Two-Dimensional Torsional Wave Propagation in Layered Transversely Isotropic Cylinders*

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In this paper, a numerical scheme based on the method of numerical integration along bicharacteristics is developed for the numerical analysis of two-dimensional torsional stress wave propagation in layered transversely isotropic cylinders. The distributions of the dynamic shear stresses in the vicinity of the junction of a layered transversely isotropic cylinder are studied. Moreover, the stability and limitations of the present scheme are investigated.

Key Words: Computational Mechanics, Numerical Technique, Stress Wave, Bicharacteristics, Transversely Isotropic Layered Cylinder

1. Introduction

The purpose of this study is to develop a numerical scheme for analyzing transient torsional stress wave propagation in layered transversely isotropic cylinders when torque is suddenly applied to its end surface. The governing equations form a hyperbolic system of partial differential equations with three independent variables. The need for solving such problems using a numerical method is dictated by the well-known difficulty of obtaining an exact solution. Among numerical procedures, the method of characteristics⁽¹⁾⁻⁽³⁾ appears to be the most suitable for analyzing the stress wave propagation problems. This method has been successfully applied to solving two-dimensional torsional wave propagation problems in isotropic solids, such as the torsional elastic-viscoplastic propagation problems in circular cylinders^{(4),(5)}, composed cylinder⁽⁶⁾ and stepped bar⁽⁷⁾. Recently, the problem of a transient torsional wave in a transversely isotropic cylinder was studied by using this

method⁽⁸⁾.

For generalizing the method of characteristics to the numerical analysis for the two-dimensional wave propagation in a layered transversely isotropic media, the key is how to determine the calculation mesh. We take the same time mesh and space mesh which is dependent on the element with the highest wave speed. So, the treatment of the mesh in the interfaces is then easy. The basic difficulty is to evaluate the stability and convergence of the method. In order to give a proper Courant-Friedrichs-Lewy (CFL) number, we present an approximate necessary stability condition by using the von Neumann necessary condition, and perform a numerical calculation for verifying the result of the stability analysis.

2. Governing Equations

The cylinder geometry and the coordinates used are shown in Fig. 1. The geometry is composed of m perfectly bonded transversely isotropic layers and subjected to torsional impact loading at its end surface $z=0$. Owing to axial symmetry, it is sufficient for the wave propagation in the plane $\theta=0$. The dynamic equation and constitutive equations for cylinder i ($i=1, \dots, m$) are

$$\sigma_{r\theta, r} + \frac{2\sigma_{r\theta}^i}{r} + \sigma_{\theta z, z} = \rho^i \nu_{\theta, t}^i,$$

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Fig. 1 Composed cylinder and coordinate

$$\sigma_{r\theta}^i = \frac{c_{11}^i - c_{12}^i}{2} \left(\nu_{\theta, r}^i - \frac{\nu_{\theta}^i}{r} \right),$$

$$\sigma_{\theta z}^i = c_{44}^i \nu_{\theta, z}^i, \quad (i=1, \dots, m), \quad (1)$$

where $\sigma_{r\theta}^i$ and $\sigma_{\theta z}^i$ denote the shear stress components, ν_{θ}^i the particle velocity component in θ direction, ρ^i the mass density, t the time, c_{11}^i , c_{12}^i and c_{44}^i the elastic coefficients. In this paper, $,r, ,z$ and $,t$ denote differentiation with respect to r, z and t , respectively. Equations (1) can be written in the following matrix form

$$A_1^i U^i, t + A_2^i U^i, r + A_3^i U^i, z + B U^i = 0 \quad (i=1, \dots, m), \quad (2)$$

where vector U^i and matrices A_1^i and B are given by

$$U^i = \begin{bmatrix} \nu_{\theta}^i \\ \sigma_{\theta z}^i \\ \sigma_{r\theta}^i \end{bmatrix}, \quad A_1^i = \begin{bmatrix} \rho^i & 0 & 0 \\ 0 & 1/c_{44}^i & 0 \\ 0 & 0 & 2/(c_{11}^i - c_{12}^i) \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & -2/r \\ 0 & 0 & 0 \\ 1/r & 0 & 0 \end{bmatrix}.$$

Here A_1^i and A_3^i denote symmetric matrices with the following nonzero components:

$$A_{r13}^i = A_{r31}^i = A_{z12}^i = A_{z21}^i = -1,$$

the other components of A_1^i and A_3^i are zero.

