Green’s Functions of an External Circular Crack in a Transversely Isotropic Piezoelectric Medium*

Weiqiu CHEN** and Tadashi SHIOYA***

This paper presents an exact analysis of the problem of an external circular crack in a transversely isotropic piezoelectric solid subjected to normal stress as well as electric charge loadings that are symmetric with respect to the crack plane. The recently proposed general solution is used and the potential theory method is employed. To take account of the effect of the electric field, a new potential of a simple layer is introduced. The derived Green’s functions for point force and point charge are completely exact and expressed in terms of elementary functions. Simple form expressions for intensity factors are also obtained. Numerical examples are finally performed.

Key Words: Piezoelectric Medium, Transverse Isotropy, General Solution, Potential Theory, External Circular Crack, Green’s Functions, Intensity Factors

1. Introduction

A number of theoretical works on piezoelectric fracture have been published for the last decade[1]. However, only a few of them were concerned with three-dimensional crack problems of piezoelectric materials (PZMs)[2-6]. Though solutions for problems related to external circular crack in isotropic and transversely isotropic elastic medium have been extensively described in Kassir and Sih[7], there is entirely no corresponding piezoelectric analysis in the literature.

The most technologically important PZMs are poled ceramics that exhibit transverse isotropy with the unique axis aligned along the poling direction. In Cartesian coordinates (with the z-axis being normal to the plane of isotropy), the linear constitutive relations of a transversely isotropic piezoelectric medium (class 6 mm) are[8]:

\[
\begin{align*}
\sigma_x &= c_{11} \frac{\partial u}{\partial x} + c_{12} \frac{\partial v}{\partial y} + c_{16} \frac{\partial w}{\partial z} + e_{31} \frac{\partial \Phi}{\partial z}, \\
\tau_{xz} &= c_{15} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) + e_{31} \frac{\partial \Phi}{\partial y}, \\
\sigma_y &= c_{15} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) + c_{16} \frac{\partial v}{\partial y} + e_{31} \frac{\partial \Phi}{\partial z}, \\
\tau_{yz} &= c_{15} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) + e_{31} \frac{\partial \Phi}{\partial x}, \\
\sigma_z &= c_{33} \frac{\partial u}{\partial z} + c_{35} \frac{\partial v}{\partial y} + c_{36} \frac{\partial w}{\partial x} + e_{33} \frac{\partial \Phi}{\partial z}, \\
\tau_{xz} &= c_{35} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right),
\end{align*}
\]

(1)

where, \( \Phi \) and \( D \) are the electric potential and electric displacement vector, respectively; \( c_{i j}, e_{i j}, \) and \( \epsilon_{ij} \) are the elastic, dielectric, and piezoelectric constants, respectively. The conventional notations of stresses and displacements are employed in Eq. (1).

This paper intends to analyze the external circular crack problem of transversely isotropic piezoelectric materials. Firstly it is assumed that arbitrarily distributed normal stresses and electric surface
charges are applied symmetrically to the upper and lower faces of the crack. The subsequent analysis is completely based on the three-dimensional piezoelectricity by employing the potential theory method. The corresponding elastic analysis for transverse isotropy was recently conducted by Fabrikant et al. In contrast to the pure elasticity, the potential theory method is generalized in the paper by the introduction of a new potential of a simple layer that corresponds to the electric field in PZMs. It is found that the resulting integro-differential equations have the same structure as that for elasticity. Thus the results of Fabrikant et al. are used to obtain the exact expressions for the elasto-electric filed for point loadings. Numerical results are presented and the phenomena of crack enhancement and crack arrest are discussed.

2. The General Solution

The governing equations of piezoelectricity for transverse isotropy can be found in Tiersten. By introducing the tangential complex displacement $U = u + iv$, these equations can be rewritten in a complex form as follows,

$$
\begin{align*}
\frac{1}{2}(c_{11} + c_{66})\Delta U + c_{16}\frac{\partial^2 U}{\partial z^2} + \frac{1}{2}(c_{11} - c_{66})\Delta^2 U \\
+ (c_{15} + c_{66})\Delta \frac{\partial U}{\partial z} + (e_{15} + e_{66})\Delta \frac{\partial \phi}{\partial z} = 0,
\end{align*}
$$

