Analytical Aspects of Cumulative Superposition Procedure for Elastic Indentation Problems*

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Indentation problems for an elastic half-space admit similarity analysis when the local shape of the contacting body can be expressed by a homogeneous function. In this situation, the solution for curved punches can be obtained by the cumulative superposition of the solution to a single auxiliary problem which amounts to indentation by a flat-ended punch. This procedure avoids treating the moving and unknown contact boundary explicitly, so that the contact region can be determined in an accurate manner. In this study, advantages of this procedure are explored from analytical and numerical points of view. Although the theoretical basis is first described for frictionless indentation of an elastic half-space by a rigid punch, the method is subsequently shown to be applicable to the contact between two elastic bodies and for more general frictional behavior. To demonstrate the use of this superposition principle, the three-dimensional indentation by a punch with elliptic cross-section, as well as the plane-strain indentation by an asymmetric punch are solved by this method. Numerical accuracy of the present procedure is verified employing some examples of plane-strain problems, together with its effectiveness in combination with the application of the boundary element analysis for the reduced flat-punch problem.

Key Words: Elasticity, Contact Problem, Computational Mechanics, Indentation, Superposition, Similarity Scaling, Flat Punch, Hertz Theory, Boundary Element Method

1. Introduction

Hertz(1) put forward one of the most classical theories of contact mechanics, which concerned the small-strain contact between elastic solids with locally smooth surface shapes. Since then, a huge number of elastic contact problems have been studied

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taking advantage of analytical means.

Hill and Storakers\textsuperscript{[7]} showed that for axisymmetric indentation of elastic half-spaces, solutions for curved punches can be constructed by so-called cumulative superposition of a single auxiliary solution for a flat-ended punch. The principle behind this is the integral expression for solutions to moving-boundary problems in terms of a solution to a simpler fixed-boundary problem. This principle is purely of a kinematical nature and applicable to problems with nonlinear material behavior, and has been used to analyze axisymmetric indentation for power-law creeping solids\textsuperscript{[8, 10]} and power-law hardening plastic solids\textsuperscript{[10]}.

In the present paper, the applicability of this method is discussed for several elastic indentation problems. In Section 2, the formulation of the problem is given, and its solution is expressed in terms of an integration (cumulative superposition) of the solution to a fixed-boundary problem. Some extensions of this principle are stated in Section 3. To explain its applicability, Sections 4 and 5 are devoted to the discussion of three-dimensional as well as plane-strain indentation problems solved by cumulative superposition of the flat-punch solution. In Section 6, BEM is employed for the reduced flat-punch problem to show that the efficiency and the accuracy of the procedure is of promise for a wider range of application.

2. Similarity Scaling and Cumulative Superposition

2.1 Statement of the problem

Consider a homogeneous, isotropic and linearly elastic half-space, monotonically indented by a rigid punch normally to its free surface. No friction is assumed temporarily between the punch and the half-space.

The shape of the punch is given by a homogeneous function using polar coordinates as

\[ f = \rho \left( \frac{r}{D} \right)^\varphi F(\theta), \quad x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad (1) \]

with \( \rho \geq 1 \), where \( \beta \) and \( D \) are constants having the dimension of length. The angular function \( F(\theta) \) representing the punch cross-section is normalized as \( F(0) = 1 \).

The boundary value problem, referred to as the original problem, Fig. 1(a), is formulated as

\[ \varepsilon = \frac{1}{2} \left( \frac{\partial u}{\partial x_j} + \frac{\partial u}{\partial x_i} \right) \frac{\partial \sigma}{\partial x_j} = 0, \quad \sigma = C_{ijkl} \varepsilon_{ij}; \quad x_3 > 0, \]

\[ u_0 = h = f(x_1, x_2), \quad \sigma_{33} = \sigma_{23} = 0; \quad x_3 = 0, \quad r \leq C(\theta), \]

\[ \sigma_{13} = \sigma_{23} = 0; \quad x_3 = 0, \quad r > C(\theta). \quad (2) \]

The notation for displacements, strains, stresses and stiffnesses are conventional, and \( h \) denotes the depth of indentation (downward displacement of the origin). The unknown contact boundary is expressed by \( C(\theta) \) using polar coordinates and contains the origin in its interior.

2.2 Incremental problem and similarity scaling

The radial coordinate of the contact boundary at \( \theta = 0 \) is taken as the reference contact size and denoted by \( a = C(0) \). The boundary is then normalized as \( r = aC(\theta) \), \( C(\theta) = C(\theta)/a, \quad C(0) = 1 \).

The monotonic indentation was assumed at the outset, so that \( a \) varies monotonically with \( h \), and the time-dependence of the relevant variables is represented by the parameter \( a \) hereafter.