3. Difference Equations

The characteristics cone for Eq.(1) can be obtained as

$$(t - t_0)^2 = \frac{(r - r_0)^2}{(c_1^i)^2} + \frac{(z - z_0)^2}{(c_2^i)^2}, \quad (3)$$

where $c_1^i = \sqrt{(c_{11}^i - c_{12}^i)/(2\rho^i)}$ and $c_2^i = \sqrt{c_{44}^i/\rho^i}$ are, respectively, torsional wave speeds in the r and the z directions. The characteristic cone has two parts extending forward and backward in time. The point (t_0, r_0, z_0) which can be a general point in the cylinder i ($i=1, \dots, m$) is the vertex of the cone.

The difference equations at the point (t_0, r_0, z_0) can be derived by integrating differential relations along bicharacteristics which are generators of the backward cone. The expressions for unknown increments $\delta\nu_{\theta}^i$, $\delta\sigma_{r\theta}^i$ and $\delta\sigma_{\theta z}^i$ are given by⁽⁸⁾

$$\delta\nu_{\theta}^i = \frac{k}{(c_1^i)^2 \rho^i r [(c_1^i)^2 k^2 + 2r^2]} \{ (c_1^i)^2 r [2r^2 (\sigma_{r\theta}^i, r + \sigma_{\theta z}^i, z) + 4r\sigma_{r\theta}^i + \rho^i (c_1^i)^2 k (r^2 \nu_{\theta, rr}^i + r \nu_{\theta, r}^i - \nu_{\theta}^i) + (c_2^i)^2 k^2 r \sigma_{r\theta, zz}^i] + (c_1^i)^4 k^2 [r^2 (\sigma_{r\theta, rr}^i + \sigma_{\theta z, rz}^i) - 2\sigma_{r\theta}^i + 2r\sigma_{r\theta, r}^i] + \rho^i k r^3 (c_1^i c_2^i)^2 \nu_{\theta, zz}^i \} + O(k^3),$$

$$\delta\sigma_{r\theta}^i = \frac{k}{2r [(c_1^i k)^2 + 2r^2]} \{ (c_1^i)^2 k [2r^2 (\sigma_{r\theta}^i, r - \sigma_{\theta z}^i, z) + 2r^3 (\sigma_{r\theta, rr}^i + \sigma_{\theta z, rz}^i) - 8r\sigma_{r\theta}^i - \rho^i (c_1^i)^2 k (r^2 \nu_{\theta, rr}^i - r \nu_{\theta, r}^i + \nu_{\theta}^i) - r (c_2^i k)^2 \sigma_{r\theta, zz}^i] + 4\rho^i (c_1^i r)^2 (r \nu_{\theta, r}^i - \nu_{\theta}^i) - \rho^i k^2 r^2 (c_1^i c_2^i)^2 \nu_{\theta, zz}^i \} + O(k^3),$$

$$\delta\sigma_{\theta z}^i = \frac{k (c_2^i)^2}{2r (c_1^i)^2} \{ k (c_1^i)^2 [r (\sigma_{r\theta}^i, rz + \sigma_{\theta z}^i, zz) + 2\sigma_{r\theta}^i, z] + 2r\rho^i (c_1^i)^2 \nu_{\theta, z}^i \} + O(k^3), \quad (4)$$

where δf denotes the quantity $f(t_0, r_0, z_0) - f(t_0 - k, r_0, z_0)$ and k is the time step.

