Scattering of Love Waves by a Surface-Breaking Crack in Piezoelectric Layered Media*

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Based on the dynamic theory of linear piezoelectricity, we investigate the scattering of Love waves by a surface-breaking crack in a layered piezoelectric solid. The piezoelectric layer is perfectly bonded to a half-space of a different elastic solid and the crack is normal to the free surface. Both cases of a partially broken layer and a completely broken layer are studied. By the use of Fourier transform techniques, the mixed boundary value problem is reduced to a singular integral equation. The singular integral equation is solved numerically to determine the dynamic stress intensity factor and the dynamic energy release rate as a function of a dimensionless frequency for various ratios of crack depth to layer thickness, and the results are presented in graphical form.

Key Words: Elasticity, Fracture Mechanics, Love Wave, Piezoelectric Material, Layered Material, Dynamic Stress Intensity Factor, $\hat{J}$ -integral, Dynamic Energy Release Rate, Stress Singularities, Integral Transforms

1. Introduction

Numerous investigators have recently demonstrated the feasibility of active structural control using active materials. The use of lead zirconate titanate (PZT) ceramics as actuators and sensors for beam and plate vibration has been studied⁽¹⁾. The strength of the piezoelectric materials is weakened by the presence of defects such as voids and cracks. The analyses of cracked piezoelectric materials have thus been performed by several researchers, due to the brittle nature of piezoelectric materials. Recently, Yu and Qin⁽²⁾ carried out a two-dimensional analysis of thermal and electric fields of a thermopiezoelectric solid damaged by cracks. Yu and Qin⁽³⁾ also studied thermoelectroelastic fields in piezoelectric materials weakened by microcracks through use of a generalized self-consistent approximate method. Shang et al.⁽⁴⁾ analyzed a thermopiezoelectric problem of a

penny-shaped crack whose surface is uniformly covered with a constant temperature different from that of the surrounding material. Gao et al.⁽⁵⁾ developed a strip saturation model for a finite crack of an infinite poled piezoelectric ceramics medium with electrical polarization reaching a saturation limit along a line segment in front of the crack. While important theoretical work has been carried out to study the mechanical lifetime of piezoceramics containing cracks, a lack of adequate analytical models has left many unexplained issues. Most published static analyses of piezoelectric crack problems adopt the assumption that the crack contains a dielectrically impermeable medium. However, the approximation has a drawback: when the defect is a slit crack the electric field becomes singular at its tip⁽⁶⁾. Shindo et al.⁽⁷⁾ discarded the impermeability approximation and calculated the stress intensity factor and the energy release rate for an anti-plane shear crack embedded in a piezoelectric ceramic strip which was subjected to both mechanical and electrical loads. Shindo and Narita⁽⁸⁾ also analyzed the dynamic electroelastic problem for a piezoelectric ceramic having a finite crack theoretically. The study presented here is different from these

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in that we study laminated structures containing a piezoelectric layered ceramic with a crack. Many piezoelectric devices are constructed of both piezoelectric and structural layers, and understanding of fracture process of piezoelectric structural systems is of great importance in order to secure the structural integrity of piezoelectric devices⁽⁹⁾.

Following the dynamic theory of linear piezoelectricity, we consider the problem of a piezoelectric layer bonded to an elastic half-plane where the layer contains a surface-breaking crack normal to its free surface, and investigate the interaction of Love waves with a surface-breaking crack. Both the partially broken layer and the completely broken layer are studied. Integral transform techniques are used to formulate the problem as a singular integral equation⁽¹⁰⁾. Numerical results for the dynamic stress intensity factor and the dynamic energy release rate corresponding to first mode are computed, and they are plotted in terms of the piezoelectric material constants, frequency, and geometrical parameters.

2. Problem Formulation

Consider a piezoelectric layer which is perfectly bonded to a half-space of a different elastic material, as shown in Fig. 1. The origin of a Cartesian coordinate system (x, y, z) is located at the mouth of the crack. The piezoelectric layer poled in the z -direction exhibiting transversely isotropic behavior (hexagonal symmetry) has thickness h and contains a surface-

Fig. 1 A piezoelectric layered medium with a surface-breaking crack and Love waves

breaking crack of depth c normal to the free surface. The crack lies in the plane $y=0$ and extends to infinity in the $\pm z$ directions. Time-harmonic antiplane shear waves (Love waves) are incident on the crack. The elastic stiffness constant measured in a constant electric field and the mass density of the piezoelectric layer are denoted by c_{44} and ρ , while those for the elastic half-space are c_{44}^E and ρ^E , respectively. The quantities designated by a superscript E refer to the elastic half-space.

The piezoelectric boundary value problem simplifies considerably if we consider only the out-of-plane displacement and the in-plane electric fields such that

$$u_x = u_y = 0, \quad u_z = u_z(x, y, t) \quad (1)$$

$$E_x = E_x(x, y, t), \quad E_y = E_y(x, y, t), \quad E_z = 0 \quad (2)$$

$$u_x^E = u_y^E = 0, \quad u_z^E = u_z^E(x, y, t) \quad (3)$$

where (u_x, u_y, u_z) and (E_x, E_y, E_z) are the components of displacement and electric field vectors. The constitutive equations can be written as

$$\left. \begin{aligned} \sigma_{xz} &= c_{44} u_{z,x} - e_{15} E_x \\ \sigma_{yz} &= c_{44} u_{z,y} - e_{15} E_y \end{aligned} \right\} \quad (4)$$

$$\left. \begin{aligned} D_x &= e_{15} u_{z,x} + \epsilon_{11} E_x \\ D_y &= e_{15} u_{z,y} + \epsilon_{11} E_y \end{aligned} \right\} \quad (5)$$

$$\left. \begin{aligned} \sigma_{xz}^E &= c_{44}^E u_{z,x}^E \\ \sigma_{yz}^E &= c_{44}^E u_{z,y}^E \end{aligned} \right\} \quad (6)$$

where σ_{xz} , σ_{yz} and D_x , D_y are the components of the stress tensor and electric displacement vector, ϵ_{11} is the dielectric constant measured at constant strain, e_{15} is the piezoelectric constant, and the subscript comma denotes a partial derivative with respect to the coordinates or the time t . The electric field components are related to the electric potential ϕ by the equation

$$\left. \begin{aligned} E_x &= -\phi_{,x} \\ E_y &= -\phi_{,y} \end{aligned} \right\} \quad (7)$$

