Deformation and Damage Characteristics of Porous Materials under Different Triaxialities*

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A new rigid plastic constitutive model for ductile materials with void evolution processes, which was developed on the base of the modified Gurson's yield function in previous research, is carefully investigated in this paper. In order to check the reasonability, reliability and accuracy of this model, detailed numerical calculations on the response of porous materials under uniaxial stress condition, biaxial equal stress condition and uniaxial strain condition are carried out respectively. The influence of triaxiality on the deformation behaviours and void evolution characteristics are revealed through such kind of calculations. Systematic comparisons of the calculation results based on both the present model and the conventional models show the correctness of the new constitutive theory. The advantages of the present model over other conventional models are discussed. This research displays the possibility of establishing the constitutive model for porous materials, such as sintered metals, on a solid theoretical basis.

Key Words: Rigid Plastic Theory, Porous Materials, Deformation and Damage Characteristics

1. Introduction

A rigid plastic constitutive theory for ductile porous materials, which is based on modified Gurson's yield function⁽¹⁾, has been established by the present authors in previous research⁽²⁾. Although this model seems to be self-consistent in theory, and close inherent relationships with classical rigid plastic theory for porous materials have been demonstrated, some problems still need to be investigated further — Is it reliable? Is it accurate? Is it able to describe the deformation and damage behaviours of porous materials correctly?

As stated in Ref. (2), the applications of Gurson-type yield function in the past are mainly concentrated on elastic-plastic analysis. Besides, all these applications have similar peculiarity: the mechanism for engendering initial void is the same, that is, voids nucleate mainly at second phase particles by deco-

hesion of the particle-matrix interface or by particle fracture under large plastic deformations. In the case of deformation-induced voids, initial void fraction depends on the particle volume fraction, and usually the particle volume percentage is very small. However, for materials with preexisting voids such as sintered metals produced by powder-metallurgy, initial void fraction may be either small or very large. It is pity that experiments with specimens made of sintered aluminium alloy⁽³⁾ revealed poor agreement with Gurson's yield function. This may partly explain why the elastic-plastic theory based on Gurson's yield function was seldom found to be applied to sintered metals. Even though Tvergaard⁽⁴⁾ tried to compensate the poor agreement by using the modified Gurson's yield function, certain doubts and suspicions still arise: within what extent is the modified Gurson's yield function able to depict the yield behaviours of sintered metals? Another interrelated query is about the present model: as a rigid plastic theory based on modified Gurson's yield function, in what scope can it describe the deformation and damage processes of sintered metals?

As is known, rigid plastic theory for sintered

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metals has been well constructed on the base of various yield functions. What is the advantage of the present model in comparison with the conventional models? Where is the limitation of conventional models?

For the purpose of answering the above questions, three loading states with different triaxialities, which are uniaxial tensile stress condition, biaxial tensile stress condition and uniaxial tensile strain condition, have been investigated on the basis of both the present model and conventional models. Numerical results about the damage parameters, deformation processes and damage behaviours are displayed and compared in this paper.

2. Summary on Present and Conventional Constitutive Models

2.1 Outline of the present model⁽²⁾

The modified Gurson-type yield function⁽¹⁾ is

$$F^*(\sigma_{ij}, \sigma_M, f) = J_2 + \frac{2}{3} q_1 f \sigma_M^2 \cosh\left(\frac{q_2}{2} \frac{\sigma_{kk}}{\sigma_M}\right) - \frac{1}{3} (1 + (q_1 f)^2) \sigma_M^2 = 0 \quad (1)$$

where σ_{ij} is Cauchy stress, σ_M is the matrix flow stress, f is the void fraction, and J_2 is the second invariant of the deviatoric stress. q_1 and q_2 are constants which will be discussed later on in this paper.

The stress-strain rate equations are

$$\dot{\epsilon}_{kk} = \frac{3\dot{\epsilon}_0}{2\bar{\sigma}_0} q_1 f q_2 \sigma_M \sinh\left(\frac{q_2}{2} \frac{\sigma_{kk}}{\sigma_M}\right) \quad (2)$$

$$\dot{\epsilon}'_{ij} = \frac{3\dot{\epsilon}_0}{2\bar{\sigma}_0} \sigma'_{ij} \quad \text{or} \quad \sigma'_{ij} = \frac{2\bar{\sigma}_0}{3\dot{\epsilon}_0} \dot{\epsilon}'_{ij} \quad (3)$$

where

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}, \quad \dot{\epsilon}'_{ij} = \dot{\epsilon}_{ij} - \frac{1}{3} \dot{\epsilon}_{kk} \delta_{ij}$$

are component of deviatoric stress and strain rate respectively. $\bar{\sigma}_0$ and $\dot{\epsilon}_0$ are respectively nominal effective stress and strain rate defined in

$$\bar{\sigma}_0^2 = 3J_2 = \frac{3}{2} \sigma'_{ij} \sigma'_{ij} \quad (4)$$

$$\dot{\epsilon}_0^2 = \frac{2}{3} \dot{\epsilon}'_{ij} \dot{\epsilon}'_{ij} \quad (5)$$

The $\bar{\sigma}_0$ - σ_M and $\dot{\epsilon}_0$ - $\dot{\epsilon}_M$ relations are represented by

$$\bar{\sigma}_0 = \sqrt{1 + (q_1 f)^2 - 2q_1 f \chi} \sigma_M \quad (6)$$

$$\dot{\epsilon}_0 = \frac{(1-f)\sqrt{1 + (q_1 f)^2 - 2q_1 f \chi}}{1 + (q_1 f)^2 - 2q_1 f \chi + q_1 f \psi \omega} \dot{\epsilon}_M \quad (7)$$

where $\dot{\epsilon}_M$ is the matrix flow strain rate, and

$$\omega = \frac{q_2}{2} \frac{\sigma_{kk}}{\sigma_M} = \ln \left[-\beta + \sqrt{(\beta + q_1 f) \left(\beta + \frac{1}{q_1 f} \right)} \right] \pm \sqrt{\beta \left(\sqrt{\beta + \frac{1}{q_1 f}} - \sqrt{\beta + q_1 f} \right)} \quad (8)$$

$$\chi = \cosh \omega = -\beta + \sqrt{(\beta + q_1 f) \left(\beta + \frac{1}{q_1 f} \right)} \quad (9)$$

$$\psi = \sinh \omega = \pm \sqrt{\chi^2 - 1}$$

$$= \pm \sqrt{\beta} \left(\sqrt{\beta + \frac{1}{q_1 f}} - \sqrt{\beta + q_1 f} \right) \quad (10)$$

The generalized triaxiality function in the above transformations is able to be defined in both strain space and stress space

$$\beta = \frac{4}{9} \frac{1}{q_1 f q_2^2} \left(\frac{\dot{\epsilon}_{kk}}{\dot{\epsilon}_0} \right)^2 = q_1 f \left(\frac{\sinh \omega}{\bar{\sigma}_0 / \sigma_M} \right)^2 = \frac{1}{4} q_1 f q_2^2 r^2 \quad (11)$$

where r is generalized triaxiality ratio. It can also be expressed either in stress space or in strain space

$$r = \frac{2}{q_2} \frac{\sinh \omega}{\bar{\sigma}_0 / \sigma_M} = \frac{4}{3} \frac{1}{q_1 f q_2^2} \frac{\dot{\epsilon}_{kk}}{\dot{\epsilon}_0} \quad (12)$$

If only consider the void growth, the void evolution law becomes

$$\dot{f} = (1-f) \dot{\epsilon}_{kk} = \frac{3}{4} q_1 f (1-f) q_2^2 \dot{\epsilon}_0 r \quad (13)$$

2.2 Summary of the renewed conventional constitutive model⁽²⁾

The conventional yield function for porous materials is

$$\Phi = A J_2 + B (\sigma_{kk})^2 - C \sigma_M^2 = 0 \quad (14)$$

where A , B and C are functions of void fraction. The reframed stress-strain rate formulas are

$$\dot{\epsilon}_{kk} = 9 \frac{B}{A} \frac{\dot{\epsilon}_0}{\bar{\sigma}_0} \sigma_{kk} \quad (15)$$

$$\dot{\epsilon}'_{ij} = \frac{3\dot{\epsilon}_0}{2\bar{\sigma}_0} \sigma'_{ij} \quad \text{or} \quad \sigma'_{ij} = \frac{2\bar{\sigma}_0}{3\dot{\epsilon}_0} \dot{\epsilon}'_{ij} \quad (16)$$