When the point (t_0, r_0, z_0) is located on the interface between cylinder i and cylinder $i+1$, for cylinder i , some bicharacteristics will intersect the plane $t=t_0 - k$ at points outside the cylinder i ; for cylinder $i+1$, the same situation occurs. Therefore, the relations along these bicharacteristics cannot be used. On the interface, the continuous conditions of the stress component $\sigma_{\theta z}$ and velocity component ν_{θ} become

$$\sigma_{\theta z}^i = \sigma_{\theta z}^{i+1},$$

$$\nu_{\theta}^i = \nu_{\theta}^{i+1}. \quad (5)$$

Taking a similar way presented in Ref.(8) and using Eq.(5), the unknown increments $\delta\nu_{\theta}^i$, $\delta\nu_{\theta}^{i+1}$, $\delta\sigma_{r\theta}^i$, $\delta\sigma_{r\theta}^{i+1}$, $\delta\sigma_{\theta z}^i$ and $\delta\sigma_{\theta z}^{i+1}$ can be obtained as

$$\delta\nu_{\theta}^i = \delta\nu_{\theta}^{i+1} = A_4 \left\{ \frac{1}{2} r A_1 A_2 c_2^{i+1} (c_1^i)^2 \left[2(c_1^{i+1} r)^2 (\sigma_{r\theta}^i, r + \sigma_{\theta z}^{i+1}, z) + 4r\sigma_{r\theta}^i (c_1^{i+1})^2 + \frac{1}{2} k A_2 B_1 (c_1^{i+1})^2 + k r B_3 c_2^{i+1} (c_1^{i+1})^2 + r (k c_1^{i+1} c_2^{i+1})^2 \sigma_{r\theta, zz}^{i+1} + r k^2 (c_1^{i+1})^4 \sigma_{\theta z, rz}^{i+1} + \frac{1}{2} k A_2 (r c_2^{i+1})^2 \nu_{\theta, zz}^{i+1} + A_2 c_2^{i+1} r^2 \nu_{\theta, z}^{i+1} \right] + \frac{r c_2^i}{4} (A_1 c_1^{i+1} c_1^i)^2 \left[2r B_4 + \frac{1}{2} k A_2 B_1 + r (k c_1^{i+1})^2 \sigma_{\theta z, rz}^{i+1} \right] + \frac{1}{4} A_1 A_2 c_2^i (c_1^i c_1^{i+1})^2 [2B_4 r^2 - B_2 (k c_1^{i+1})^2] + \frac{1}{8} A_1 B_1 k r c_2^i (A_1 c_1^i c_1^{i+1})^2 + \frac{1}{4} r c_2^{i+1} (A_1 c_1^i)^2 \left[\frac{1}{2} k A_2 B_1 (c_1^{i+1})^2 + 2r^2 (c_1^{i+1})^2 \sigma_{\theta z, z}^{i+1} + k r B_3 c_2^{i+1} (c_1^{i+1})^2 + r (k c_2^{i+1} c_1^{i+1})^2 \sigma_{r\theta, zz}^{i+1} + \frac{1}{2} k A_2 (r c_2^{i+1})^2 (c_1^{i+1} - c_1^{i+1}) \nu_{\theta, zz}^{i+1} + A_2 c_2^{i+1} r^2 \nu_{\theta, z}^{i+1} \right] - \frac{1}{4} k A_1 (r c_1^i c_1^{i+1} c_2^i)^2 (A_1 + A_2) [(r \sigma_{r\theta}^i, rz + r \sigma_{\theta z}^i, zz + 2\sigma_{r\theta}^i, z) - k c_2^i \sigma_{r\theta, zz}^i] + A_3 \right\},$$