The governing equations have the form

$$c_{44} \nabla^2 u_z + e_{15} \nabla^2 \phi = \rho \frac{\partial^2 u_z}{\partial t^2} \quad (0 \leq x \leq h) \quad (8)$$

$$e_{15} \nabla^2 u_z - \epsilon_{11} \nabla^2 \phi = 0 \quad (0 \leq x \leq h) \quad (9)$$

$$c_{44}^E \nabla^2 u_z^E = \rho^E \frac{\partial^2 u_z^E}{\partial t^2} \quad (h \leq x < \infty) \quad (10)$$

where $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ is the two-dimensional Laplace operator in the variables x and y . In a vacuum, the constitutive equations (5) and the governing equation (9) become

$$\left. \begin{aligned} D_x &= \epsilon_0 E_x \\ D_y &= \epsilon_0 E_y \end{aligned} \right\} \quad (11)$$

$$\nabla^2 \phi = 0 \quad (12)$$

where ϵ_0 is the electric permittivity of vacuum.

If it is assumed that the piezoelectric layered composite is subjected to a periodic displacement disturbance arriving from an infinitely large distance

and propagating in the y direction, then the appropriate solutions of the governing equations can be expressed as

$$u_z^i = \frac{\epsilon_{11}}{e_{15}} \{a_2 \cosh(\lambda kx) + b_2 \sinh(\lambda kx)\} \exp\{-i(ky + \omega t)\} \quad (13)$$

$$\phi^i = \{a_1 \cosh(kx) + b_1 \sinh(kx) + a_2 \cosh(\lambda kx) + b_2 \sinh(\lambda kx)\} \exp\{-i(ky + \omega t)\} \quad (14)$$

$$u_z^{Ei} = a_3 \exp(-\lambda^E kx) \exp\{-i(ky + \omega t)\} \quad (15)$$

where $a_i (i=1,2,3)$, $b_i (i=1,2)$ are the constants, k is the wavenumber, ω is the circular frequency, and a superscript i stands for the incident component of the field quantity. The nondimensional parameters λ and λ^E are given by

$$\lambda^2 = 1 - \left(\frac{\omega}{kC_{SH}}\right)^2, (\lambda^E)^2 = 1 - \left(\frac{\omega}{kC_{SH}^E}\right)^2 \quad (16)$$

where

$$C_{SH} = \left(\frac{\bar{c}_{44}}{\rho}\right)^{1/2}, C_{SH}^E = \left(\frac{C_{44}^E}{\rho^E}\right)^{1/2} \quad (17)$$

are the shear wave velocities and $\bar{c}_{44} = c_{44} + e_{15}^2/\epsilon_{11}$ is the piezoelectrically stiffened elastic constant.

The boundary conditions can be written as

$$\sigma_{xz}^i(0, y) = 0 \quad (18)$$

$$D_x^i(0, y) = 0 \quad (19)$$

$$\sigma_{xz}^i(h, y) = \sigma_{xz}^{Ei}(h, y) \quad (20)$$

$$u_z^i(h, y) = u_z^{Ei}(h, y) \quad (21)$$

$$D_x^i(h, y) = 0 \quad (22)$$

The boundary conditions (19), (22) were discussed in Shindo et al.⁽⁷⁾. The conditions (18), (19) together with Eqs. (4), (5) and (7) yield

$$\frac{e_{15}^2}{\epsilon_{11}} b_1 + \bar{c}_{44} \lambda b_2 = 0 \quad (23)$$

$$b_1 = 0 \quad (24)$$

Equations (20)–(22) together with Eqs. (4), (5) and (7) lead to the following relations:

$$\begin{aligned} \frac{e_{15}^2}{\epsilon_{11}} \sinh(kh) a_1 + \frac{e_{15}^2}{\epsilon_{11}} \cosh(kh) b_1 + \bar{c}_{44} \lambda \sinh(\lambda kh) a_2 \\ + \bar{c}_{44} \lambda \cosh(\lambda kh) b_2 \\ + \frac{C_{44}^E e_{15}}{\epsilon_{11}} \lambda^E \exp(-\lambda^E kh) a_3 = 0 \end{aligned} \quad (25)$$

$$\begin{aligned} \frac{\epsilon_{11}}{e_{15}} \cosh(\lambda kh) a_2 + \frac{\epsilon_{11}}{e_{15}} \sinh(\lambda kh) b_2 \\ - \exp(-\lambda^E kh) a_3 = 0 \end{aligned} \quad (26)$$

$$\sinh(kh) a_1 + \cosh(kh) b_1 = 0 \quad (27)$$

By combining these five results we find the following equation for the phase velocity ω/k :

$$\frac{\bar{c}_{44} \epsilon_{11}}{e_{15}^2} \lambda \sinh(\lambda kh) + \frac{C_{44}^E \epsilon_{11}}{e_{15}^2} \lambda^E \cosh(\lambda kh) = 0 \quad (28)$$

It is noted that Eq. (28) is the well-known frequency equation for Love waves. The displacement u_z^i and electric potential ϕ^i generated by incident Love waves can be expressed in terms of $a_i (i=1, 2)$ and are given by

$$u_z^i = \frac{\epsilon_{11}}{e_{15}} a_2 \cosh(\lambda kx) \exp\{-i(ky + \omega t)\} \quad (29)$$

$$\phi^i = \{a_1 \cosh(kx) + a_2 \cosh(\lambda kx)\} \exp\{-i(ky + \omega t)\} \quad (30)$$

The shear stress σ_{yz}^i corresponding to u_z^i and ϕ^i may be computed from Eqs. (4) and (7), and this is

$$\begin{aligned} \sigma_{yz}^i = i p_0 \frac{C_{SH}}{\omega/k} \left[\left\{ \lambda \frac{\sinh(\lambda kh)}{\sinh(kh)} + \frac{C_{44}^E}{\bar{c}_{44}} \lambda^E \frac{\cosh(\lambda kh)}{\sinh(kh)} \right\} \right. \\ \left. \times \cosh(kx) - \cosh(\lambda kx) \right] \exp\{-i(ky + \omega t)\} \end{aligned} \quad (31)$$

where

$$p_0 = \omega \rho C_{SH} \frac{\epsilon_{11}}{e_{15}} a_2 \quad (32)$$

The time factor $\exp(-i\omega t)$, which is common to all field variables in a steady-state regime, is omitted throughout this paper. Observe now that the scattered field is physically antisymmetric with respect to the plane $y=0$. Hence, we need only to consider the quarter-space region defined by $x \geq 0, y \geq 0$.