where,

$$
\begin{align*}
\frac{1}{2}(c_{11} + c_{66})\Delta (\Delta U + \Delta U) + c_{15}\Delta w \\
+ c_{16}\frac{\partial^2 w}{\partial z^2} + e_{15}\Delta \phi + e_{66}\frac{\partial \phi}{\partial z} = 0,
\end{align*}
$$

and,

$$
\begin{align*}
\frac{1}{2}(c_{11} + c_{66})\Delta (\Delta U + \Delta U) + e_{15}\Delta w \\
+ e_{16}\frac{\partial^2 w}{\partial z^2} - e_{66}\Delta \phi - e_{15}\frac{\partial \phi}{\partial z} = 0,
\end{align*}
$$

where, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\Lambda = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$, and the overbar indicates the complex conjugate value. The general solution of Eq. (2) proposed by Ding et al. can also be rewritten in the following form:

$$U = A \left( \sum_{i=1}^{2} F_i + iF_i \right), \\
\phi = \sum_{i=1}^{3} \alpha_i \frac{\partial F_i}{\partial z_i},$$

where,

$$
\begin{align*}
a_i &= c_{11} e_{11} - m_{11} s_{11}^2 + c_{44} e_{44} s_{11}^4 \\
&\quad (m_{11} - m_{33} s_{11}^2) s_{11}^4,
\end{align*}
$$

$$
\begin{align*}
a_{i1} &= c_{11} e_{11} - m_{11} s_{11}^2 + c_{44} e_{44} s_{11}^4 \\
&\quad (m_{11} - m_{33} s_{11}^2) s_{11}^4,
\end{align*}
$$

$$
\begin{align*}
m_i &= e_{15}(c_{15} + c_{66}) + e_{66}(e_{15} + e_{66}),
\end{align*}
$$

and $s_i = s_{i2} = c_{66}/c_{44}$, and $s_i^2 (i=1, 2, 3)$ are the roots of the following algebraic equation:

$$as^6 - bs^4 + cs^2 - d = 0,$$

where,

$$a = c_{44}(e_{33}^2 + c_{22} e_{33}),$$

$$b = c_{44} m_2 + e_{33}[c_{44}(c_{11} + c_{66})] + e_{33} m_4 - c_{11} e_{33},$$

$$c = c_{44} m_3 + e_{33}[c_{44}(c_{11} + c_{66})],$$

$$d = c_{11}(e_{33}^3 + c_{22} e_{33}).$$

It is noted here that the general solution given in Eq. (3) is only valid for distinct $s_i$, while different forms should be adopted for other cases.

Moreover, $F_i(x)$ satisfies the following quasi harmonic equation,

$$(\Delta + \frac{\partial^2}{\partial z^2})F_i = 0, \quad (i=1, 2, 3, 4). \hspace{1cm} (6)$$

From Eqs. (1) and (3), the following expressions for stresses and electric displacements are derived:

$$\sigma_i = 2\sum_{i=1}^{3} \frac{\partial}{\partial z_i}[(c_{33} - c_{11}) + c_{33}s_{11}a_{i1} + e_{33}s_{11}a_{i2}]F_i,$$

$$\tau_i = 2\sum_{i=1}^{3} \frac{\partial}{\partial z_i}a_{i1}F_i + i\sum_{i=1}^{3} e_{33}a_{i2}F_i,$$

$$\gamma_{12} = \sum_{i=1}^{3} \left( \sum_{i=1}^{3} \gamma_{12i} \right) F_i,$$

$$\gamma_{13} = \sum_{i=1}^{3} \left( \sum_{i=1}^{3} \gamma_{13i} \right) F_i,$$

$$\gamma_{23} = \sum_{i=1}^{3} \left( \sum_{i=1}^{3} \gamma_{23i} \right) F_i,$$

where,

$$\gamma_{12i} = \gamma_{13i} = \gamma_{23i} = c_{33} s_{11} a_{i1} + c_{33} s_{11} a_{i2},$$

and,

$$\tau_{12} = \gamma_{12i} = \gamma_{13i} = \gamma_{23i} = c_{33} s_{11} a_{i1} + c_{33} s_{11} a_{i2}.$$

(8)

