on internal wall Fig. 3 Error convergence $\partial^2 u_r / \partial^2 r_1$ on external wall

Packard Workstation HP9000 Model 712 and QR-decomposition method of LAPACK software⁽¹⁶⁾. Besides, the computational error of the physical quantity q at arbitrary point is estimated by

$$e_q = \frac{\|q - q^{ex}\|^{1/2}}{|q^{ex}|} \quad (34)$$

where $\|\cdot\|$ means square norm and the superscript ex theoretical one. Besides, the total error on the part of the boundary is estimated by

$$TE = \frac{1}{M|q^{ex}|} \sum_k [\|q - q^{ex}\|^{1/2}] \quad (35)$$

where L is the number of the collocation points taken on the boundary.

5.2 Second-order sensitivity analysis

We shall consider the second-order sensitivity analysis with respect to the internal radius r_1 . The convergence property of the second-order sensitivity is shown in Figs. 2 and 3.

In the case of 34 collocation points, the error decreases with the increase of the number of the T-complete functions and then, almost zero at 7 T-complete functions. In 64 collocation points, the error decreases quicker than that in 34 collocation points and then, almost zero at 13 T-complete functions. However, it turns to increase at 20 T-complete func-

Fig. 4 Condition number of coefficient matrix

Fig. 5 Distribution of $\partial^2 u_r / \partial^2 r_1$

tions. This is because the coefficient matrix becomes ill-posed. Figure 4 shows the condition number of the coefficient matrix. The gradient of the condition number slightly decreases at 20 function, which indicates the ill-posedness of the coefficient matrix. In the second-order analysis, the ill-posedness of the coefficient matrix strongly affects the computational accuracy. Therefore, we have to apply new additional schemes for solving the ill-posed equations. Figure 5 shows the distribution of the second-order sensitivities of the r -directional displacement u_r with respect to the internal radius r_1 . 14 T-complete functions are employed for the computation. Numerical results well agree with theoretical one.

6. Conclusion

This paper presented the second-order sensitivity analysis scheme of the two-dimensional elastic problem by using the Trefftz-type solution procedure. Since the Trefftz method is known as a boundary-type solution procedure, the input data generation is easier than the domain-type solution procedure such as the finite element method. Besides, the displacement and the traction components are approximated by the superposition of the regular T-complete func-

tions. Therefore, direct differentiation of the approximate expressions of the quantities leads to the regular expressions of the sensitivities.

The present methods were applied to the two-dimensional elastic problem in order to confirm the validity. The numerical results well agreed with the theoretical one. Therefore, we can conclude that the validity of the present method is confirmed. Now, we plan the application of the present method to the actual optimization problems.

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