The boundary conditions for the scattered field become

$$\begin{cases} \sigma_{yz}^s(x, 0) = -\sigma_{yz}^i(x, 0) & (0 \leq x < c) \\ u_z^s(x, 0) = 0 & (c \leq x \leq h) \end{cases} \quad (33)$$

$$\begin{cases} E_x^s(x, 0) = E_x^{cs}(x, 0) & (0 \leq x < c) \\ \phi^s(x, 0) = 0 & (c \leq x \leq h) \end{cases} \quad (34)$$

$$D_y^s(x, 0) = D_y^{cs}(x, 0) \quad (0 \leq x < c) \quad (35)$$

$$\sigma_{xz}^s(0, y) = 0 \quad (36)$$

$$D_x^s(0, y) = 0 \quad (37)$$

$$\sigma_{xz}^s(h, y) = \sigma_{xz}^{Es}(h, y) \quad (38)$$

$$u_z^s(h, y) = u_z^{Es}(h, y) \quad (39)$$

$$D_x^s(h, y) = 0 \quad (40)$$

where superscripts s and cs stand for the scattered components of the field quantity and the electric field quantity in the void inside the crack, respectively.

3. Solution Procedure

The Fourier transform is applied to Eqs. (8)–(10) and the results are

$$\begin{aligned} u_z^s(x, y) = \frac{2}{\pi} \int_0^\infty A_1(a) e^{-\gamma y} \cos(ax) da \\ + \frac{2}{\pi} \int_0^\infty \{A_2(a) \cosh(\gamma x) + A_3(a) \sinh(\gamma x)\} \\ \times \sin(\alpha y) da \end{aligned} \quad (41)$$

$$\begin{aligned} \phi^s(x, y) = \frac{2}{\pi} \int_0^\infty \left\{ \frac{e_{15}}{\epsilon_{11}} A_1(a) e^{-\gamma y} + B_1(a) e^{-\alpha y} \right\} \\ \times \cos(ax) da \\ + \frac{2}{\pi} \int_0^\infty \left\{ \frac{e_{15}}{\epsilon_{11}} A_2(a) \cosh(\gamma x) + B_2(a) \cosh(\alpha x) \right. \\ \left. + \frac{e_{15}}{\epsilon_{11}} A_3(a) \sinh(\gamma x) + B_3(a) \sinh(\alpha x) \right\} \\ \times \sin(\alpha y) da \end{aligned} \quad (42)$$

$$u_z^{Es}(x, y) = \frac{2}{\pi} \int_0^\infty C_1(a) e^{-\gamma^E x} \sin(\alpha y) da \quad (43)$$

where $A_i(a), B_i(a) (i=1,2,3), C_1(a)$ are the unknowns to be solved and $\gamma(a), \gamma^E(a)$ are

$$\gamma(\alpha) = \left\{ \alpha^2 - \left(\frac{\omega}{c_{SH}} \right)^2 \right\}^{1/2}, \quad \gamma^E(\alpha) = \left\{ \alpha^2 - \left(\frac{\omega}{c_{SH}^E} \right)^2 \right\}^{1/2} \quad (44)$$

The functions $\gamma(\alpha)$ and $\gamma^E(\alpha)$ should be restricted as

$$\begin{aligned} \operatorname{Re} \gamma(\alpha) > 0, \operatorname{Im} \gamma(\alpha) < 0 \\ \operatorname{Re} \gamma^E(\alpha) > 0, \operatorname{Im} \gamma^E(\alpha) < 0 \end{aligned} \quad (45)$$

in the upper half space $y \geq 0$, because of a radiation condition at infinity and an edge condition near the crack tip. Application of the Fourier transform on Eq. (12) yields

$$\phi^{cs} = \frac{2}{\pi} \int_0^\infty D(\alpha) \sinh(\alpha y) \cos(\alpha x) d\alpha \quad (0 \leq x < c) \quad (46)$$

where $D(\alpha)$ is also unknown. The stresses, electric field intensities and electric displacements can be obtained by making use of Eqs. (4)–(7) and (41)–(43). The electric field intensities and electric displacements in the void inside the crack can also be obtained from Eqs. (7), (11) and (46).

The boundary condition of Eq. (34) leads to the following relation between unknown functions:

$$\frac{e_{15}}{\varepsilon_{11}} A_1(\alpha) + B_1(\alpha) = 0 \quad (47)$$

Application of the mixed boundary conditions (33) gives rise to a pair of dual integral equations:

$$\left\{ \begin{aligned} & \frac{2}{\pi} \int_0^\infty \left(\bar{c}_{44} \gamma - \frac{e_{15}^2}{\varepsilon_{11}} \alpha \right) A_1(\alpha) \cos(\alpha x) d\alpha \\ & - \frac{2}{\pi} \int_0^\infty \bar{c}_{44} \alpha \{ A_2(\alpha) \cosh(\gamma x) \\ & + A_3(\alpha) \sinh(\gamma x) \} d\alpha \\ & - \frac{2}{\pi} \int_0^\infty e_{15} \alpha \{ B_2(\alpha) \cosh(\alpha x) \\ & + B_3(\alpha) \sinh(\alpha x) \} d\alpha \\ & = i p_0 \frac{c_{SH}}{\omega/k} \left[\left\{ \lambda \frac{\sinh(\lambda k h)}{\sinh(k h)} + \frac{c_{44}^E}{\bar{c}_{44}} \lambda^E \frac{\cosh(\lambda k h)}{\sinh(k h)} \right\} \right. \\ & \quad \times \cosh(k x) - \cosh(\lambda k x) \left. \right] \quad (0 \leq x < c) \\ & \int_0^\infty A_1(\alpha) \cos(\alpha x) d\alpha = 0 \quad (c \leq x \leq h) \end{aligned} \right. \quad (48)$$