The $\bar{\sigma}_0$ - σ_M and $\dot{\epsilon}_0$ - $\dot{\epsilon}_M$ relations are given by

$$\bar{\sigma}_0 = \sqrt{3 \frac{C}{A} \frac{1}{1 + \beta_t} \sigma_M} \quad (17)$$

$$\dot{\epsilon}_0 = (1-f) \sqrt{\frac{1}{3} \frac{A}{C} \frac{1}{1 + \beta_t} \dot{\epsilon}_M} \quad (18)$$

$$\sigma_{kk} = \sqrt{\frac{C}{B} \frac{\beta_t}{1 + \beta_t} \sigma_M} \quad (19)$$

The triaxiality function β_t indicated in strain space and stress space is

$$\beta_t = \frac{1}{27} \frac{A}{B} \left(\frac{\dot{\epsilon}_{kk}}{\dot{\epsilon}_0} \right)^2 = 3 \frac{B}{A} \left(\frac{\sigma_{kk}}{\bar{\sigma}_0} \right)^2 = 3 \frac{B}{A} r_t^2 \quad (20)$$

The triaxiality ratio r_t stated in stress space and strain space is defined as

$$r_t = \frac{\sigma_{kk}}{\bar{\sigma}_0} = \frac{1}{9} \frac{A}{B} \frac{\dot{\epsilon}_{kk}}{\dot{\epsilon}_0} \quad (21)$$

The void growth law is of the form

$$\dot{f} = (1-f) \dot{\epsilon}_{kk} = 9(1-f) \frac{B}{A} \dot{\epsilon}_0 r_t \quad (22)$$

3. Conventional Models to Be Compared

The material investigated here is sintered copper. Suppose that the matrix copper has not experienced any hardening or initial deformation at the beginning of the loading.

The Cauchy stress σ_{ij} in present model is the true stress defined per unit area in current configuration. The plastic work per unit volume in current configuration, as designated by $\dot{W}$ in Ref. (2), is

formed by the inner product of the deformation rate tensor (or the so-called strain rate tensor) $\dot{\epsilon}_{ij}$ with its work conjugate, Cauchy stress σ_{ij} . In accordance with the above background, the flow stress-flow strain curve should be selected as the uniaxial true stress-natural strain curve of matrix material. This relation is usually taken as

$$\sigma_M = \sigma_Y \left(1 + \frac{\epsilon_M}{\epsilon_Y} \right)^n \quad (23)$$

where for matrix copper

$$\sigma_Y = 70.3 \text{ Mpa}, \quad \epsilon_Y = 0.022, \quad n = 0.49$$

As for porous copper, the parameters q_1 and q_2 in Eq. (1), which reflect the influence of void interaction, are suggested as

$$q_1 = 2, \quad q_2 = 1.1482 \quad (24)$$

Thus the corresponding ultimate void fraction is

$$f_u = 1/q_1 = 50\% \quad (25)$$

The present model will be compared with other three representative models: the first order approximate model⁽²⁾, Doraivelu et al model⁽³⁾ and Shima et al. model⁽⁵⁾. The yield function for the three models has the same quadratic shape as Eq.(14). Thus the rigid plastic constitutive equations and damage evolutions of all three models are able to be unitedly indicated by Eqs.(15)-(22). The only diversity among the three models is that the detailed structures of the coefficient functions $A(f)$, $B(f)$ and $C(f)$ in Eq.(14) are different.

The first order approximate model is obtained by expanding the $\sinh(\cdot)$ term in the present model into Taylor series and just keeping the first term. Such an approximate treatment leads to the following formulations for $A(f)$, $B(f)$ and $C(f)$

$$A=1, \quad B=q_1 f q_2^2 / 12, \quad C=(1-q_1 f)^2 / 3 \quad (26)$$

As mentioned in Ref.(2), this treatment may be valid within a certain extent. In consideration of the model's simplicity in engineering, it is under the necessity of making sure the model's scope of application.

The Shima model was originated twenty years ago and is still widely used today in plastic processing for sintered metals. An important distinguishing feature of the model is that the yield function and the constitutive equations were mainly determined by a large number of elaborately designed experiments with specimens made of sintered copper. Eqs.(14)-(22) are absolutely consistent with the original Shima model if $A(f)$, $B(f)$ and $C(f)$ are chosen as

$$A=1, \quad B=0.2296 f^{1.028}, \quad C=(1-f)^5 / 3 \quad (27)$$

Expression (27) should be regarded as the quintessence of experiments. Therefore the comparison with the Shima model is roughly equivalent to the comparison with experiments. The ultimate void fraction for Shima model is obtained by $C=0$

$$f_u = 100\% \quad (28)$$

Generally speaking, Eq.(28) is not reasonable.

The Doraivelu model is unique because it was created partly by a energy-form yield criterion and partly by experimental results. Also this model directly includes the concept of ultimate relative density of sintered metals. Eqs.(14)-(22) are precisely equivalent to the original Doraivelu model if $A(f)$, $B(f)$ and $C(f)$ are selected as

$$\begin{aligned} A &= (3-2f+f^2)/3, \quad B = (2f-f^2)/9, \\ C &= (1-4f+2f^2)/3 \end{aligned} \quad (29)$$

The ultimate void fraction for the Doraivelu model is $f_u = 29.29\%$ (30)

It is necessary to indicate that Doraivelu et al believed Eq.(14) should converge to a real condition which exists at the yield point under uniaxial state of stress. In other words, the applied stress should equal the yield stress of the aggregate. In order to satisfy this condition, $A(f)$ and $B(f)$ should be contracted by

$$A = 3(1-B) \quad (*)$$

The above viewpoint is surely correct, but condition (*) may not be so. A misunderstanding on "the yield stress of the aggregate" seems to exist. In Ref.(3), the yield stress of the aggregate under uniaxial stress state is considered to be $\sqrt{C} \sigma_M$. In reality, however, this value should be $\sqrt{3C/(A+3B)} \sigma_M$ (see Eqs.(45) and (46) in this paper). Hence condition (*) is superfluous. It is of surprise that the Doraivelu model showed nice agreement with experiment for powder compacted aluminium alloy, though such a possible misunderstanding exists. Because the same experiment disclosed poor agreement with Gurson's yield function, this paper also selects the Doraivelu model for detailed comparison.

Although the current theory aims at numerical rigid plastic analysis, this paper is just confined to test and verify the theory's reliability and accuracy; thus only simple and uniform deformation states are to be studied. Under such loading cases, the stress-strain curves, damage evolutions, and all the physical quantities defined in the model are able to be calculated without difficulty.