$$\begin{aligned}
\delta\sigma_{r\theta}^i &= \frac{k}{r^2(A_1+A_2)} \left\{ \frac{1}{2} A_1 r [A_2(r\nu_{\theta}^i, r - \nu_{\theta}^i) + kr(c_1^{i+1})^2 \sigma_{\theta z, rz}^{i+1}] + \frac{1}{2} k A_1 (c_1^i)^2 (r^2 \sigma_{\theta z, rz}^i - B_2) \right. \\
&\quad \left. - \frac{1}{4} A_1 r (A_1 + A_2) \delta\nu_{\theta}^{i+1} - \frac{1}{2} k A_2 B_2 (c_1^{i+1})^2 + \frac{1}{2} r (A_1)^2 (r\nu_{\theta}^i, r - \nu_{\theta}^i) \right\}, \\
\delta\sigma_{r\theta}^{i+1} &= \frac{A_2}{A_1} \delta\sigma_{r\theta}^i, \\
\delta\sigma_{\theta z}^i &= \delta\sigma_{\theta z}^{i+1} = A_5 \left\{ \frac{1}{4} A_1 A_2 r \left[2r(c_1^{i+1})^2 (r\sigma_{r\theta}^i, r + r\sigma_{\theta z, z}^{i+1} + 2\sigma_{r\theta}^i) + \frac{1}{2} A_2 B_1 k (c_1^{i+1})^2 \right. \right. \\
&\quad \left. \left. + krB_3 c_2^{i+1} (c_1^{i+1})^2 + r(kc_1^{i+1} c_2^{i+1})^2 \sigma_{r\theta, zz}^{i+1} + rk^2 (c_1^{i+1})^4 \sigma_{\theta z, rz}^{i+1} + \frac{1}{2} A_2 k (rc_2^{i+1})^2 \nu_{\theta, z}^{i+1} \right. \right. \\
&\quad \left. \left. + A_2 r^2 c_2^{i+1} \nu_{\theta, z}^{i+1} \right] + \frac{1}{4} r (A_1)^2 \left[\frac{1}{2} k A_2 B_1 (c_1^{i+1})^2 + 2r^2 (c_1^{i+1})^2 \sigma_{\theta z, z}^{i+1} + krB_3 c_2^{i+1} (c_1^{i+1})^2 \right. \right. \\
&\quad \left. \left. + r(kc_1^{i+1} c_2^{i+1})^2 \sigma_{r\theta, zz}^{i+1} + \frac{k}{2} A_2 (rc_2^{i+1})^2 \nu_{\theta, z}^{i+1} + A_2 r^2 c_2^{i+1} \nu_{\theta, z}^{i+1} \right] - \frac{1}{4} B_2 (c_1^{i+1})^4 (kA_2)^2 \right. \\
&\quad \left. + \frac{1}{2} (rA_2 c_1^{i+1})^2 (2\sigma_{r\theta}^i + r\sigma_{r\theta}^i, r) - \frac{1}{2} A_1 A_2 k^2 (c_1^i)^4 (r^2 \sigma_{\theta z, rz}^i - B_2) - \frac{1}{8} rc_2^i A_1 A_2 (A_1 \right. \\
&\quad \left. + A_2) (kc_2^i \nu_{\theta, z}^i - 2\nu_{\theta, z}^i) [(kc_1^{i+1})^2 + 2r^2] + \frac{1}{2} r A_1 A_2 (kc_1^i c_1^{i+1})^2 (2\sigma_{r\theta}^i, r + \sigma_{\theta z, z}^{i+1} - \sigma_{\theta z, z}^i) \right. \\
&\quad \left. + \frac{1}{2} A_1 A_2 (kr c_1^i c_1^{i+1})^2 (\sigma_{r\theta}^i, rr - \sigma_{\theta z, rz}^{i+1} + \sigma_{\theta z, rz}^i) - A_1 A_2 (kc_1^i c_1^{i+1})^2 \sigma_{r\theta}^i - A_1 A_2 B_4 r^2 (c_1^i)^2 \right. \\
&\quad \left. + \frac{1}{4} krc_2^{i+1} A_1 A_2 B_3 (kc_1^i c_1^{i+1})^2 + \frac{1}{4} krc_2^{i+1} A_1 A_2 B_3 (kc_1^i c_1^{i+1})^2 + \frac{1}{4} A_1 A_2 k^4 (c_1^i c_1^{i+1} c_2^{i+1})^2 \sigma_{r\theta, zz}^{i+1} \right. \\