By defining $A_1(\alpha)$ in the following manner:

$$A_1(\alpha) = \frac{1}{\alpha} \int_0^c b(t) \sin(\alpha t) dt \quad (49)$$

where $b(t)$ represents a dislocation density for the crack sliding displacement⁽¹¹⁾. The application of the Fourier transforms to the conditions, Eqs. (36)–(40), leads to the following definition for $A_i(\alpha)$, $B_i(\alpha)$ ($i=2, 3$) and $C_1(\alpha)$:

$$A_2(\alpha) = \int_0^c \frac{L(\alpha)}{\Delta(\alpha)} b(t) dt \quad (50)$$

$$A_3(\alpha) = 0 \quad (51)$$

$$B_2(\alpha) = -\frac{e_{15}}{\varepsilon_{11}} \frac{\exp(-\alpha h)}{\alpha \sinh(\alpha h)} \int_0^c b(t) \sinh(\alpha t) dt \quad (52)$$

$$B_3(\alpha) = 0 \quad (53)$$

$$\begin{aligned} C_1(\alpha) = & \left[\left\{ \frac{\bar{c}_{44} \varepsilon_{11}}{e_{15}^2} - \frac{c_{44}^E \varepsilon_{11}}{e_{15}^2} \frac{\gamma^E}{\gamma} \right\} \frac{\cosh(\gamma h)}{\Delta(\alpha)} + \frac{1}{\gamma} \right] \\ & \times \frac{\alpha}{\gamma} \exp(\gamma^E h - \gamma h) \\ & \times \int_0^c b(t) \sinh(\gamma t) dt \end{aligned} \quad (54)$$

where

$$L(\alpha) = \left(\frac{\bar{c}_{44} \varepsilon_{11}}{e_{15}^2} - \frac{c_{44}^E \varepsilon_{11}}{e_{15}^2} \frac{\gamma^E}{\gamma} \right) \frac{\alpha}{\gamma} \exp(-\gamma h) \sinh(\gamma t) \quad (55)$$

$$\Delta(\alpha) = \frac{\bar{c}_{44} \varepsilon_{11}}{e_{15}^2} \gamma \sinh(\gamma h) + \frac{c_{44}^E \varepsilon_{11}}{e_{15}^2} \gamma^E \cosh(\gamma h) \quad (56)$$

The dual integral equations (48), together with Eqs. (49)–(53), lead to the following singular integral equation:

$$\begin{aligned} & \frac{1}{\pi} \int_0^c \frac{b(t)}{t-x} dt + \frac{1}{\pi} \int_0^c \left\{ \frac{1}{t+x} + 2K(t, x) \right\} b(t) dt \\ & = i \frac{1}{c_{44}} p_0 \frac{c_{SH}}{\omega/k} \left[\left\{ \lambda \frac{\sinh(\lambda k h)}{\sinh(k h)} + \frac{c_{44}^E}{\bar{c}_{44}} \lambda^E \frac{\cosh(\lambda k h)}{\sinh(k h)} \right\} \right. \\ & \quad \times \cosh(k x) - \cosh(\lambda k x) \left. \right] \end{aligned} \quad (57)$$

where the kernel $K(t, x)$ is given by

$$\begin{aligned} K(t, x) = & \int_0^\infty \frac{\bar{c}_{44}}{c_{44}} \left(\frac{\gamma}{\alpha} - 1 \right) \sin(\alpha t) \cos(\alpha x) d\alpha \\ & - \int_0^\infty \frac{\bar{c}_{44}}{c_{44}} \frac{\alpha L(\alpha)}{\Delta(\alpha)} \cosh(\gamma x) d\alpha + \int_0^\infty \frac{e_{15}^2}{c_{44} \varepsilon_{11}} \\ & \times \frac{\exp(-\alpha h)}{\sinh(\alpha h)} \sinh(\alpha t) \cosh(\alpha x) d\alpha + F_M \end{aligned} \quad (58)$$

The singularities in the integrands of the integrals that define the kernel function $K(t, x)$ in Eq. (58) are poles that occur in the complex α -plane at the zeros of $\Delta(\alpha)$. The technique merely deforms the contour of integration below the real α -axis so that no poles occur on the path of integration. The contour made up of C_m ($m=1, \dots, M$) is shown in Fig. 2. In Eq. (58) the terms F_M are given as

$$F_M = -i\pi \sum_{m=1}^M \frac{\bar{c}_{44}}{c_{44}} \frac{\alpha_m L(\alpha_m)}{\Delta'(\alpha_m)} \cosh(\gamma_m x) \quad (59)$$

where M is the number of roots from the equation $\Delta(\alpha)=0$, α_m are roots of $\Delta(\alpha)=0$, and $\Delta'(\alpha_m)$ is $d\Delta(\alpha)/d\alpha$ evaluated at $\alpha=\alpha_m$. These last terms represent the addition of a free wave solution that will ensure that the scattered wave solution does not contain standing waves⁽¹⁰⁾. Note that $K(t, x)$ will also have principal value integrals.

The numerical analysis is concerned with the

Fig. 2 Integration contour for $K(t, x)$ in the complex plane

solution of Eq.(57). It is convenient to separate the analysis into two cases: (A) unbroken layer ($h > c$) and (B) broken layer ($h = c$). When the crack tips have not yet reached the interfaces ($h > c$), we introduce dimensionless variables as follows:

$$\left. \begin{aligned} t_0 &= \frac{t}{c}, x_0 = \frac{x}{c}, \alpha = \frac{\omega}{c_{SH}} \eta, \frac{\omega}{(c_{44}/\rho)^{1/2}} = \frac{\Omega}{c} \\ \frac{c_{SH}}{c_{SH}^E} &= \delta, \phi(t_0) = \phi(t/c) = c_{44} b(t) \end{aligned} \right\} \quad (60)$$

The integral equation (57) then becomes the following singular integral equation:

$$\begin{aligned} & \frac{1}{\pi} \int_0^1 \frac{\phi(t_0)}{t_0 - x_0} dt_0 + \frac{1}{\pi} \int_0^1 \left[\frac{1}{t_0 + x_0} + 2\{K_0^\delta(t_0, x_0) \right. \\ & \quad \left. + K_\delta^1(t_0, x_0) + K_I^\infty(t_0, x_0)\} \right] \phi(t_0) dt_0 \\ & = i p_0 \frac{c_{SH}}{\omega/k} \left[\left\{ \lambda \frac{\sinh(\lambda k h)}{\sinh(k h)} + \frac{c_{44}^E}{\bar{c}_{44}} \lambda^E \frac{\cosh(\lambda k h)}{\sinh(k h)} \right\} \right. \\ & \quad \left. \times \cosh(k h x_0 c/h) - \cosh(\lambda k h x_0 c/h) \right] \end{aligned} \quad (61)$$