4. Response of Porous Material Element under Uniaxial Stress Condition

Suppose a porous material element is loaded by uniaxial tensile stress in z axis, that is

$$\sigma_x = 0, \quad \sigma_y = 0, \quad \sigma_z \geq 0 \quad (31)$$

$$\sigma_{kk} = \sigma_z \quad (32)$$

$$\dot{\epsilon}_x = \dot{\epsilon}_y = \dot{\epsilon}_r \leq 0, \quad \dot{\epsilon}_z \geq 0 \quad (33)$$

$$\dot{\epsilon}_{kk} = \dot{\epsilon}_z + 2\dot{\epsilon}_r \quad (34)$$

$$\dot{\epsilon}_0 = \frac{2}{3}(\dot{\epsilon}_z - \dot{\epsilon}_r)$$

4.1 Deduction of present model under uniaxial tensile stress condition

By Eq. (8) and Eq. (31), parameter ω becomes

$$\omega = \frac{q_2}{2} \frac{\sigma_z}{\sigma_M} \quad (35)$$

From Eqs. (12), (32) and (35), the generalized triaxiality ratio is rewritten as

$$r = \frac{\sinh \omega}{\omega} \quad (36)$$

The generalized triaxiality function in Eq. (11) may be changed into

$$\beta = \frac{1}{4} q_1 f q_2^2 \left(\frac{\sinh \omega}{\omega} \right)^2 \quad (37)$$

The yield condition is determined from Eqs. (6), (9), (32) and (35)

$$\sigma_z = \sqrt{1 + (q_1 f)^2 - 2 q_1 f \chi} \sigma_M \quad \text{or} \quad \omega = \frac{q_2}{2} \sqrt{1 + (q_1 f)^2 - 2 q_1 f \chi} \quad (38)$$

By using Eqs. (2) and (31)–(36), the $\dot{\epsilon}_r$ and $\dot{\epsilon}_z$ can be related by

$$\dot{\epsilon}_r = -\frac{1 - q_1 f q_2^2 r/2}{2 + q_1 f q_2^2 r/2} \dot{\epsilon}_z \quad (39)$$

Define Poisson's ratio, μ , as

$$\dot{\epsilon}_r = -\mu \dot{\epsilon}_z \quad (40)$$

then

$$\mu = \frac{1 - q_1 f q_2^2 r/2}{2 + q_1 f q_2^2 r/2} \quad (41)$$

Different from void-free materials, for porous materials, the Poisson's ratio in Eq. (41) is affected not only by void fraction f , void interaction (q_1 and q_2), but also by generalized triaxiality ratio r .

The strain rate relation $\dot{\epsilon}_z - \dot{\epsilon}_M$ may be obtained from Eqs. (7), (34) and (40)

$$\dot{\epsilon}_z = \frac{3}{2} \frac{1}{1 + \mu} \frac{(1 - f) \sqrt{1 + (q_1 f)^2 - 2 q_1 f \chi} + q_1 f \psi \omega}{1 + (q_1 f)^2 - 2 q_1 f \chi + q_1 f \psi \omega} \dot{\epsilon}_M \quad (42)$$

where χ and ψ are defined in Eq. (9) and (10) respectively. By combining Eqs. (13), (33) and (40), the void evolution law will be

$$\dot{f} = (1 - f)(1 - 2\mu) \dot{\epsilon}_z \quad (43)$$

4.2 Evolvement of conventional model under uniaxial tensile stress condition

Under uniaxial stress condition, the conventional model's specific formulations, such as the triaxiality ratio, triaxiality function, yield condition, Poisson's ratio, strain rate relation and void evolution law, can be degenerated directly from the general forms of Eqs. (15)–(22) as follows:

$$r_t = 1 \quad (44)$$

$$\beta_t = 3 \frac{B}{A} \quad (45)$$

$$\sigma_z = \sqrt{\frac{3C}{A} \frac{1}{1 + \beta_t}} \sigma_M \quad (46)$$

$$\mu_t = \frac{1 - 2\beta_t}{2(1 + \beta_t)} \quad (47)$$

$$\dot{\epsilon}_z = \frac{3}{2} \frac{1 - f}{1 + \mu_t} \sqrt{\frac{A}{3C} \frac{1}{1 + \beta_t}} \dot{\epsilon}_M \quad (48)$$

$$\dot{f} = (1 - f)(1 - 2\mu_t) \dot{\epsilon}_z \quad (49)$$

4.3 Calculation results

Under uniaxial stress condition, parameter ω occupies an important place in the present model. At a loading step, for a given ω on the yield surface, the generalized triaxiality ratio, corresponding void fraction, generalized triaxiality function, Poisson's ratio, uniaxial strain, and uniaxial stress, can be directly computed by the combination of Eq. (23) with Eqs. (35)–(38) and (41)–(43). Similar computational processes for conventional models can also be put into effect by combining Eq. (23) with Eqs. (44)–(49).

1) Triaxiality ratio

From Eqs. (32), (35) and (36), the following limit values are obtained

$$f \rightarrow 0, \quad \sigma_z = \bar{\sigma}_0 \rightarrow \sigma_M, \quad \omega \rightarrow \omega_0 = q_2/2 = 0.5741, \quad r \rightarrow r_0 = 1.0558 \quad (50)$$

$$f \rightarrow f_u, \quad \omega \rightarrow \omega_u = 0, \quad r \rightarrow r_u = 1 \quad (51)$$

Under uniaxial stress condition, the triaxiality ratio in three conventional models are unity (Eq. (44)). Fig. 1 shows that the generalized triaxiality ratio in the present model just slightly diverges from unity when the void fraction is small. Therefore the statement that Eq. (12) is a relatively successful definition of the concept generalized triaxiality ratio is not exaggerated.

2) Triaxiality function

Applying Eq. (37) and expression (51), the following limit values of the generalized triaxiality function are derived

$$f \rightarrow 0, \quad \beta \rightarrow \beta_0 = 0 \quad (52)$$

$$f \rightarrow f_u, \quad \beta \rightarrow \beta_u = q_2^2 r_u^2 / 4 = 0.3296 \quad (53)$$

Fig. 1 Variation in triaxiality ratio vs. void fraction under uniaxial tensile stress condition

In Fig. 2, the (generalized) triaxiality function also increases when the void fraction increases, which means the ductility of the material will decrease. Under uniaxial stress condition, the curves for the four models in Fig. 2 coincide very well with each other. This is an encouraging result. At least two conclusions can be drawn: (a) plastic deformation, plastic work hardening, growing void fraction and void interaction will cut down the ductility of porous materials. The generalized triaxiality function defined in the present model is able to predict such tendencies quantitatively and successfully; (b) under the theoretical framework of three conventional models, the triaxiality function borrowed from present model is also able to predict the variation of material's ductility with small divergence.

3) Poisson's ratio

Combination of Eqs.(41) and expression (50) leads to the following limit values of Poisson's ratio

$$f \rightarrow 0, \quad \mu \rightarrow \mu_0 = 0.5 \quad (54)$$

$$f \rightarrow f_u, \quad \mu \rightarrow \mu_u = \frac{1 - q_2^2/2}{2 + q_2^2/2} = 0.1282 \quad (55)$$

Fig. 3 demonstrates that if the void fraction increases, Poisson's ratio decreases. Furthermore, the curve for the present model agrees very well with those for other models. According to Ref.(5), the predicted μ - f curve in the Shima model seems to overlap the experimental result. In Ref.(3), the Doraivelu model was established directly on the basis of experimental μ - f curve. This means that the μ - f curve predicted by the present model is also able to coincide with experiment.

The present authors maintain such a standpoint, that is, the μ - f curves for different sintered metal,

such as sintered copper, sintered aluminium and sintered iron, may well overlap each other. But this does not signify that the yield function and constitutive equations fitting one porous material are also suitable for another. Poisson's ratio on the whole reflects only the compressibility of porous materials. However, the factors that influence the yield and deformation behaviours of porous materials far exceed the compressibility. In a word, the experimental μ - f curve is not appropriate to be taken as an judgement on a model.

4) Yield curves

Fig. 4 displays the characteristic yield curves predicted by four yield conditions. When $f < 25\%$, all

Fig. 3 Variation in Poisson's ratio vs. void fraction under uniaxial tensile stress condition

Fig. 2 Variation in triaxiality function vs. void fraction under uniaxial tensile stress condition

Fig. 4 Influence of void fraction on yield behaviours under uniaxial tensile stress condition

the four curves coincide very well; when $f < 35\%$, the divergence among modified Gurson's yield criterion, first order approximate criterion, and the Shima criterion is small. Note that after $f > 25\%$, materials described by the Doraivelu criterion lose load carrying capacity rapidly. This phenomenon is not strange, because the ultimate void fraction for sintered aluminium is smaller than that of sintered copper. From Eq. (28) and Fig. 4, the Shima criterion shows unreasonable tendency after $f > 35\%$.

As mentioned above, the Shima model and the Doraivelu model were based on the experiments of sintered copper and sintered aluminium respectively. A reasonable inference from Fig. 4 may be as follows: The yield behaviours under uniaxial stress condition are close for different sintered materials if void fraction is not high. Disparity occurs at relative high void fraction, because the ultimate void fraction and intensity of void interaction are different for different sintered materials.