&\quad \left. - \frac{1}{4} kr A_1 B_1 (A_2 c_1^i)^2 + \frac{1}{8} A_1 kr (A_2 kc_1^i c_2^{i+1})^2 \nu_{\theta, z}^{i+1} + \frac{1}{4} A_1 rc_2^{i+1} (A_2 kc_1^i)^2 \nu_{\theta, z}^{i+1} \right. \\
&\quad \left. + \frac{1}{4} kc_2^{i+1} B_3 (A_1 kc_1^i c_1^{i+1})^2 + \frac{k}{4} (A_1 kc_1^i c_1^{i+1} c_2^{i+1})^2 \sigma_{r\theta, zz}^{i+1} + \frac{r}{4} c_2^{i+1} (A_1 kc_1^i c_1^{i+1})^2 \sigma_{\theta z, z}^{i+1} \right. \\
&\quad \left. - \frac{1}{4} kr A_2 B_1 (A_1 c_1^i)^2 + \frac{1}{8} kr A_2 (A_1 kc_1^i c_2^{i+1})^2 \nu_{\theta, z}^{i+1} + \frac{1}{4} rc_2^{i+1} A_2 (A_1 kc_1^i)^2 \nu_{\theta, z}^{i+1} \right. \\
&\quad \left. - \frac{1}{2} (A_2 kc_1^i c_1^{i+1})^2 (r\sigma_{\theta z, z}^i - B_2) - B_4 (A_2 rc_1^i)^2 + \frac{1}{4} A_2 kc_2^i (c_1^i)^2 (A_1 + A_2) (r\sigma_{r\theta}^i, rz + r\sigma_{\theta z, zz}^i \right. \\
&\quad \left. + 2\sigma_{r\theta}^i, z - kc_2^i \sigma_{r\theta}^i, zz) [(kc_1^{i+1})^2 + 2r^2] \right\}, \tag{6}
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= c_{11}^i - c_{12}^i, \quad A_2 = c_{11}^{i+1} - c_{12}^{i+1}, \quad B_1 = r^2 \nu_{\theta, rr}^i + r\nu_{\theta, r}^i, r - \nu_{\theta}^i, \quad B_2 = 2\sigma_{r\theta}^i - 2r\sigma_{r\theta}^i, r - r^2 \sigma_{r\theta, rr}^i, \\
B_3 &= r\sigma_{r\theta}^{i+1}, rz + r\sigma_{\theta z, zz}^{i+1} + 2\sigma_{r\theta}^{i+1}, z, \quad B_4 = r\sigma_{r\theta}^i, r + r\sigma_{\theta z, z}^i + 2\sigma_{r\theta}^i, \\
A_3 &= \frac{1}{4} (kc_1^{i+1})^2 (c_1^i)^4 A_1 (A_1 c_2^i + A_2 c_2^{i+1}) (r^2 \sigma_{\theta z, rz}^i - B_2) + \frac{1}{8} r^3 (A_1 + A_2) (A_1 c_1^{i+1} c_2^i)^2 (kc_2^i \nu_{\theta, z}^i - 2\nu_{\theta, z}^i) \\
&\quad - \frac{c_2^{i+1}}{4} B_2 (kA_2 c_1^i)^2 (c_1^{i+1})^4 + \frac{r^2 c_2^{i+1}}{2} (2\sigma_{r\theta}^i + r\sigma_{r\theta}^i, r) (A_2 c_1^i c_1^{i+1})^2, \\
A_4 &= \frac{4k}{rA_1(A_1+A_2)} \left[\frac{c_2^{i+1}}{2} (kA_2 c_1^{i+1} c_1^i)^2 + A_2 c_2^{i+1} (rc_1^i)^2 + \frac{1}{2} A_1 c_2^i (kc_1^i c_1^{i+1})^2 + A_1 c_2^i (rc_1^{i+1})^2 \right]^{(-1)}, \\
A_5 &= \frac{kc_2^i c_2^{i+1}}{r(A_1+A_2)} \left[\frac{1}{2} A_1 c_2^i (kc_1^i c_1^{i+1})^2 + \frac{1}{2} A_2 c_2^{i+1} (kc_1^i c_1^{i+1})^2 + A_2 c_2^{i+1} (rc_1^i)^2 + A_1 c_2^i (rc_1^{i+1})^2 \right]^{(-1)}.
\end{aligned}$$