In Eq.(61) the kernels $K_0^\delta(t_0, x_0)$, $K_\delta^1(t_0, x_0)$ and $K_I^\infty(t_0, x_0)$ are given in Appendix A. By the introduction of the nondimensional variables

$$\tau = 2t_0 - 1, s = 2x_0 - 1 \quad (62)$$

the integral equation (61) may be written as

$$\begin{aligned} & \frac{1}{\pi} \int_{-1}^1 \frac{\phi^*(\tau)}{\tau - s} d\tau + \frac{1}{\pi} \int_{-1}^1 \left\{ \frac{1}{\tau + s + 2} + K_0^\delta(\tau^*, s^*) \right. \\ & \quad \left. + K_\delta^1(\tau^*, s^*) + K_I^\infty(\tau^*, s^*) \right\} \phi^*(\tau) d\tau \\ & = i p_0 \frac{c_{SH}}{\omega/k} \left[\left\{ \lambda \frac{\sinh(\lambda k h)}{\sinh(k h)} + \frac{c_{44}^E}{\bar{c}_{44}} \lambda^E \frac{\cosh(\lambda k h)}{\sinh(k h)} \right\} \right. \\ & \quad \left. \times \cosh(k h s^* c/h) - \cosh(\lambda k h s^* c/h) \right] \end{aligned} \quad (63)$$

where

$$\phi^*(\tau) = \phi\left(\frac{\tau+1}{2}\right), \tau^* = \frac{\tau+1}{2}, s^* = \frac{s+1}{2} \quad (64)$$

It is expedient to introduce a function $\Psi(\cdot)$ by

$$\phi^*(\tau) = \frac{\Psi(\tau)}{(1-\tau^2)^{1/2}} \quad (65)$$

Now, using the Gauss-Chebyshev formula⁽¹²⁾, the following system of $N-1$ equations for N unknown values of $\Psi(\tau)$ follows:

$$\begin{aligned} & \sum_{k=1}^N \frac{1}{N} \Psi(\tau_k) \left\{ \frac{1}{\tau_k - s_r} + \frac{1}{(\tau_k + 1) + (s_r + 1)} \right. \\ & \quad \left. + K_0^\delta(\tau_k^*, s_r^*) + K_\delta^1(\tau_k^*, s_r^*) + K_I^\infty(\tau_k^*, s_r^*) \right\} \\ & = i p_0 \frac{c_{SH}}{\omega/k} \left[\left\{ \lambda \frac{\sinh(\lambda k h)}{\sinh(k h)} + \frac{c_{44}^E}{\bar{c}_{44}} \lambda^E \frac{\cosh(\lambda k h)}{\sinh(k h)} \right\} \right. \\ & \quad \left. \times \cosh(k h s_r^* c/h) - \cosh(\lambda k h s_r^* c/h) \right] \end{aligned} \quad (r=1, \dots, N-1) \quad (66)$$

where

$$\tau_k = \cos\left(\pi \frac{2k-1}{2N}\right), s_r = \cos\left(\pi \frac{r}{N}\right) \quad (67)$$

$$\tau_k^* = \frac{\tau_k + 1}{2}, s_r^* = \frac{s_r + 1}{2} \quad (68)$$

The N th unknown, $\Psi(\tau_N)$, is assumed to be zero since

τ_N is the closest of the τ_k to -1 . This assumption is in fact true as $N \rightarrow \infty$ ⁽¹¹⁾. Eliminating this unknown yields $N-1$ equations for the $N-1$ remaining unknowns. The linear algebraic system given in Eq. (66), where the sum is taken only to $N-1$, is solved using Gaussian elimination for $N-1$ values of $\Psi(\tau_k)$.

The dynamic stress intensity factor is obtained as

$$K_{III} = \lim_{x \rightarrow c^+} \{2\pi(x-c)\}^{1/2} \sigma_{yz}^s(x, 0) = (\pi c)^{1/2} \Psi(1) \quad (69)$$

Obtained in Shindo and Narita⁽⁸⁾ is the dynamic energy release rate $\bar{J}$ for antiplane shear as given by

$$\bar{J} = \frac{1}{2c_{44}} K_{III}^2 \quad (70)$$

When the piezoelectric layer is completely cracked through, $h = c$, the singularity is no longer of the square root type, but will have a power dependence. We express the solution of Eq.(63) by

$$\phi^*(\tau) = \frac{\Psi(\tau)}{(1-\tau)^\beta} \quad (71)$$

where β is given in Appendix B. The solution can be carried out by a procedure similar to the one given above but using the Gauss-Jacobi integration formula⁽¹³⁾. Whence the integral equations may be expressed as

$$\begin{aligned} & \sum_{j=1}^N \frac{W_j}{N} \Psi(\tau_j) \left\{ \frac{1}{\tau_j - s_r} + \frac{1}{(\tau_j + 1) + (s_r + 1)} + K_0^\delta(\tau_j^*, s_r^*) \right. \\ & \quad \left. + K_\delta^1(\tau_j^*, s_r^*) + K_I^\infty(\tau_j^*, s_r^*) \right\} \\ & = i p_0 \frac{c_{SH}}{\omega/k} \left[\left\{ \lambda \frac{\sinh(\lambda k h)}{\sinh(k h)} + \frac{c_{44}^E}{\bar{c}_{44}} \lambda^E \frac{\cosh(\lambda k h)}{\sinh(k h)} \right\} \right. \\ & \quad \left. \times \cosh(k h s_r^* c/h) - \cosh(\lambda k h s_r^* c/h) \right] \end{aligned} \quad (r=1, \dots, N-1) \quad (72)$$

In Eq. (72) W_j and τ_j are, respectively, the weights and roots of the Jacobi polynomial

$$P_N^{(0, -\beta)}(\tau_j) = 0, (j=1, \dots, N) \quad (73)$$

and s_r are the roots of

$$P_{N-1}^{(1, 1-\beta)}(s_r) = 0, (r=1, \dots, N-1) \quad (74)$$

The weighting constants W_j are given by

$$\begin{aligned} W_j & = -\frac{2N - \beta + 2}{(N+1)!(N - \beta + 1)} \Gamma(N+1) \\ & \quad \times \frac{2^{-\beta}}{P_N^{(0, -\beta)}(\tau_j) P_{N+1}^{(0, -\beta)}(\tau_j)} \end{aligned} \quad (75)$$

where $\Gamma(\cdot)$ is the Gamma function. The N th unknown, $\Psi(\tau_N)$, is assumed to be zero thus yielding a system of $N-1$ equations in $N-1$ unknowns which are solved using Gaussian elimination.