The original Gurson's yield function did not contain the effects of void interaction. As a result, poor agreement with experiment is normal. In the modified Gurson's yield function, void interaction is reckoned by parameters q_1 and q_2 . Computations indeed bring to light the importance of these two parameters: q_1 mainly controls the locations of the yield curve in Fig. 4 and q_2 intensely dominates the tendency of the μ - f curve in Fig. 3.

Now one of the queries raised in the introduction can be answered. Under uniaxial stress condition, the modified Gurson's yield function is capable of portraying the yield behaviours of sintered metals in a wide range of void fraction if q_1 and q_2 are properly selected. Further explorations on q_1 and q_2 will be made later in this paper.

Because of the lower triaxiality, the first order approximate yield function agrees well with the modified Gurson's yield function under uniaxial stress condition.

5) Uniaxial strain rate ratio

The ratio of uniaxial strain rate to matrix flow strain rate increases slowly from unity in Fig. 5 when the void fraction rises. If $f < 25\%$, all the four models coincide with high accuracy. After $f > 25\%$, deformation in the Doraivelu model tends to infinity. Until $f = 35\%$, the present model, approximate model and Shima model agree well.

6) Predicted stress-strain curves and void growth curves

Figs. 6-9 are stress-strain curves and corresponding void growth curves for the four models. The hardening property of matrix copper in Eq. (23) is also indicated by a dotted line for comparison. The follow-

Fig. 5 Variation in strain rate ratio vs. void fraction under uniaxial tensile stress condition

ing conclusions may be drawn:

- (a) The void growth is driven by the deformation increase.
- (b) The larger the initial void fraction, the faster the void growth.
- (c) The larger the initial void fraction, the lower the initial yield stress.
- (d) Two opposite tendencies are developed in sintered copper at every loading step. On one hand, the plastic deformation drives the hardening tendency in matrix copper; on the other hand, it also strengthens the void growing tendency which may cause porous copper softening. At the primary deformation stage, the void fraction is relatively small and the hardening in matrix copper is dominative. Hence sintered copper emerges global hardening behaviour. Gradually, the void fraction becomes larger and larger, while the global hardening tendency slows down. After the deformation and void fraction approach certain values, the global hardening property of porous copper is overcome. Once the peak point is surpassed, global softening process occurs and takes dominative position, though the hardening still continues in matrix copper.
- (e) The larger the initial void fraction, the earlier the appearance of the peak point and the lower the peak stress.

The differences in the deformation and damage behaviours predicted by the four models can be found in Fig. 14 and Fig. 15.

(f) The smaller the initial void fraction, the better the consistence among the σ_z - ϵ_z curves.

(g) The larger the initial void fraction, the better the coincidence among the f - ϵ_z curves.

Fig. 6 Effect of initial void fraction on stress-strain curves under uniaxial tensile stress condition for present model

Fig. 8 Comparison between the stress-strain curves under uniaxial tensile stress condition ($f_0 = 0.00858$ & 0.0525)

Fig. 7 Effect of initial void fraction on void evolution under uniaxial tensile stress condition for present model

Fig. 9 Comparison between the void evolution curves under uniaxial tensile stress condition ($f_0 = 0.00858$ & 0.0525)

(h) The disparities between the predicted peak stress by four models are small.

(i) The predicted positions of the peak points by four models are very close.

At this point another query from the introduction may be partially answered. The present model seems to possess the capability of describing the deformation and damage behaviours of sintered metals. Especially at lower void fraction under uniaxial stress condition, a high degree of consistency with other experimentally-based models (such as the Shima model) can be attained by the present model.

5. Response of Porous Material under Biaxial Equal Stress Condition

Suppose a porous material element is loaded by biaxial equal tensile stress in x and y axes, that is

$$\sigma_z = 0, \quad \sigma_x = \sigma_y = \sigma_r \geq 0 \quad (56)$$

$$\bar{\sigma}_0 = \sigma_r \quad (57)$$

$$\dot{\epsilon}_x = \dot{\epsilon}_y = \dot{\epsilon}_r \geq 0, \quad \dot{\epsilon}_z \leq 0 \quad (58)$$

$$\dot{\epsilon}_{kk} = \dot{\epsilon}_z + 2\dot{\epsilon}_r \quad (59)$$

$$\dot{\epsilon}_0 = \frac{2}{3}(\dot{\epsilon}_r - \dot{\epsilon}_z)$$

5.1 Deduction of the present model under biaxial equal stress state

Under biaxial equal stress condition, the parameter ω , generalized triaxiality ratio, generalized triaxiality function, yield condition can be deduce as

$$\omega = q_2 \frac{\sigma_r}{\sigma_M} \quad (60)$$

$$r = 2 \frac{\sinh \omega}{\omega} \quad (61)$$

$$\beta = q_1 f q_2^2 \left(\frac{\sinh \omega}{\omega} \right)^2 \quad (62)$$

$$\sigma_r = \sqrt{1 + (q_1 f)^2 - 2 q_1 f \chi} \sigma_M \quad \text{or} \quad \omega = q_2 \sqrt{1 + (q_1 f)^2 - 2 q_1 f \chi} \quad (63)$$

The restriction between out-plane and in-plane strain rates is

$$\dot{\epsilon}_z = - \frac{2 - q_1 f q_2^2 r / 2}{1 + q_1 f q_2^2 r / 2} \dot{\epsilon}_r \quad (64)$$

Define the proportional coefficient in Eq. (64) as

$$\dot{\epsilon}_z = -\nu \dot{\epsilon}_r \quad (65)$$

then

$$\nu = \frac{2 - q_1 f q_2^2 r / 2}{1 + q_1 f q_2^2 r / 2} \quad (66)$$

The strain rate relation and void evolution law are respectively specified by

$$\dot{\epsilon}_r = \frac{3}{2} \frac{1}{1 + \nu} \frac{(1 - f) \sqrt{1 + (q_1 f)^2 - 2 q_1 f \chi}}{1 + (q_1 f)^2 - 2 q_1 f \chi + q_1 f \psi \omega} \dot{\epsilon}_M \quad (67)$$

$$\dot{f} = (1 - f)(2 - \nu) \dot{\epsilon}_r \quad (68)$$

where χ and ψ are defined in Eq. (9) and (10) respectively.

5.2 Evolvement of conventional models under biaxial equal stress condition

The specific formulations of the conventional model under biaxial equal stress condition can be degenerated directly from the general forms of Eqs. (15)–(22) as follows:

The triaxiality ratio, triaxiality function and yield condition are respectively

$$r_t = 2 \quad (69)$$

$$\beta_t = 12 \frac{B}{A} \quad (70)$$

$$\sigma_r = \sqrt{\frac{3C}{A} \frac{1}{1 + \beta_t}} \sigma_M \quad (71)$$

Define the proportional coefficient between out-plane and in-plane strain rates

$$\dot{\epsilon}_z = -\nu_t \dot{\epsilon}_r \quad (72)$$

then

$$\nu_t = \frac{2 - \beta_t}{1 + \beta_t} \quad (73)$$

The strain rate relation and void evolution law can be respectively written as

$$\dot{\epsilon}_r = \frac{3}{2} \frac{1 - f}{1 + \nu_t} \sqrt{\frac{A}{3C} \frac{1}{1 + \beta_t}} \dot{\epsilon}_M \quad (74)$$

$$\dot{f} = (1 - f)(2 - \nu_t) \dot{\epsilon}_r \quad (75)$$

5.3 Calculation results

The numerical processes for both the present

model and conventional models under biaxial equal stress condition are the same as those under uniaxial stress condition.

1) Triaxiality ratio

In the present model, from Eqs. (57), (60) and (61), the following limit values are obtained

$$f \rightarrow 0, \quad \sigma_r = \bar{\sigma}_0 \rightarrow \sigma_M, \quad \omega \rightarrow \omega_0 = q_2 = 1.1482, \quad r \rightarrow r_0 = 2.4693 \quad (76)$$

$$f \rightarrow f_u, \quad \omega \rightarrow \omega_u = 0, \quad r \rightarrow r_u = 2 \quad (77)$$

In Fig. 10, r gradually approaches r_t when f reaches f_u under biaxial equal stress condition.