3. The Potential Theory Method

It is firstly considered that an external circular crack, Fig. 1, is located in the plane $z=0$ of an infinite transversely isotropic piezoelectric medium and subjected to arbitrarily distributed normal forces and surface electric charges acting symmetrically on its upper and lower faces. The interior radius of the crack is

![Fig. 1 External circular crack in an infinite piezoelectric body](image_url)
denoted as \( a \). Using the symmetric condition with respect to the crack face, the problem can be turned to a mixed boundary value problem of a piezoelectric half-space \( z \geq 0 \), subject to the following conditions on the plane \( z = 0 \):

\[
\begin{align*}
\omega &= \Phi = 0, \quad \text{for } \rho < a ; \\
\sigma_{e} &= -p, \quad D_{e} = q, \quad \text{for } a < \rho < \infty ; \\
\tau_{n} &= 0, \quad \text{for } 0 \leq \rho < \infty .
\end{align*}
\]

(9)

Hereafter, the cylindrical coordinates \((\rho, \phi, z)\) are alternatively used for the sake of convenience. Conditions (9) can be satisfied by a representation in terms of two harmonic functions \( G \) and \( H \), i.e. \( F_{i}(z) = c_{i} G(z) + d_{i} H(z) \), \((i = 1, 2, 3) ; F_{i}(z) = 0, \)

\[
(10)
\]

where, \( c_{i} \) and \( d_{i} \) are undetermined constants. To satisfy the third condition in Eq. (9), it is assumed \( \sum_{i=1}^{3} c_{i} r_{1} s_{i} = 0, \frac{1}{2} \sum_{i=1}^{3} d_{i} r_{1} s_{i} = 0, \)

(11)

and the two functions \( G \) and \( H \) are given as:

\[
\begin{align*}
G(\rho, \phi, z) &= \int_{0}^{2\pi} \int_{0}^{\infty} \frac{\omega(r, \phi)}{[\rho^{2} + r^{2} - 2\rho r \cos(\phi - \phi')]^{1/2}} r dr d\phi, \\
H(\rho, \phi, z) &= \int_{0}^{2\pi} \int_{0}^{\infty} \frac{\phi(r, \phi)}{[\rho^{2} + r^{2} - 2\rho r \cos(\phi - \phi')]^{1/2}} r dr d\phi,
\end{align*}
\]

(12)

where \( \omega \) and \( \phi \) stand for the crack face displacement \( w(\rho, \phi, 0) \) and electric potential \( \Phi(\rho, \phi, 0) \), respectively. Making use of the property of the potential of a simple layer, the first condition in Eq. (9) is identically satisfied, while for \( \rho > a \) one has

\[
\begin{align*}
\frac{\partial G}{\partial \rho} |_{\rho = a} &= -2\pi \omega = -2\pi \omega(\rho, \phi, 0), \\
\frac{\partial H}{\partial \rho} |_{\rho = a} &= -2\pi \phi = -2\pi \phi(\rho, \phi, 0).
\end{align*}
\]

(13)

The following relations are then obtained from Eqs. (3), (10) and (13):

\[
\begin{align*}
\sum_{i=1}^{3} c_{i} a_{1} &= -\frac{1}{2\pi}, \\
\sum_{i=1}^{3} c_{i} a_{2} &= 0, \\
\sum_{i=1}^{3} d_{i} a_{1} &= 0, \\
\sum_{i=1}^{3} d_{i} a_{2} &= -\frac{1}{2\pi}.
\end{align*}
\]

(14)

Now \( c_{i} \) and \( d_{i} \) are solved from Eqs. (11) and (14) as follows:

\[
\begin{align*}
\left[ \begin{array}{c}
c_{1} \\
c_{2} \\
c_{3} \\
d_{1} \\
d_{2} \\
d_{3}
\end{array} \right] &= \frac{1}{2\pi} \left[ \begin{array}{ccc}
\gamma_{11} & \gamma_{12} & \gamma_{13} \\
\gamma_{21} & \gamma_{22} & \gamma_{23} \\
\gamma_{31} & \gamma_{32} & \gamma_{33}
\end{array} \right]^{-1} \left[ \begin{array}{c}
0 \\
0 \\
0
\end{array} \right] = \left[ \begin{array}{c}
c_{1} \\
c_{2} \\
c_{3}
\end{array} \right] = \left[ \begin{array}{c}
0 \\
0 \\
0
\end{array} \right].
\end{align*}
\]