If we define the dynamic stress intensity factor by

$$K_{III} = \lim_{x \rightarrow c^+} (2\pi)^{1/2} (x-c)^\beta \sigma_{yz}^s(x, 0) \quad (76)$$

we obtain

$$K_{III} = (2\pi)^{1/2} \mu^* c^\beta \Psi(1) \quad (77)$$

where

$$\mu^* = -\frac{1}{\sin(\pi\beta)} \frac{c_{44}^E}{c_{44}} \frac{\bar{c}_{44}}{\bar{c}_{44} + c_{44}^E} \quad (78)$$

4. Numerical Results and Discussion

Numerical calculations were carried out for two different material pairs: PZT-4/aluminum and BaTiO₃/aluminum. The material properties for PZT-4, BaTiO₃ and aluminum are listed in Table 1^{(14)–(16)}. We first consider wave propagation in the piezoelectric layered media. Figure 3 is a plot of the dimensionless phase velocity $\omega/k(c_{44}/\rho)^{1/2}$ versus the dimensionless wave number kh for the first mode. The phase velocity is found to depend on the combination of the material.

Next, to examine the effect of electroelastic interactions for the first mode on the dynamic stress intensity factor and the dynamic energy release rate, a singular integral equation was solved numerically.

Table 1 Material properties of PZT-4, BaTiO₃, aluminum

		PZT-4	BaTiO ₃	Aluminum	
c_{44}	($\times 10^{10}\text{N/m}^2$)	2.56	4.4	c_{44}^E	2.65
e_{15}	(C/m ²)	12.7	11.4		—
ϵ_{11}	($\times 10^{-10}\text{C/Vm}$)	64.6	128.3		—
ρ	(kg/m ³)	7500	5700	ρ^E	2706

Fig. 3 Phase velocities of Love waves for piezoelectric layered media

Fig. 4 Dynamic stress intensity factor versus frequency (PZT-4/aluminum)

Figure 4 shows the normalized dynamic stress intensity factor $|K_{III}/p_0(\pi c)^{1/2}|$ as a function of the normalized frequency $\Omega = c\omega(c_{44}/\rho)^{-1/2}$ for various values of the layer-thickness to crack-depth ratio h/c for the case of a PZT-4 layer on an aluminum substrate. Three different ratios of $h/c = 1.2, 1.4$ and 1.8 are considered. The dynamic stress intensity factor tends to increase with frequency reaching a peak and then decrease in magnitude. These peaks decrease in magnitude as the crack depth c is increased relative to the thickness h of the piezoelectric layer. The peak values of $|K_{III}/p_0(\pi c)^{1/2}|$ for the PZT-4/aluminum are listed in Table 2. Figure 5 displays the variation of

Table 2 Peak values of $|K_{III}/p_0(\pi c)^{1/2}|$

	Ω	$ K_{III}/p_0(\pi c)^{1/2} $
$h/c = 1.2$		
PZT-4/Aluminum	0.72	0.833
BaTiO ₃ /Aluminum	0.65	1.196
$h/c = 1.4$		
PZT-4/Aluminum	0.72	0.896
BaTiO ₃ /Aluminum	0.68	1.162
$h/c = 1.8$		
PZT-4/Aluminum	0.74	0.963
BaTiO ₃ /Aluminum	0.76	1.145

Fig. 5 Dynamic energy release rate versus frequency (PZT-4/aluminum)

Fig. 6 Dynamic stress intensity factor versus frequency (BaTiO₃/aluminum)

Fig. 7 Dynamic energy release rate versus frequency (BaTiO₃/aluminum)

Fig. 8 Dynamic energy release rate versus frequency ($h/c=1.5$)

the normalized dynamic energy release rate $\dot{J} / p_0^2 \pi c (2c_{44})^{-1}$ with frequency Ω for the PZT-4/aluminum. Figures 6 and 7 show the corresponding results for a BaTiO₃ layer on the aluminum substrate. The peak values of $|K_{III} / p_0 (\pi c)^{1/2}|$ for the BaTiO₃/aluminum are also listed in Table 2. Figures 4 to 7 show that the amplitude of the dynamic stress intensity factor and the dynamic energy release rate can be raised or lowered depending on the combination of the material and geometric parameters. Figure 8 shows the effect of the piezoelectric properties and the frequency on the normalized dynamic energy release rate for $h/c=1.5$. Figure 9 presents the variation of the normalized energy release rate with the ratio h/c for the PZT-4/aluminum and BaTiO₃/aluminum while the frequency is varied. When the crack lies in PZT-4 layer, we observe much smaller dynamic energy release rate than when the BaTiO₃ layer contains the crack.

Also, since the case of $c < h$ has been treated, results for $c = h$ are presented. Figure 10 shows the variation of the stress intensity factor ratio $|K_{III} / p_0 \pi^{1/2} c^\beta|$ with respect to the frequency Ω for the PZT-4/aluminum and BaTiO₃/aluminum. Their

Fig. 9 Dynamic energy release rate versus ratio h/c

Fig. 10 Dynamic stress intensity factor versus frequency ($h/c=1.0$)

amplitude rises reaching a peak and then declines. The order of stress singularities at the tip of a crack which ends at an interface is also determined. The power β of the stress singularity and the dynamic stress intensity factor depend on the piezoelectric material properties.

5. Conclusions

The dynamic electroelastic problem for piezoelectric layered media having a surface-breaking crack has been analyzed theoretically. The results are expressed in terms of the dynamic stress intensity factor and the dynamic energy release rate. The equation for determining the order of singularities at the tip of a crack which is normal to and ends at an interface are also formulated. The peak values of the dynamic stress intensity factor and the dynamic energy release rate can either be raised or lowered by varying the piezoelectric material properties and h/c . It is possible to adjust these material and geometric parameters so as to reduce the magnitudes of the dynamic stress intensity factor and the dynamic energy release rate.