2) Triaxiality function

By Eq. (62) and expression (77), the limit values of the generalized triaxiality function may be calculated

Fig. 10 Variation in triaxiality ratio vs. void fraction under biaxial equal tensile stress condition

Fig. 11 Variation in triaxiality function vs. void fraction under biaxial equal tensile stress condition

$$f \rightarrow 0, \quad \beta \rightarrow \beta_0 = 0 \quad (78)$$

$$f \rightarrow f_u, \quad \beta \rightarrow \beta_u = q_2^2 r_u^2 / 4 = 1.3184 \quad (79)$$

Comparison between Fig. 11 and Fig. 2 reveals that at a given void fraction, the (generalized) triaxiality function under biaxial equal stress condition is about four times those under uniaxial stress condition. Hence a material element under biaxial equal stress condition is much more brittle than that under uniaxial stress condition.

3) Predicted ν - f curves

Parameter ν in Eq.(65) is very similar to Poisson's ratio in Eq.(40). Both of them are characteristic parameters representing the compressibility

Fig. 12 Variation in the ratio of out-plane to in-plane strain rate vs. void fraction under biaxial equal tensile stress condition

Fig. 13 Influence of void fraction on yield behaviours under biaxial equal tensile stress condition

of porous materials. At limit conditions, Eq. (66) and expression (77) lead to

$$f \rightarrow 0, \quad \nu \rightarrow \nu_0 = 2 \quad (80)$$

$$f \rightarrow f_u, \quad \nu \rightarrow \nu_u = \frac{2 - q_2^2}{1 + q_2^2} = 0.2940 \quad (81)$$

Only small divergences exist in the predicted ν - f curves in Fig. 12.

4) Yield curves and strain rate ratio

The yield behaviours (Fig. 13) and strain rate ratio (Fig. 14) under biaxial equal stress condition are very similar to those under uniaxial stress condition (Figs. 4 and 5).

5) Predicted stress-strain curves and void growth curves

Figs. 15 - 18 are stress-strain curves and corresponding void growth curves for the four models. If the σ_M - ϵ_M relation in Eq.(23) is considered to be the hardening curve of matrix material under uniaxial stress condition, then the hardening curve of matrix material under biaxial equal stress condition can be represented by σ_M - $\epsilon_M/2$ relation. The σ_M - $\epsilon_M/2$ relation is also indicated by a dotted line in stress-strain curves for comparison.

Figs. 15 - 18 resemble Figs. 6 - 9 respectively to a large extent. Hence the corresponding conclusions in case of uniaxial stress condition keep valid in case of biaxial equal stress condition. But biaxial equal stress condition has its own features. Because of the higher triaxiality ratio, the biaxial equal stress condition has the following features in comparison with uniaxial stress condition :

- (a) much lower peak stress ;
- (b) much earlier appearance of the peak point ;
- (c) much more rapid void growth ;

Fig. 14 Variation in strain rate ratio vs. void fraction under biaxial equal tensile stress condition

Fig. 15 Influence of initial void fraction on stress-strain curves under biaxial equal tensile stress condition for present model

Fig. 17 Comparison between the stress-strain curves under biaxial equal tensile stress condition ($f_0 = 0.00583$ & 0.0507)

Fig. 16 Influence of initial void fraction on void evolution under biaxial equal tensile stress condition for present model

Fig. 18 Comparison between the void evolution curves under biaxial equal tensile stress condition ($f_0 = 0.00583$ & 0.0507)

(d) much faster softening processes.

Now keep an eye on the curves with lower initial void fraction ($f_0 = 0.00583$) in Figs. 17 and 18. In Fig. 18, the void growth rate given by the present model is larger than those of three conventional models. In Fig. 17, extremely nice coincidence is reached both at the initial yield point and in the hardening processes of four models. Especially for the three conventional models, nice agreement can be kept even at the peak point and initial softening processes. However, the peak point and the peak value predicted by the present model are earlier and much lower than those predicted by conventional models. This is an interest-

ing phenomenon that is worthy of further testing by experiment. Though the Shima model possesses a sound experimental basis, the experimental data was mainly obtained under compressive stress states. Under relative higher triaxial tensile stress states, experimental results were scarcely found in the past. The authors in this paper tend to hold the view that conventional models based on quadratic yield functions (Eq. (14)) overestimate the peak stress of porous materials with relative lower initial void fraction and under higher tensile triaxiality.

6. Response of Porous Material under Uniaxial Tensile Strain Condition

Suppose a porous material element is loaded in z axis with the deformation in x and y directions constrained completely, that is

$$\sigma_z \geq 0, \quad \sigma_x = \sigma_y = \sigma_r \leq 0 \quad (82)$$

$$\bar{\sigma}_0 = \sigma_z - \sigma_r \quad (83)$$

$$\dot{\epsilon}_x = \dot{\epsilon}_y = 0, \quad \dot{\epsilon}_z \geq 0 \quad (84)$$

$$\dot{\epsilon}_{kk} = \dot{\epsilon}_z \quad (85)$$

In practice, uniaxial strain compression does exist, such as closed-die compaction of porous materials. But uniaxial tensile strain condition is difficult or even impossible to achieve. In spite of lacking direct engineering background, this imaginary loading state is not useless. High triaxial tensile states, such as the stress states in a necking area or near a crack tip, are vital to the brittle fracture of porous materials. Whether a model can manage such extreme loading states is of utmost importance, while uniaxial tensile strain condition indeed provides a simple example to deal with the response of different models under changing triaxiality — from relatively low level to extremely high level.

6.1 Deduction of present model under uniaxial tensile strain condition

Under uniaxial tensile strain condition, the generalized triaxiality ratio and generalized triaxiality function can be deduced from the general formulations of the present model as follows:

$$r = \frac{2}{q_1 f q_2^2} \quad (86)$$

$$\beta = \frac{1}{q_1 f q_2^2} = \frac{1}{2} r \quad (87)$$

Note that both the generalized triaxiality ratio and generalized triaxiality function are inverse proportional to the void fraction. For simplification, let

$$\eta = \bar{\sigma}_0 / \sigma_M = \sqrt{1 + (q_1 f)^2 - 2 q_1 f \chi} \quad (88)$$

Considering Eqs. (9), (87) and (88)

$$\eta = \sqrt{1 + (q_1 f)^2 + 2/q_2^2 - 2\sqrt{1/q_2^4 + 1/q_2^2 + (1 + 1/q_2^2)(q_1 f)^2}} \quad (89)$$

The yield condition will be

$$\sigma_z = \frac{2}{3} \left(\frac{1}{q_2} \omega + \eta \right) \sigma_M \quad (90)$$

or

$$\sigma_r = \frac{1}{3} \left(\frac{2}{q_2} \omega - \eta \right) \sigma_M \quad (91)$$

Define the proportional coefficient between circumferential and axial stresses as

$$\sigma_r = \zeta \sigma_z \quad (92)$$

then

$$\zeta = \frac{1 - q_2 \eta / 2 \omega}{1 + q_2 \eta / \omega} \quad (93)$$

The strain rate relation and void growth law are respectively written as

$$\dot{\epsilon}_z = \frac{3}{2} \frac{(1-f)\eta}{\eta^2 + q_1 f \psi \omega} \dot{\epsilon}_M \quad (94)$$

$$\dot{f} = (1-f) \dot{\epsilon}_z \quad (95)$$

Different from uniaxial stress and biaxial stress conditions, the constitutive equation under uniaxial strain condition can be written in a more compact form as follows: Integrate Eq.(95) and assume that there is no initial deformation in a porous material element

$$f = 1 - (1 - f_0) \exp(-\epsilon_z) \quad (96)$$

Let

$$\xi = \frac{2}{3} \frac{\eta^2 + q_1 f \psi \omega}{(1-f)\eta} \quad (97)$$

In the light of Eqs.(87), (8), (9), (10) and (89), $\omega = \omega(f)$, $\chi = \chi(f)$, $\psi = \psi(f)$, $\eta = \eta(f)$, and $\xi = \xi(f) = \xi(\epsilon_z)$. Integrate Eq.(94) and presume that initial deformation in matrix material does not exist

$$\epsilon_M = \int_0^{\epsilon_z} \xi(\epsilon_z) d\epsilon_z \quad (98)$$

By Eqs.(90), (23) and (98), the σ_z - ϵ_z relation will be

$$\sigma_z = \frac{2}{3} \sigma_Y \left(\frac{1}{q_2} \omega + \eta \right) \left(1 + \int_0^{\epsilon_z} \xi(\epsilon_z) d\epsilon_z / \epsilon_Y \right)^n \quad (99)$$