(15)

Combining Eqs. (7) and (10) with Eq. (12), the expressions of stresses and electric displacements are derived. Taking consideration of the first condition in Eq. (9), the following integro-differential equations are obtained:

\[
\begin{align*}
\rho(\rho_{0}, \phi_{0}) &= -g_{a} A \int_{0}^{2\pi} \int_{0}^{\infty} \frac{\omega(r, \phi)}{[\rho^{2} + r^{2} - 2\rho r \cos(\phi - \phi')]^{1/2}} r dr d\phi, \\
\gamma(\rho_{0}, \phi_{0}) &= -g_{a} A \int_{0}^{2\pi} \int_{0}^{\infty} \frac{\phi(r, \phi)}{[\rho^{2} + r^{2} - 2\rho r \cos(\phi - \phi')]^{1/2}} r dr d\phi.
\end{align*}
\]

(16)

The newly introduced constants \( g_{i} \), \((i = 1, 2, 3, 4)\) are:

\[
\begin{align*}
g_{1} &= -\sum_{i=1}^{3} c_{i} r_{1} s_{i}, \\
g_{2} &= -\sum_{i=1}^{3} c_{i} r_{1} s_{i}, \\
g_{3} &= \sum_{i=1}^{3} c_{i} r_{1} s_{i}, \\
g_{4} &= \sum_{i=1}^{3} d_{i} r_{1} s_{i}.
\end{align*}
\]

(17)

From Eq. (16), it is obtained,

\[
\begin{align*}
g_{a} \rho(\rho_{0}, \phi_{0}) &= -g_{a} \gamma(\rho_{0}, \phi_{0}) \\
&= -\frac{1}{4\pi A^{2}} \int_{0}^{2\pi} \int_{0}^{\infty} \frac{\omega(r, \phi)}{[\rho^{2} + r^{2} - 2\rho r \cos(\phi - \phi')]^{1/2}} r dr d\phi, \\
&= -\frac{1}{4\pi A^{2}} \int_{0}^{2\pi} \int_{0}^{\infty} \frac{\phi(r, \phi)}{[\rho^{2} + r^{2} - 2\rho r \cos(\phi - \phi')]^{1/2}} r dr d\phi,
\end{align*}
\]

(18)

where \( \omega = \frac{1}{4\pi^{2}(\rho_{0} a_{0} - \rho_{0} a_{0})} \). Eqs. (18) and (19) have the similar form as that reported in Fabrikant et al. \(^{10} \) so that the following solutions are obtained,

\[
\begin{align*}
\omega &= \frac{2A}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{\sqrt{R}} \tan^{-1}(\frac{x}{R})[g_{a} \rho(\rho_{0}, \phi_{0}) - g_{a} \gamma(\rho_{0}, \phi_{0})], \\
\gamma &= \frac{2A}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{\sqrt{R}} \tan^{-1}(\frac{x}{R})[g_{a} \rho(\rho_{0}, \phi_{0}) - g_{a} \gamma(\rho_{0}, \phi_{0})],
\end{align*}
\]

(20)

where,

\[
R = [\rho^{2} - \rho_{0}^{2} - 2\rho \rho_{0} \cos(\phi - \phi')]^{1/2}, \\
\xi = (\rho^{2} - a^{2})^{1/2}(\rho_{0}^{2} - a^{2})^{1/2}/a.
\]

(21)

To obtain the complete elastoelectric field, the substitution of Eq. (20) into Eq. (12) gives

\[
\begin{align*}
G(\rho, \phi, z) &= \frac{2A}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} \left[ K(\rho, \phi, z; \rho_{0}, \phi_{0}) \right] \\
&= \left[ g_{a} \rho(\rho_{0}, \phi_{0}) - g_{a} \gamma(\rho_{0}, \phi_{0}) \right] \rho a_{0} dB d\phi, \\
H(\rho, \phi, z) &= \frac{2A}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} \left[ K(\rho, \phi, z; \rho_{0}, \phi_{0}) \right] \\
&= \left[ g_{a} \rho(\rho_{0}, \phi_{0}) - g_{a} \gamma(\rho_{0}, \phi_{0}) \right] \rho a_{0} dB d\phi,
\end{align*}
\]