The quantities on the right hand side of Eqs. (4) and (6) are all the known data at the point $(t_0 - k, r_0, z_0)$. Note that the partial differences $f, r, f, z, f, rr, f, zz, f, rz$ at the point $(t_0 - k, r_0, z_0)$ are approximated by central differences⁽²⁾.

If the point (t_0, r_0, z_0) is located on the boundary, we can make use of admissible boundary conditions instead of unused differential relations which are outside the region of interest.

4. Numerical Examples

In order to investigate the feature of the transient torsional wave propagation in the layered transversely isotropic cylinder, we calculate the problem of the torsional wave in a semi-infinite transversely

isotropic cylinder composed of two different materials, subjected to dynamic shear stress on its end surface. The material constants of material A (stainless-aluminum composite) are: $(c_{11} - c_{12})/2 = 48.79$ GPa, $c_{44} = 49.95$ GPa, $\rho = 5.76 \times 10^3$ kg/m³, and the material constants of material B (stainless-epoxy composite) are: $(c_{11} - c_{12})/2 = 4.23$ GPa, $c_{44} = 4.44$ GPa, $\rho = 5.18 \times 10^3$ kg/m³.

The initial and boundary conditions are given by

$$\begin{cases} \sigma_{r\theta}^i = \sigma_{\theta z}^i = \nu_{\theta}^i = 0 & (i=1, 2) \text{ for } t=0 \\ \sigma_{\theta z} = Q(r, t) & \text{for } z=0, \\ \sigma_{r\theta}^i = 0 & (i=1, 2) \text{ for } r=R \end{cases}$$

$$Q(r, t) = \begin{cases} \frac{\mu r}{R_i} P_0 H(t) & R_i \geq r \geq 0, \\ 0 & R \geq r > R_i \end{cases},$$

where $H(t)$ is the Heaviside function, R the cylinder radius, R_l the radius of the loading area and μ an adjustable parameter. The following values were used: $R=0.02$ m, $L_1=0.075$ m, $R_l=R$, $\mu=1$, $P_0=1.0$ MPa. The spatial mesh size is taken as $h=0.0005$ m, and the time mesh size is taken as $k=0.5h/\max(c_1^i, c_2^i)$.

Figures 2-4 display the distribution of the shear stress $\sigma_{\theta z}$ in the vicinity of the interface of the bar when $t=25.5 \mu\text{s}$, $29.7 \mu\text{s}$, and $34.0 \mu\text{s}$, respectively. When the shear wave front arrives at the interface, the distribution of the torsional stresses $\sigma_{\theta z}$ near the interface is shown in Fig. 2. From Fig. 2, we can see

Fig. 2 Distribution of stress $\sigma_{\theta z}$ near the interface of the composed cylinder at $t=25.5 \mu\text{s}$

Fig. 3 Distribution of stress $\sigma_{\theta z}$ near the interface of the composed cylinder at $t=29.7 \mu\text{s}$

Fig. 4 Distribution of stress $\sigma_{\theta z}$ near the interface of the composed cylinder at $t=34.0 \mu\text{s}$

that the distribution of the torsional stress $\sigma_{\theta z}$ along the radial direction becomes linear if the distance between the cross-section and input wave front is larger than $0.9R$. When $t > 25.5 \mu\text{s}$, the shear wave begins to transmit and reflect from the interface. From Figs. 3-4, it can be seen that while the shear wave continuously reflects, the value of the torsional stress $\sigma_{\theta z}$ of bar A becomes small and its variation gradually approaches a straight line except for that of the torsional stress $\sigma_{\theta z}$ adjacent to the reflection wave front. This phenomenon can be explained by the fact that the sign of the torsional stress $\sigma_{\theta z}$ produced by the input wave is opposite to that produced by the reflection wave. Moreover, after the transmission wave front goes through the interface, the distribution of its torsional stress $\sigma_{\theta z}$ along the radial direction becomes linear.