Appendix A

The kernels in Eq.(61) are

$$K_0^0(t_0, x_0) = \frac{\Omega}{z_p} \int_0^{\delta} \left(-i \frac{p_1}{\eta} - 1 \right) \sin(z_p \Omega \eta t_0) \times \cos(z_p \Omega \eta x_0) d\eta \\ - \frac{\Omega}{z_p} \int_0^{\delta} \frac{\eta L_0^0(\eta)}{\Delta_0^0(\eta)} \cos(z_p \Omega \eta t_0) \cos(z_p \Omega \eta x_0) d\eta \\ + z_p \Omega \int_0^{\delta} \frac{e_{15}^2}{C_{44} \epsilon_{11}} \frac{\exp(-z_p \Omega \eta h/c)}{\sinh(z_p \Omega \eta h/c)} \sinh(z_p \Omega \eta t_0) \times \cosh(z_p \Omega \eta x_0) d\eta \quad (A 1)$$

$$K_s^1(t_0, x_0) = \frac{\Omega}{z_p} \int_{\delta}^1 \left(-i \frac{p_1}{\eta} - 1 \right) \sin(z_p \Omega \eta t_0) \times \cos(z_p \Omega \eta x_0) d\eta \\ - \frac{\Omega}{z_p} \int_{\delta}^1 \frac{\eta L_s^1(\eta)}{\Delta_s^1(\eta)} \cos(z_p \Omega \eta t_0) \cos(z_p \Omega \eta x_0) d\eta \\ + z_p \Omega \int_{\delta}^1 \frac{e_{15}^2}{C_{44} \epsilon_{11}} \frac{\exp(-z_p \Omega \eta h/c)}{\sinh(z_p \Omega \eta h/c)} \sinh(z_p \Omega \eta t_0) \times \cosh(z_p \Omega \eta x_0) d\eta + F_{\delta M}^1 \quad (A 2)$$

$$K_i^{\infty}(t_0, x_0) = \frac{\Omega}{z_p} \int_1^{\infty} \left(\frac{p_1}{\eta} - 1 \right) \sin(z_p \Omega \eta t_0) \times \cos(z_p \Omega \eta x_0) d\eta \\ - \frac{\Omega}{z_p} \int_1^{\infty} \frac{\eta L_i^{\infty}(\eta)}{\Delta_i^{\infty}(\eta)} \cosh(z_p \Omega \eta t_0) \cosh(z_p \Omega \eta x_0) d\eta \\ + z_p \Omega \int_1^{\infty} \frac{e_{15}^2}{C_{44} \epsilon_{11}} \frac{\exp(-z_p \Omega \eta h/c)}{\sinh(z_p \Omega \eta h/c)} \sinh(z_p \Omega \eta t_0) \times \cosh(z_p \Omega \eta x_0) d\eta \quad (A 3)$$

where the constant $z_p = (c_{44}/\bar{c}_{44})^{1/2}$, and

$$F_{\delta M}^1 = -i\pi \frac{\Omega}{z_p} \sum_{m=1}^M \frac{\eta_m L_{\delta}^1(\eta_m)}{\Delta_{\delta}^1(\eta_m)} \cos(z_p \Omega \eta_m t_0) \quad (A 4)$$

$$L_0^0(\eta) = \left(\frac{\bar{c}_{44} \epsilon_{11}}{e_{15}^2} - \frac{c_{44}^E \epsilon_{11}}{e_{15}^2} \frac{p_2}{p_1} \right) \frac{\eta}{p_1} \cos(z_p \Omega \eta h/c) \\ \times \sin(z_p \Omega \eta t_0) + i \left(\frac{\bar{c}_{44} \epsilon_{11}}{e_{15}^2} - \frac{c_{44}^E \epsilon_{11}}{e_{15}^2} \frac{p_2}{p_1} \right)$$

$$\times \frac{\eta}{p_1} \sin(z_p \Omega \eta h/c) \sin(z_p \Omega \eta t_0)$$

$$L_s^1(\eta) = \left(\frac{\bar{c}_{44} \epsilon_{11}}{e_{15}^2} - i \frac{c_{44}^E \epsilon_{11}}{e_{15}^2} \frac{p_2}{p_1} \right) \frac{\eta}{p_1} \cos(z_p \Omega \eta h/c) \\ \times \sin(z_p \Omega \eta t_0) + i \left(\frac{\bar{c}_{44} \epsilon_{11}}{e_{15}^2} - i \frac{c_{44}^E \epsilon_{11}}{e_{15}^2} \frac{p_2}{p_1} \right)$$

$$\times \frac{\eta}{p_1} \sin(z_p \Omega \eta h/c) \sin(z_p \Omega \eta t_0)$$

$$L_i^{\infty}(\eta) = \left(\frac{\bar{c}_{44} \epsilon_{11}}{e_{15}^2} - \frac{c_{44}^E \epsilon_{11}}{e_{15}^2} \frac{p_2}{p_1} \right) \\ \times \frac{\eta}{p_1} \exp(-z_p \Omega \eta h/c) \sinh(z_p \Omega \eta t_0) \quad (A 5)$$

$$\Delta_0^0(\eta) = -\frac{\bar{c}_{44} \epsilon_{11}}{e_{15}^2} p_1 \sin(z_p \Omega \eta h/c) - i \frac{c_{44}^E \epsilon_{11}}{e_{15}^2} p_2 \\ \times \cos(z_p \Omega \eta h/c) \quad (A 6)$$

$$\Delta_s^1(\eta) = -\frac{\bar{c}_{44} \epsilon_{11}}{e_{15}^2} p_1 \sin(z_p \Omega \eta h/c) + \frac{c_{44}^E \epsilon_{11}}{e_{15}^2} p_2 \\ \times \cos(z_p \Omega \eta h/c) \quad (A 7)$$

$$\Delta_i^{\infty}(\eta) = \frac{\bar{c}_{44} \epsilon_{11}}{e_{15}^2} p_1 \sinh(z_p \Omega \eta h/c) + \frac{c_{44}^E \epsilon_{11}}{e_{15}^2} p_2 \\ \times \cosh(z_p \Omega \eta h/c) \quad (A 8)$$

In Eqs.(A 1)-(A 8) the following notation is used:

$$p_1(\eta) = |\eta^2 - 1|^{1/2}, \quad p_2(\eta) = |\eta^2 - \delta^2|^{1/2} \quad (A 9)$$