(22)

where function \( K \) reads

\[
K(M; N_{0}) = K(\rho, \phi, z; \rho_{0}, \phi_{0})
\]

JSME International Journal

Series A, Vol. 42, No. 1, 1999
Here $R$ denotes the distance between respective points: $N(a, \phi, \rho, v, \rho_0, \phi_0, 0, 0)$. The derivatives of $G$ and $H$ with respect to $z$ are obtained from Eq. (22):

$$\frac{\partial G}{\partial z} = -4A \int_0^z \int_0^z \frac{1}{R(M, N)} \tan^{-1} \left[ \frac{(x^2 - a^2)^{1/2}(\rho_0^2 - \rho^2)^{1/2}}{aR(N, N_0)} \right] \cdot \frac{\partial R}{\partial z},$$

$$\frac{\partial H}{\partial z} = -4A \int_0^z \int_0^z \frac{1}{R(M, N)} \left[ g_0 q(\rho_0, \phi_0)\rho_0 d\rho_0 d\phi_0 - g_0 q(\rho_0, \phi_0)\rho_0 d\rho_0 d\phi_0 \right],$$

where

$$j = (x^2 - a^2)^{1/2}(\rho_0^2 - \rho^2)^{1/2}/a \quad \text{and} \quad k = \frac{1}{2} \left[ (x^2 - a^2)^{1/2} + \rho_0^2 - \rho^2 \right]^{1/2}.$$

The derivatives of the function $K$ can be obtained in elementary functions from Eq. (23) by differentiation; these have been calculated by Fabricant et al.\(^{10}\) and are given in the Appendix for the sake of integrity of this paper.

4. Green’s Functions and Intensity Factors

It is supposed that the external circular crack is subjected to a couple of normal concentrated forces $P$ applied in opposite directions at the points $(\rho_0, \phi_0, 0^\circ)$, $\rho_0 > a$ and a couple of point charges $Q$ acting at the points $(\rho_0, \phi_0, \phi_0^\circ)$, $\rho_0 > a$. By utilizing Eqs. (A1)–(A5) in the Appendix, the following exact expressions of Green’s functions for the elastoelectric field are obtained:

$$U = 4A \int \left[ \beta_1 f_1(z)P + \beta_2 a_2 f_2(z)Q \right],$$

$$w = -4A \int \left[ \beta_1 f_1(z)P + \beta_2 a_2 f_2(z)Q \right],$$

$$\Phi = -4A \int \left[ \beta_1 f_1(z)P + \beta_2 a_2 f_2(z)Q \right],$$

$$\sigma_1 = 8A \sum a_{m} \left[ \beta_1 f_1(z)P + \beta_2 a_2 f_2(z)Q \right] + \left[ \beta_1 f_1(z)P + \beta_2 a_2 f_2(z)Q \right],$$

$$\sigma_2 = 8A c_{m} \left[ \beta_1 f_1(z)P + \beta_2 a_2 f_2(z)Q \right],$$

$$\tau_2 = 4A \sum a_{m} \left[ \beta_1 f_1(z)P + \beta_2 a_2 f_2(z)Q \right],$$

$$D = 4A \sum a_{m} \left[ \beta_1 f_1(z)P + \beta_2 a_2 f_2(z)Q \right],$$

where

$$f_1(z) = \frac{1}{a} \left[ \frac{x}{R_k} \tan^{-1} \left( \frac{j}{R_k} \right) - \tan^{-1} \left( \frac{(\rho_k^2 - a^2)^{1/2}}{a} \right) \right]$$

and

$$\tau_2 = \frac{1}{R_k} \tan^{-1} \left( \frac{j}{R_k} \right),$$

$$D = \frac{1}{R_k} \tan^{-1} \left( \frac{j}{R_k} \right).$$

It is interesting now to give the stress and electric displacement intensity factors. The stress intensity factor and electric displacement intensity factor are defined, respectively, as follows

$$k_\sigma = \lim_{\rho \to a}(\rho - a)^{1/2}\sigma_2|_{z=0}, \quad k_D = \lim_{\rho \to a}(\rho - a)^{1/2}D_2|_{z=0}.$$

(27)