6.2 Evolvement of the conventional model under uniaxial tensile strain condition

The triaxiality ratio, triaxiality function, yield condition and strain rate relation are deduced from the general formulations of the conventional model as follows:

$$r_t = \frac{1}{6} \frac{A}{B} \quad (100)$$

$$\beta_t = \frac{1}{12} \frac{A}{B} = \frac{1}{2} r_t \quad (101)$$

$$\sigma_z = 2(1 + \beta_t) \sqrt{\frac{C}{3A} \frac{1}{1 + \beta_t} \sigma_M} \quad (102)$$

or

$$\sigma_r = (2\beta_t - 1) \sqrt{\frac{C}{3A} \frac{1}{1 + \beta_t} \sigma_M} \quad (103)$$

$$\dot{\epsilon}_z = \frac{1-f}{2} \sqrt{\frac{3A}{C} \frac{1}{1 + \beta_t}} \dot{\epsilon}_M \quad (104)$$

Define the proportional coefficient between circumferential and axial stresses as

$$\sigma_r = \zeta_t \sigma_z \quad (105)$$

then

$$\zeta_t = \frac{2\beta_t - 1}{2(1 + \beta_t)} \quad (106)$$

The void growth law in the conventional model is the same as Eq.(96).

6.3 Calculation results

The numerical analysis is easier under uniaxial strain condition, because all quantities can be directly expressed as functions of the void fraction f . In the present model, Eqs.(86), (87), (90), (94), (95), (8), (10) and (23) are theoretically sufficient to portray

the deformation and damage behaviours of porous materials. Making use of Eqs.(96) and (99) will render the calculation more straightforward. At every deformation step, void fraction can be computed from Eq.(96) and all other quantities can be calculated at once by Eqs.(87), (89), (8), (10) and (97). The stress-strain curves may be obtained by Eq.(99) without any problems. For conventional models, the same procedures can be applied.

1) Triaxiality ratio and triaxiality function

Check Eqs.(86) and (87). Under uniaxial strain condition, the triaxiality function is one half of the triaxiality ratio, and the β - f or r - f curve in the present model is exactly a hyperbola where the following limit cases exist:

$$f \rightarrow 0, \quad r \rightarrow \infty, \quad \beta \rightarrow \infty \quad (107)$$

$$f \rightarrow f_u, \quad r \rightarrow r_u = 2/q_2^2 = 1.5170,$$

$$\beta \rightarrow \beta_u = r_u/2 = 0.7585 \quad (108)$$

From Fig. 19, the following conclusions may be drawn.

(a) In common sense, the higher the void fraction, the larger the triaxiality function and the more brittle the porous materials, for example, in the cases of uniaxial stress condition and biaxial equal stress condition. But under uniaxial strain state, Eq.(87) and calculated curves show a completely inverse tendency: the higher the void fraction, the smaller the triaxiality function and the more ductile the porous materials. This is strange at first look. But in consideration of the fact that the uniaxial strain condition is a very special case with strict constraint on circumferential deformation, this conclusion becomes very reasonable. In brief, if and only if the restraint on plastic deformation (especially plastic volume dilatation) is small, higher void fraction is likely to lead to

Fig. 19 Variation in triaxiality function vs. void fraction under uniaxial tensile strain condition

higher brittleness of porous materials. On the contrary, lower void fraction is bound to cause higher brittleness of porous materials at high constraint. During the plastic processing of porous materials with lower void fraction, high triaxial stress state is the most dangerous factor that results in brittle rupture.

(b) The so-called ductility of porous material is just a relative concept. For continuous medium, the ductility merely belongs to the mechanical properties of the material itself. For porous material, the ductility hinges on not only the property of the matrix material, but also the void fraction, void interaction and external loading state. In pace with the variation of void fraction and loading state, the porous material may experience the change from ductile deformation to brittle deformation or in reverse.

(c) The β - f curves predicted by four models agree extremely well. Especially the curves for the approximate model and the present model coincide with each other completely. Fig. 19 confirms further that the generalized triaxiality function β , which is able to synthesize the influences of void fraction, void interaction and generalized triaxiality ratio, is readily capable of depicting quantitatively the variation of the ductility of porous materials under the theoretical frames of conventional models as well as the present model.

2) Predicted ζ - f curves

To get the limit values accurately in present model, solve the simultaneous Eqs.(2), (6), (84), (85) and (88)

$$\eta = q_1 f q_2 \sinh \omega \quad (109)$$

then ω can be expressed by η as

$$\omega = \ln \left[\frac{\eta}{q_1 f q_2} + \sqrt{\left(\frac{\eta}{q_1 f q_2} \right)^2 + 1} \right] \quad (110)$$

Check Eqs.(89), (93) and (110)

$$f \rightarrow 0, \quad \eta \rightarrow \eta_0 = \sqrt{1 + 2(1 - \sqrt{1 + q_2^2})/q_2^2} = 0.4552, \quad (111)$$

$$\omega \rightarrow \omega_0 = \infty, \quad \zeta \rightarrow \zeta_0 = 1$$

$$f \rightarrow f_u, \quad \eta \rightarrow \eta_u = 0, \quad \omega \rightarrow \omega_u = 0,$$

$$\zeta \rightarrow \zeta_u = \frac{1 - 0.5 q_2 (\eta/\omega)_u}{1 + q_2 (\eta/\omega)_u} \quad (112)$$

Both η and ω approach zero when f tends to f_u ; hence at the ultimate state

$$(\eta/\omega)_u = (\eta'/\omega')_u = (d\eta/d\omega)_u \quad (113)$$

Eqs.(89) and (110) mark that $\omega = \omega(\eta, f(\eta))$, so that

$$\frac{d\omega}{d\eta} = \frac{\partial \omega}{\partial \eta} + \frac{\partial \omega}{\partial f} \frac{df}{d\eta} \quad \text{or} \quad \left(\frac{d\omega}{d\eta} \right)_u = \left(\frac{\partial \omega}{\partial \eta} \right)_u + \left(\frac{\partial \omega}{\partial f} \right)_u \left(\frac{df}{d\eta} \right)_u \quad (114)$$

From Eq.(110), the following partial derivatives are obtained

$$\frac{\partial \omega}{\partial \eta} = \frac{1 + \left(\frac{\eta}{q_1 f q_2} \right) / \sqrt{\left(\frac{\eta}{q_1 f q_2} \right)^2 + 1}}{\eta + q_1 f q_2 \sqrt{\left(\frac{\eta}{q_1 f q_2} \right)^2 + 1}} \quad (115)$$

$$\frac{\partial \omega}{\partial f} = -\frac{\eta}{f} \frac{1 + \left(\frac{\eta}{q_1 f q_2} \right) / \sqrt{\left(\frac{\eta}{q_1 f q_2} \right)^2 + 1}}{\eta + q_1 f q_2 \sqrt{\left(\frac{\eta}{q_1 f q_2} \right)^2 + 1}} = -\frac{\eta}{f} \frac{\partial \omega}{\partial \eta} \quad (116)$$

At the ultimate state, Eqs.(115) and (116) reduce to

$$\left(\frac{\partial \omega}{\partial \eta} \right)_u = \frac{1}{q_2} \quad (117)$$

$$\left(\frac{\partial \omega}{\partial f} \right)_u = 0 \quad (118)$$

Differentiate with respect to f at the both sides of Eq. (89)

$$\begin{aligned} \frac{1}{q_1^2 f} \frac{d\eta}{df} \\ = \frac{1 - (1 + 1/q_1^2) / \sqrt{1/q_2^4 + 1/q_2^2 + (1 + 1/q_2^2)(q_1 f)^2}}{\eta} \end{aligned} \quad (119)$$