Appendix B

When the crack extends to the interfaces, the kernel in Eq.(61) is no longer bounded and contains point singularities at $t_0=1$ and $x_0=1$. To extract these singularities, we need to study the infinite integral in the expression for $K_0^0(t_0, x_0) + K_s^1(t_0, x_0) + K_i^{\infty}(t_0, x_0)$. Following Gupta⁽¹³⁾ the part of the kernel contributing these singularities may be written as

$$K_{\infty}(t_0, x_0) = -\frac{1}{2} \left(\frac{1}{2+x_0-t_0} + \frac{1}{2-x_0-t_0} \right) f_{\infty} \quad (B 1)$$

where

$$f_{\infty} = \frac{\bar{c}_{44}}{C_{44}} \frac{\bar{c}_{44} - C_{44}^E}{\bar{c}_{44} + C_{44}^E} - \frac{e_{15}^2}{C_{44} \epsilon_{11}} \quad (B 2)$$

To analyze the behavior of the unknown function $\phi(t_0)$ near the end points, we must consider the dominant part of the singular integral equation which can be expressed as

$$\frac{1}{\pi} \int_0^1 \frac{\phi(t_0)}{t_0 - x_0} dt_0 + \frac{1}{\pi} \int_0^1 \left\{ \frac{1}{t_0 + x_0} - \left(\frac{1}{2+x_0-t_0} + \frac{1}{2-x_0-t_0} \right) f_{\infty} \right\} \phi(t_0) dt_0 \\ = i p_0 \frac{C_{SH}}{\omega/k} \left[\left\{ \lambda \sin \frac{\sinh(\lambda k h)}{\sinh(k h)} + \frac{c_{44}^E}{\bar{c}_{44}} \lambda^E \frac{\cosh(\lambda k h)}{\sinh(k h)} \right\} \right. \\ \left. \times \cosh(k h x_0 c/h) - \cosh(\lambda k h x_0 c/h) \right] \quad (B 3)$$

To examine the singular behavior of the solution of Eq.(B3), we first express the unknown function $\phi(t_0)$ as follows⁽¹⁷⁾:

$$\phi(t_0) = \frac{g(t_0)}{t_0^{\alpha}(1-t_0)^{\beta}} = \frac{g(t_0)e^{\pi i \beta}}{t_0^{\alpha}(t_0-1)^{\beta}}, \quad 0 < t_0 < 1 \quad (B 4)$$

where $g(t_0)$ is Hölder-continuous in $0 \leq t_0 \leq 1$ and $t_0^{-\alpha}(t_0-1)^{-\beta}$ is any definite branch which varies continuously in $0 < t_0 < 1$. In Eq.(B 4) it is assumed that

$$g(0) \neq 0, \quad g(1) \neq 0, \quad 0 < \text{Re}(\alpha, \beta) < 1 \quad (B 5)$$

Technique of determining α and β has been treated by Cook and Erdogan⁽¹⁷⁾. Consider the following sectionally holomorphic function⁽¹⁷⁾

$$G(z) = \frac{1}{\pi} \int_0^1 \frac{\phi(t_0)}{t_0 - z} dt_0 = \frac{1}{\pi} \int_0^1 \frac{g(t_0)e^{\pi i \beta}}{t_0^{\alpha}(t_0-1)^{\beta}(t_0-z)} dt_0 \\ = \frac{g(0)e^{\pi i \alpha}}{\sin(\pi \alpha)} \frac{1}{z^{\alpha}} - \frac{g(1)}{\sin(\pi \beta)} \frac{1}{(z-1)^{\beta}} + G_0(z) \quad (B 6)$$

where the function $G_0(z)$ is bounded everywhere with a possible exception of the end points c_j , ($c_1=0$, $c_2=1$) near which

$$|G_0(z)| < \frac{C_j}{|z - c_j|^{\gamma_j}}, \quad \gamma_j < \text{Re}(\alpha, \beta), \quad (j=1, 2) \quad (B 7)$$

C_j and γ_j being real constants. By using the Plemelj formulas, from Eq.(B 6) it can be shown that

$$\frac{1}{\pi} \int_0^1 \frac{\phi(t_0)}{t_0 - x_0} dt_0 = \frac{g(0)}{x_0^{\alpha}} \cot(\pi \alpha) - \frac{g(1)}{(1-x_0)^{\beta}} \cot(\pi \beta) \\ + G_1(x_0), \quad 0 < x_0 < 1 \quad (B 8)$$

$$\frac{1}{\pi} \int_0^1 \frac{\phi(t_0)}{t_0 + x_0} dt_0 = G(-x_0) = \frac{g(0)}{\sin(\pi \alpha) x_0^{\alpha}} + G_2(x_0),$$

$$0 < x_0 < 1 \quad (\text{B } 9)$$

$$\frac{1}{\pi} \int_0^1 \frac{\psi(t_0)}{t_0 - (x_0 + 2)} dt_0 = G(x_0 + 2) = G_3(x_0),$$

$$0 < x_0 < 1 \quad (\text{B } 10)$$

$$\frac{1}{\pi} \int_0^1 \frac{\psi(t_0)}{t_0 - (2 - x_0)} dt_0 = G(2 - x_0)$$

$$= -\frac{g(1)}{\sin(\pi\beta)(1-x_0)^\beta} + G_4(x_0), 0 < x_0 < 1$$

$$(\text{B } 11)$$

where at the end points the behavior of $G_j(x_0)$ ($j=1, \dots, 4$) is the same as that of $G_0(z)$.

Substituting now from Eqs. (B 8)–(B 11) into Eq. (B 3), multiplying both sides first by x_0^α and letting $x_0=0$, and then by $(1-x_0)^\beta$ and letting $x_0=1$, it is found that

$$\frac{g(0)}{\sin(\pi\alpha)} \{\cos(\pi\alpha) + 1\} = 0 \quad (\text{B } 12)$$

$$\frac{g(1)}{\sin(\pi\beta)} \{\cos(\pi\beta) + f_\infty\} = 0 \quad (\text{B } 13)$$

Equations (B 5) and (B 13) give $\beta = \cos^{-1}(-f_\infty)/\pi$. For $g(0) \neq 0$ the only possible root of Eq. (B 12) is $\alpha=1$ which is unacceptable. It is clear that the function $\psi(t_0)$ does not have a power singularity at the end point $t_0=0$ ⁽¹⁸⁾. Since the function $\psi(t_0)$ must have an integrable singularity at $t_0=1$, and since the foregoing analysis indicates that it cannot have a power singularity at $t_0=0$, the next step is to look for a solution of the following form:

$$\psi(t_0) = \frac{g_0(t_0)}{(1-t_0)^\beta} \quad (\text{B } 14)$$

It is easy to show that the solution of Eq. (B 3) is of the form (B 14).

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