These two intensity factors correspond to the Mode I type. It is obtained from Eq. (26) that

$$\lim_{\rho \to a} f_\sigma(z) = -\frac{1}{\rho^2 + \rho_k^2 - 2\rho\rho_k \cos(\phi - \phi_k)} \left(\frac{\rho_k - \rho}{\rho_k - a}\right)^{1/2}. \quad \text{(28)}$$

Making use of it, the intensity factors defined in Eq. (27) are obtained from Eq. (25)

$$k_\sigma = \frac{1}{(2a)^{1/2}} \left[ k_1 P (\rho - a)^{1/2} \right],$$

$$k_D = \frac{1}{(2a)^{1/2}} \left[ k_1 P (\rho - a)^{1/2} \right].$$

(29)

where $k_{\sigma m} = -4A \sum \gamma_{m} \beta_{m}$ ($l, m = 1, 2$). From Eq. (29), it seems as if either the concentrated force or point charge will lead to both the singularities of stress and electric displacement at the crack tip. However, it is not the case; in fact, by utilizing Eq. (17), it can be verified that $K_{11} = K_{33} = 0$ and $K_{13} = -K_{31}$.
\( \frac{1}{\pi^2} \). Therefore, Eq. (29) becomes,

\[
\begin{align*}
  k_\sigma &= \frac{P}{\pi^2(2a)^{3/2}} \frac{(\rho_0^2 - \rho^2)^{1/2}}{\alpha^2 + \rho_0^2 - 2\alpha\rho_0 \cos(\phi - \phi_0)}, \\
  k_\Phi &= \frac{Q}{\pi^2(2a)^{3/2}} \frac{(\rho_0^2 - \rho^2)^{1/2}}{\alpha^2 + \rho_0^2 - 2\alpha\rho_0 \cos(\phi - \phi_0)}.
\end{align*}
\]

(30)

It can be seen immediately that the stress intensity factor is identically the same as that of the pure elasticity both for isotropy and transverse isotropy. The corresponding curves of the nondimensional stress intensity factor \( 100\pi \kappa k_{\sigma} \sqrt{2a} / P \) (or the nondimensional electric displacement intensity factor \( -100\pi \kappa k_{\Phi} \sqrt{2a} / Q \)) versus the nondimensional force position coordinate \( b/a \), Fig. 1, for different values of \( \phi \) are thus exactly the same as those in Fig. A.2.b on page 68 of Kassir and Sih[13]. For the sake of simplicity, these are not repeated in this paper.

The corresponding intensity factors for arbitrarily distributed pressure \( \rho \) and surface charge \( \eta \) can obviously be obtained by integrating Eq. (30) over the crack domain, i.e.

\[
\begin{align*}
  k_\sigma &= \frac{1}{\pi^2(2a)^{3/2}} \int_0^{2\pi} \int_0^\infty \frac{\rho_0^2(\rho_0^2 - \rho^2)^{1/2}}{\alpha^2 + \rho_0^2 - 2\alpha\rho_0 \cos(\phi - \phi_0)} \cdot \rho_0 d\rho d\phi_0, \\
  k_\Phi &= \frac{1}{\pi^2(2a)^{3/2}} \int_0^{2\pi} \int_0^\infty \frac{\eta_0^2(\rho_0^2 - \rho^2)^{1/2}}{\alpha^2 + \rho_0^2 - 2\alpha\rho_0 \cos(\phi - \phi_0)} \cdot \rho_0 d\rho d\phi_0.
\end{align*}
\]  

(31)

5. Numerical Examples

The distributions of the normal stress \( \sigma \) and electric displacement \( D_e \) in the neck of the crack (\( z = 0, \rho < a \)) and the normal displacement \( w \) and electric potential \( \Phi \) at the crack face (\( z = 0, \rho > a \)) are of particular importance in practice because they are directly related to the fail criterion of the components containing external circular crack. For the sake of computational convenience, the following nondimensional quantities are introduced,

\[
\Sigma = a^2 \pi^2 \sigma / P, \quad w = wcM / aP, \quad \xi = x/a, \quad \eta = y/a
\]  

(32)