Let f tend to the ultimate value, the right side of Eq. (119) will approach "0/0", and both sides become

$$\frac{1}{q_1} \left(\frac{d\eta}{df} \right)_u = \frac{q_1}{(1 + 1/q_2^2)^2} \left(\frac{df}{d\eta} \right)_u \quad (120)$$

Solve Eq.(120)

$$\left(\frac{d\eta}{df} \right)_u = \pm q_1 / (1 + 1/q_2^2) \quad (121)$$

Inspect Eq.(119)

$$\frac{d\eta}{df} \leq 0, \quad \forall f \in [0, f_u]$$

so the "+" in Eq.(121) should be abandoned. Substitute Eqs.(114), (117), (118) and (121) into Eq.(113)

$$(\eta/\omega)_u = q_2 \quad (122)$$

Combine Eqs.(112) and (122)

Fig. 20 Variation in stress ratio vs. void fraction under uniaxial tensile strain condition

$$\zeta_u = \frac{1 - q_2^2/2}{1 + q_2^2} = 0.1470 \quad (123)$$

Calculated ζ - f curves in Fig. 20 display that the difference between the present model and conventional models occurs within the scope of high triaxiality. Curves for all of the conventional models overlap each other.

3) Yield curves and strain rate ratio

Similar conclusions can also be drawn from Fig. 21. Within the scope of high triaxiality, porous materials' capacity to resist yielding is much weaker in the present theory than those in conventional theory. Furthermore, in spite of the nice agreement among the three conventional models, their extremely drastic variation in yield behaviours under high

Fig. 21 Influence of void fraction on yield behaviours under uniaxial tensile strain condition

Fig. 22 Variation in strain rate ratio vs. void fraction under uniaxial tensile strain condition

triaxiality seems to be unbelievable.

Though the above divergence is obvious, the predicted $\dot{\epsilon}_z/\dot{\epsilon}_M-f$ curves (Fig. 22) of four models are duplicated with high accuracy, and are very similar to those under uniaxial stress and biaxial equal stress conditions (Figs. 5 and 14).

4) Predicted stress-strain curves and void growth curves

Figs. 23 - 26 are stress-strain curves and void growth curves for four models. Among the three loading cases above, the uniaxial strain condition has the following characteristics:

(a) It is the most dangerous case with the fastest

void growth process (Fig. 24). The void growth curves are upper convex in the case of uniaxial strain condition but concave in the previous two cases.

(b) It has the shortest hardening process and the fastest softening process (Figs. 23, 25 and 26).

(c) The three conventional models are in harmony with each other (Figs. 25 and 26).

(d) At relative higher initial void fraction ($f_0 > 5\%$, $r_0 < 3.8$), the $\sigma_z-\epsilon_z$ curves predicted by the four models are nearer to each other. At lower initial void fraction (or higher triaxiality) ($f_0 < 1\%$, $r_0 > 75.8$), the initial yield stress and peak stress predicted by present model are much lower than those predicted by conventional models. The hardening processes

Fig. 23 Effect of initial void fraction on stress-strain curves under uniaxial tensile strain condition for present model

Fig. 24 Effect of initial void fraction on void evolution under uniaxial tensile strain condition for present model

Fig. 25 Comparison between the stress-strain curves under uniaxial tensile strain condition ($f_0=0.01$)

Fig. 26 Comparison between the stress-strain curves under uniaxial tensile strain condition ($f_0=0.05$ & 0.1)

predicted by conventional models are much shorter and much steeper than those predicted by the present model.

Although lacking experimental data, the present authors suspect that the steep peaks of the conventional model is not a normal mirror of porous materials' behaviours under high triaxiality (Fig. 25). Hence the speculation appeared in biaxial equal stress condition may be further expanded as follows: under high triaxiality, conventional constitutive theories based on quadratic yield function (Eq.(14)) overestimate the initial yield point as well as the peak point of porous materials with low (initial) void fraction.

This is not a speculation without any grounds. As mentioned in Ref. (2), necking process of a tensile bar, where triaxiality is extremely high, is very complicated. It is the elastic-plastic theory based on the Gurson-type yield function rather than any other yield condition that is capable of simulating such a complex process successfully. As it were, the capability to keep valid under high triaxiality is the advantage of the Gurson-type yield function; the capacity to depict the deformation behaviours of porous materials under high triaxiality is the advantage of the constitutive theories based on the Gurson-type yield function.

7. Further Investigation on Parameters q_1 and q_2

In the original version of Gurson's yield function, each individual void is treated separately, with the effect of void interaction neglected. To reveal this interaction, detailed finite element calculations were carried out by Tvergaard for a rigid plastic body with double periodic void array. He found that if two constants q_1 and q_2 were introduced into the Gurson's yield surface, the corrected yield surface in Eq.(1) would fit better to finite element results with consideration on void interaction. Eq.(1) will reduce to the original Gurson's yield function when $q_1=q_2=1$. The numerical data of Tvergaard suggested that $q_1=1.5$ and $q_2=1$.

A few questions may be asked. (a) What is the physical meaning of q_1 and q_2 ? (b) Tvergaard investigated the interaction among voids through complicated micromechanical studies. Is it possible to approach the same goal by macromechanical method? (c) Are the values of q_1 and q_2 calculated by Tvergaard able to represent the void interaction mechanisms of different porous materials? The present research may provide an opportunity to manage such kind of problems.

7.1 Reconsideration on q_1 and q_2

For any ductile porous material, Eqs.(25), (55), (81) and (123) can be changed into

$$q_1 = \frac{1}{f_u} \quad (124)$$

$$q_2 = \sqrt{\frac{2(1-2\mu_u)}{1+\mu_u}} = \sqrt{\frac{2-\nu_u}{1+\nu_u}} = \sqrt{\frac{2(1-\xi_u)}{1+2\xi_u}} \quad (125)$$

Eqs.(124) and (125) may provide a new point of view to understand q_1 and q_2 :

(a) q_1 is a parameter that relates to the ultimate void fraction. Fig. 4 manifests that it may be one of the most important parameters that control the yield behaviours of porous materials.

(b) q_2 is a parameter that correlates the characteristic quantities such as ultimate Poisson's ratio. Fig. 3 shows that it may be one of the most important parameters that regulate the compressibility of porous materials.

(c) Because f_u , μ_u , ν_u and ξ_u are all material constants, q_1 and q_2 are also material parameters. This means, for different porous materials, the values of q_1 and q_2 are different.

(d) The above viewpoints furnish the possibility to measure q_1 and q_2 by macromechanical method, which will be discussed later on in this paper.

7.2 Rerecognition on modified Gurson's yield function

Substitute Eq.(124) and the first equality in expression (125) into Eq.(1)

$$F^*(\sigma_{ij}, \sigma_M, f) = J_2 + \frac{2}{3} \left(\frac{f}{f_u} \right) \sigma_M^2 \cosh \left(\sqrt{\frac{1-2\mu_u}{2(1+\mu_u)}} \times \frac{\sigma_{kk}}{\sigma_M} \right) - \frac{1}{3} \left[1 + \left(\frac{f}{f_u} \right)^2 \right] \sigma_M^2 = 0 \quad (126)$$

Eq.(126) may be understood as follows:

(a) It makes the physical meaning of the modified Gurson's yield function extremely clear. Eq. (126) also reveals the same ideas raised above: it is the ultimate void fraction that mainly restricts the yield behaviours of porous materials; it is the ultimate Poisson's ratio that mainly influences the compressibility of material elements.

(b) Distinct from the original version of Gurson's yield function, the modified version in Eq. (126) is much more material dependent. For the same void fraction, the yield surfaces are different for different porous materials.

7.3 Suggestions on the measurement of q_1 and q_2

q_1 and q_2 may be identified by macroscopic experimental methods. A few of them are picked and listed as follows.