5. Stability of Difference Equations

It is well known that difference methods for initial value problems described by hyperbolic systems of partial differential equations with constant coefficients are stable in the L_2 -norm if the associated amplification matrix satisfies the uniform bound condition⁽⁹⁾. A uniform bound condition of the amplification matrix given by the von Neumann necessary condition stated that the spectral radius of the matrix be less than or equal to unity. If the amplification matrix is Hermitian, then the von Neumann condition is also sufficient for stability. Since matrix B in Eq.(2) is not constant, the von Neumann test is not strictly applicable to the proposed numerical scheme. However, in order to determine some indication of the stability properties of the technique, we can obtain an approximately useful convergence condition by making each element of B as zero.

The amplification matrix associated with the difference equations for Eq.(2) can be written as

$$S(\theta_1, \theta_2) = R + iJ, \quad (7)$$

where R and J are the real matrices

$$\begin{aligned} R &= I + \left(\frac{k}{h}\right)^2 \left[(\bar{A}_1^i)^2 (\cos \theta_1 - 1) + (\bar{A}_2^i)^2 (\cos \theta_2 - 1) \right] \\ &\quad - \frac{1}{2} \left(\frac{k}{h}\right)^2 (\bar{A}_1^i \bar{A}_2^i + \bar{A}_2^i \bar{A}_1^i) \sin \theta_1 \sin \theta_2, \\ J &= -\left(\frac{k}{h}\right) [\bar{A}_1^i \sin \theta_1 + \bar{A}_2^i \sin \theta_2], \end{aligned}$$

and

$$\bar{A}_1^i = (A_1^i)^{-1} A_{r_1}^i, \quad \bar{A}_2^i = (A_2^i)^{-1} A_{z_2}^i,$$

and θ_1, θ_2 are arbitrary parameters corresponding to arbitrary wavelengths in the r, z directions. For the special case $\theta_1 = \theta_2 = \pi$, the condition that the spectral radius be less than or equal to unity is assured if

Fig. 5 Results of the stability numerical experiment

$$k/h \leq \min_i \left\{ \frac{1}{[(c_1^i)^2 + (c_2^i)^2]^{1/2}} \right\}. \quad (8)$$

It should be noted that the above stability analysis is only a rough approximate result as it does not concern the effect of boundary condition and the matrix B . For the above numerical examples, the CFL number is taken as $\max_i (c_1^i, c_2^i) k/h \leq 0.71$ by Eq. (8). However, the numerical results shown in Fig. 5 display that, for CFL=0.6, the numerical solution is convergent; for CFL=0.68 or 0.70, the numerical solution is divergent.

6. Conclusions

Theoretically, the numerical scheme developed here is applicable to a wide variety of problems for simulating torsional stress wave propagation in layered transversely isotropic solids. However, in view of the fact that the treatment for general curved boundaries and interfaces is difficult using the present technique, the computer program is only made for analyzing the impact problems of which the computational model is shown in Fig. 1.

For evaluating the stability of the numerical method, the stability analysis is performed, as a result, the approximate necessary condition for stability is obtained by means of the von Neumann necessary condition. However, this condition can only give us a guiding rule to select the value of k/h . According to

our experience, we recommend a formula for evaluating k/h as follows

$$k/h \leq \min_i \left\{ \frac{0.9}{[(c_1^i)^2 + (c_2^i)^2]^{1/2}} \right\}.$$

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