As one can see from Eq. (25), the concentrated force \( P \) has completely no effect on the distribution of \( D_e \) in the neck. In other words, the distribution of \( D_e \) in the neck is solely due to the application of the concentrated electric charge \( Q \). This is also true when considering the distribution of \( \sigma \) in the neck, which in fact is only caused by \( P \). It can be further seen that the distribution of the nondimensional stress \( \Sigma \) in the neck due to \( P \) applied at point \( (\rho_0, \phi_0, 0) \). However, in regard to giving the distributions of \( w \) and \( \Phi \) at the crack face, the material constants should be primarily known. The following calculation is carried out for the piezoelectric ceramic PZT-6B, whose constants are:

![Fig. 2 Distribution of the nondimensional stress \( \Sigma \) in the neck along the \( x \)-axis due to the concentrated force \( P \) applied at the point \((\lambda a, 0, 0)\)]

![Fig. 3 Distribution of the nondimensional stress \( \Sigma \) inside the neck plane due to the concentrated force \( P \) applied at the point \((1.5a, 0, 0)\)]
Fig. 4  Distribution of the nondimensional displacement $\omega$ at the crack face along the $x$-axis due to the concentrated force $P$ applied at the point $(\lambda a, 0, 0)$

Elastic stiffness ($10^{10}$ Nm$^{-2}$):

$\begin{align*}
c_{11} &= 16.8, & c_{12} &= 6.0, & c_{22} &= 16.3, & c_{44} &= 2.71; \\
\end{align*}$

Piezoelectric coefficients (Cm$^{-2}$):

$\begin{align*}
c_{31} &= -0.9, & c_{32} &= 4.6, & c_{33} &= 7.1; \\
\end{align*}$

Dielectric constants ($10^{-10}$ Fm$^{-1}$):

$\begin{align*}
e_{11} &= 36, & e_{33} &= 34. \\
\end{align*}$

Figure 4 shows the distribution of the nondimensional displacement $\omega$ at the crack face along the positive $x$-axis when the concentrated force $P$ is applied at the point $(\lambda a, 0, 0)$. It can be shown that $\omega$ is singular at the point the force applied. The corresponding surface distribution of $\omega$ at the crack face is shown in Fig. 5, where $P$ is applied at the point $(1.5a, 0, 0)$. The symmetry of the distribution about the $x$-axis is also observed.

The electric potential $\Phi$ due to $P$ is not given here because its distribution is likely the same as the $\omega$'s. In fact, the electric potential is different from the normal distribution just by a constant factor. For example, there is a simple relation $\Phi_{\text{equal}}/P = 0.284\omega$ for PZT-6B. The nondimensional displacement $\omega = \Phi_{\text{equal}}/Q$ due to the concentrated electric charge $Q$ applied at one point can also be obtained from $\omega$ due to $P$ applied at the same point by the relation $\omega' = -0.284\omega$ (for PZT-6B). Simple relation can also be established between the electric potentials due to mechanical and electric point loadings. Based on above discussion, it is clearly that if the concentrated force $P$ and electric charge $Q$ are applied simultaneously in the manner as shown in Fig. 1, the crack will experience a less serious situation than the pure elastic case. This is known as the crack arrest phenomenon. However, the situation changes when either the force is applied in an opposite direction or the electric charge is negative. This will lead to the enhanced effect that has been reported earlier.

Fig. 5  Distribution of the nondimensional displacement $\omega \times 10^6$ at the crack face due to the concentrated force $P$ applied at the point $(1.5a, 0, 0)$

6. Conclusion

The potential theory method has been generalized in the paper to analyze the symmetric problem of an external circular crack in an infinite piezoelectric medium. A new potential is introduced to take account of the effect of electric field. Exact Green's functions for point loadings are obtained in terms of elementary functions, which have not been reported before. The corresponding stress and electric displacement intensity factors, defined in a usual manner, are also given in the paper. It is found that the stress intensity factor is independent of the material constants and is identically the same as that of the pure elasticity. Numerical results are presented for the distributions of relevant quantities in the neck or at the crack surface. The crack enhancement and crack arrest phenomena are discussed.

Acknowledgments

The financial support from the Japanese Ministry of Culture, Education and Science is acknowledged. The work of CWQ was also supported by the Natural Science Foundation of China (No. 19872060) and partly by the Zhejiang Provincial Natural Science Foundation.