1) Pure shear experiment

As mentioned above, q_1 is a parameter that mainly relates to the yield behaviours of porous materials. Generally, for porous materials, yield and damage evolution are coupled with each other. If such an experiment is designed that yield is uncoupled from damage evolution, the measurement of q_1 may reach

the highest accuracy. This requirement can be satisfied by pure shear experiment. Provided that a porous material element is loaded by pure shear stress τ , then

$$\sigma_{kk}=0, \quad \bar{\sigma}_0^2=3\tau^2, \quad J_2=\tau^2, \quad r=0, \quad \beta=0$$

Let

$$\tau_M^2 = \sigma_M^2 / 3 \quad (127)$$

where τ_M is the shear flow stress of matrix material. The yield curve is obtained from Eq.(126)

$$\frac{\tau}{\tau_M} = 1 - \frac{f}{f_u} \quad (128)$$

Under pure shear deformation, there is no triaxiality, and the damage evolution disappears. The yield curve expressed by Eq.(126) reduces to a line on τ - f plane. Because point $(0, \sigma_M/\sqrt{3})$ is already on the line, the position of the line and the point $(f_u, 0)$ will be known at once if another point (f_0, τ_0) is measured. Theoretically speaking, just "one time" experiment on just "one piece" of specimen with known initial void fraction are necessary to obtain the ultimate void fraction.

2) Triaxial equal stress experiment

Provided that a porous material element is loaded by triaxial equal stress σ , then

$$\sigma_{kk}=3\sigma, \quad \bar{\sigma}_0=0, \quad J_2=0, \quad r=\infty, \quad \beta=\infty$$

The modified Gurson's yield function in Eq.(1) becomes

$$\cosh\left(\frac{3q_2}{2} \frac{\sigma}{\sigma_M}\right) = \frac{1}{2} \left(q_1 f + \frac{1}{q_1 f}\right) \quad (129)$$

Solve q_2 from Eq.(129)

$$q_2 = \frac{2}{3} \frac{\sigma_M}{\sigma} \ln(p \pm \sqrt{p^2 - 1}) \quad (130)$$

where

$$p = \frac{1}{2} \left(q_1 f + \frac{1}{q_1 f}\right) \quad (131)$$

In Eq.(130), "+" is selected when $\sigma > 0$, and "-" is chosen if $\sigma < 0$. Under triaxial equal stress condition, the matrix material does not yield. The so-called yield is caused by the volume change of the voids inside the porous material element and is controlled by Eq.(129). For a given initial void fraction, if corresponding yield stress σ is measured and q_1 is known, q_2 can be calculated by Eq.(130).

3) Uniaxial stress experiment

For porous materials, both pure shear experiment and triaxial equal stress experiment may be too idealistic. Especially for the triaxial equal stress experiment, immense technical difficulties exist. The uniaxial stress experiment may be more practical and simple. Combining Eqs.(35), (36), (38) and (41) will lead to

$$2q_1 f \cosh\left(\frac{q_2}{2} \frac{\sigma_z}{\sigma_M}\right) - (q_1 f)^2 = 1 - \left(\frac{\sigma_z}{\sigma_M}\right)^2 \quad (132)$$

$$q_1 f q_2 \sinh\left(\frac{q_2}{2} \frac{\sigma_z}{\sigma_M}\right) = \frac{1 - 2\mu}{1 + \mu} \frac{\sigma_z}{\sigma_M} \quad (133)$$

For a porous specimen with known initial void fraction, it is easy to measure the initial yield stress σ_z and Poisson's ratio μ in uniaxial stress experiment. Hence q_1 and q_2 can be solved from Eqs.(132) and (133) without difficulty.

As analyzed above, the generalized triaxiality ratio under uniaxial stress condition is very near unity. So the approximate form of Eq.(41) may be valuable

$$\mu \doteq \frac{1 - q_1 f q_2^2 / 2}{2 + q_1 f q_2^2 / 2} \quad (134)$$

If the void fraction in the specimen is large enough, for example near the ultimate value, Eq.(134) can be further written as

$$\mu \doteq \mu_u = \frac{1 - q_2^2 / 2}{2 + q_2^2 / 2} \quad (135)$$

If the required accuracy is not very high, Eq.(134) or (135) can be used directly for the measurement of q_2 .

4) Biaxial equal stress experiment

This method is very similar with the uniaxial stress experiment and is not explained here.

8. Conclusion

Based on present and conventional models, detailed investigations on the deformation and damage characteristics of porous materials under different triaxialities have been carried out. For convenience, the loading and damage states appeared in the above calculation examples are delimited qualitatively and relatively into four categories: (a) low triaxiality ($r \leq 2$) and low void fraction ($f \leq 0.01$); (b) low triaxiality and high void fraction; (c) high triaxiality and high void fraction; (d) high triaxiality and low void fraction. From the theoretical and numerical analysis above, the following conclusions can be drawn.

1) To a large extent, the modified Gurson's yield function is able to predict the yield behaviours of sintered materials in the categories (a), (b) and (c), provided that the void interaction mechanism is well taken into account and modification coefficients q_1 and q_2 are properly selected. Category (d) is an exception and will be further commented below.

2) The present model does possess the capacity to depict the deformation and damage behaviours of porous materials with sufficient accuracy and reliability. Extremely nice agreement with the experimentally-based conventional model has been revealed especially in categories (a), (b) and (c). Exceptional category (d) will be summarized below.

3) The approximate model, as the first order approximation to the present model, is accordant with other conventional models in all the four categories. The nice agreement in category (d) particularly

discloses that this model has either the characteristics or the limitations of all other conventional models based on quadratic yield function. It can be applied to any porous materials to which all other conventional models are suitable to be applied.

4) The physical meanings of q_1 and q_2 in the modified Gurson's yield function have been clarified under the present theoretical framework, which makes it possible to identify these two parameters for different sintered metals by simple experiment.

5) Numerical results expose the great vitality of the concepts of generalized triaxiality ratio and generalized triaxiality function. Under the frameworks of either present theory or conventional theory, the (generalized) triaxiality functions are all able to estimate unanimously the variation in ductility of porous materials in all of the four categories.

6) One of the most important advantages of the present model is that it creates a united theoretical frame which may be suitable for different kind of sintered metals. One of the most intense limitations of conventional models is that one model is usually only suitable for one kind of sintered material.

For different sintered metals, the mechanism and intensity of void interaction may be different. As a result, the yield behaviour and compressibility may also be different. Logically speaking, it is divorced from reality to expect one conventional model, which was usually based on the experimental data of a given sintered metal, to be suitable for all other kind of sintered metals. This viewpoint has been clearly supported by the large divergence between the Shima model and the Doraivelu model in categories (b) (Figs. 4 and 8) and (c) (Figs. 13 and 17).

The advantage of present model is directly inherited from the modified Gurson's yield function which naturally unites the generality with the particularity of different kind of porous materials. The so-called generality means that the influence of hydrostatic stress on yield of all kinds of porous materials can be simply regulated by a cosh ($\cdot$) law, whereas the so-called particularity signifies that q_1 and q_2 can be adjusted to represent the difference in intensity of void interaction for different porous materials. It is the q_1 and q_2 that bring about the present model's flexibility to be applied to different sintered metals.

The key to conventional models is to determine the three functions $A(f)$, $B(f)$ and $C(f)$ in Eq.(14) by experiment, which means that a large number of experiments must be carried out. But in present model, only a few limited experiments are necessary to identify the two constants q_1 and q_2 .

7) Category (d) may be the only exception which shows notably the divergence between the present model and conventional models. In essence, this divergence may be also imputed to the difference between the modified Gurson's yield function and the quadratic yield function under the condition of high triaxiality and low void fraction. The authors speculate in this paper that the quadratic yield function overestimates the yield point within category (d), and that the conventional models overestimate the peak point of sintered metals. Although the first-hand experimental data are not sufficient, this judgement is still derived in the light of second-hand fact and numerical results. If this speculation is correct, another important advantage of the present model, or another important limitation of conventional models, may be revealed. If it is wrong, the numerical analysis in this paper may lay the foundation for further improvement on both the modified Gurson's yield function and the present model.

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