Appendix

The following derivatives of the function $K$ can be found in Fabrikanet al.:(30):

$\frac{\partial K}{\partial z} = -2\pi \tan^{-1} \left( \frac{j}{R_0} \right).$     \hspace{1cm}  (A1)

$\Delta K = \frac{2\pi}{i} \left[ \frac{\tilde{z}}{R_0} \tan^{-1} \left( \frac{j}{R_0} \right) \right. \\
\left. - \left( \frac{\tilde{R} - \tilde{d}}{\tilde{s}} \right)^{1/2} \frac{\tan^{-1} \left( \frac{\tilde{y}}{(\tilde{R} - \tilde{d})^{1/2}} \right) - \tan^{-1} \left( \frac{\tilde{y}}{a} \right)}{a} \right].$     \hspace{1cm}  (A2)

$\frac{\partial^2 K}{\partial z^2} = 2\pi \left[ \frac{\tilde{z}}{R_0} \tan^{-1} \left( \frac{j}{R_0} \right) \right.$
\[
\frac{1}{z^2} \left( \frac{x^2}{R^2} - \frac{y^2}{R^2 - R^2} \right)
\]
\[+ \frac{1}{z} \left( \frac{x}{R^2} - \frac{y}{R^2 - R^2} \right) \frac{\partial^2}{\partial y^2} \frac{1}{z^2} \left( \frac{x^2}{R^2} - \frac{y^2}{R^2 - R^2} \right) \] \[= 2\pi \frac{1}{z^2} \left( \frac{x}{R^2} - \frac{y}{R^2 - R^2} \right) \frac{\partial^2}{\partial y^2} \frac{1}{z^2} \left( \frac{x^2}{R^2} - \frac{y^2}{R^2 - R^2} \right) \frac{1}{z^2} \left( \frac{x}{R^2} - \frac{y}{R^2 - R^2} \right) \frac{\partial^2}{\partial y^2} \frac{1}{z^2} \left( \frac{x^2}{R^2} - \frac{y^2}{R^2 - R^2} \right)
\]
\[= 2\pi \left( \frac{(\alpha^2 - a^2)^{1/2}}{t} \frac{2}{l} + \frac{\partial a e^{i\phi_0}}{\partial a} \right) \left( \frac{(\alpha^2 - a^2)^{1/2}}{t} \frac{2}{l} + \frac{\partial a e^{i\phi_0}}{\partial a} \right)
\]
\[+ \frac{1}{t^2 R^2} \tan^{-1} \left( \frac{t}{R_0} \right)
\]
\[+ \frac{(\alpha^2 - a^2)^{1/2}}{t^2 R^2} \left( \frac{x^2}{R^2} - \frac{y^2}{R^2} \right) \frac{\partial a e^{i\phi_0}}{\partial a} \left( \frac{x^2}{R^2} - \frac{y^2}{R^2} \right) \frac{\partial a e^{i\phi_0}}{\partial a}
\]
\[= 2\pi \left( \frac{(\alpha^2 - a^2)^{1/2}}{t} \frac{2}{l} + \frac{\partial a e^{i\phi_0}}{\partial a} \right) \left( \frac{(\alpha^2 - a^2)^{1/2}}{t} \frac{2}{l} + \frac{\partial a e^{i\phi_0}}{\partial a} \right)
\]
\[+ \frac{1}{t^2 R^2} \tan^{-1} \left( \frac{t}{R_0} \right)
\]
\[+ \frac{(\alpha^2 - a^2)^{1/2}}{t^2 R^2} \left( \frac{x^2}{R^2} - \frac{y^2}{R^2} \right) \frac{\partial a e^{i\phi_0}}{\partial a} \left( \frac{x^2}{R^2} - \frac{y^2}{R^2} \right) \frac{\partial a e^{i\phi_0}}{\partial a}
\]

where
\[R_0 = \sqrt{\rho^2 + 2\rho \cos(\phi - \phi_0) + \rho^2}
\]
\[l = \frac{1}{2} \left( [(\rho + a)^2 + x^2]^{1/2} - [(\rho - a)^2 + x^2]^{1/2} \right),
\]
\[t = \rho e^{i\phi} - \rho e^{i\phi_0}, \quad s = [\rho e^{i\phi} - (\rho - a)^2]^{1/2}.
\]

References