The original problem given by Eq. (2) is rewritten in an incremental form, Fig. 1(b), as

\[ \dot{\varepsilon} = \frac{1}{2} \left( \frac{\partial \dot{u}}{\partial x_j} + \frac{\partial \dot{u}}{\partial x_i} \right) \frac{\partial \dot{\sigma}}{\partial x_j} = 0, \]

\[ \dot{u}_0 = h = f(x_1, x_2), \quad \dot{\sigma}_{33} = \dot{\sigma}_{23} = 0; \quad x_3 = 0, \quad r \leq aC(\theta), \]

\[ \dot{\sigma}_{13} = \dot{\sigma}_{23} = 0; \quad x_3 = 0, \quad r > aC(\theta), \quad (4) \]

where dots denote infinitesimal time increments. The following transformation is carried out using the reference contact size \( a \):

\[ x_1 = a \bar{x}_1, \quad \dot{u}_0(x_1, a) = h \bar{u}(\bar{x}_1), \]

\[ \varepsilon = (h/a) \bar{\varepsilon}(\bar{x}_1). \quad (5) \]

Fig. 1 Solution of elastic indentation problems by cumulative superposition

Furthermore, a positive scalar $\sum$ with dimension of stiffness is chosen, e.g. $\sum = C_{ijkl}$, for normalization as
\[
\sigma(x, a) = \mathcal{C}(h/a) \sigma(x, x_s), \quad C_{ijkl} = \mathcal{C}_{ijkl}.
\]
(6)

The non-dimensional variables introduced here are assumed to depend only on the normalized spatial coordinates, as turns out true in the sequel.

2.3 Cumulative superposition and the reduced problem

Based on the above variable transformation, the original problem is reduced to
\[
\varepsilon = \frac{1}{2} \frac{\partial \bar{u}}{\partial x} \frac{\partial \bar{u}}{\partial x}, \quad \sigma = \mathcal{C}(h/a) \varepsilon,
\]
\[
\sigma = \mathcal{C}_{ijkl} \varepsilon_{ij}, \quad \varepsilon_{ij} = \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_j}{\partial x_i},
\]
where $\bar{u}_i = \bar{u}(x, a)$, $\bar{u}_i$ is normalized strains, and $\varepsilon_{ij}$ is normalized stresses.

As is clear from the foregoing discussion, solutions to the original problem for different contact sizes $a$ all reduce to a single solution of the flat-punch problem after normalization. Due to this similarity nature, information of time history at any material point is contained on the radial ray extending from the corresponding point to infinity in the reduced problem. The spatial integrations in Eqs. (13)–(15) naturally reflect this idea.

2.4 Application to numerical analysis

The original task is to solve an indentation problem for a given curved punch with the angular function $F(\theta)$. As shown above, this problem is reduced to that for a flat punch, which has a fixed boundary and is easier to solve by numerical means. Once its solution is determined, the original solution can be computed by cumulative superposition of Eqs. (13)–(15). Therefore, with this procedure there is no need to explicitly cope with unknown and moving boundaries typically encountered in contact problems. For general angular functions $F(\theta)$, however, the flat-punch solution in the reduced problem is unknown, and it should be determined in an iterative way based on Eqs. (11) and (12).

3. Extensions

3.1 Contact between two bodies

When the resulting deformation is small, problems for two elastic bodies with smooth shapes in normal contact, Fig. 2, are equivalent to those for two half-spaces governed by the following displacement condition;
\[
\begin{align*}
\mathcal{D}(x_1, x_2, 0) + \mathcal{D}(x_2, x_1, 0) \\
= h - \mathcal{D}(x_1, x_2) - \mathcal{D}(x_2, x_1),
\end{align*}
\]
(16)

where $h$ denotes the mutual approach distance taken from their incipient contact, and two bodies are distinguished by the indices (1) and (2), respectively.

If the shape functions of the two bodies $\mathcal{F}(x_1, x_2)$ and $\mathcal{F}(x_2, x_1)$ have the same degree of homogeneity $\rho,$
the problem possesses a similarity solution. The proof to this is lengthy and omitted here, but its essence can be found elsewhere\textsuperscript{(11),(13)}. Further, when the two bodies have the same stiffness tensor but for the scaling factors $\Sigma_{k\ell}^{(1)} = \Sigma_{k\ell}^{(2)}$, that is,
\begin{equation}
C_{k\ell}^{(1)} / \Sigma_{k\ell}^{(1)} = C_{k\ell}^{(2)} / \Sigma_{k\ell}^{(2)} = C_{k\ell},
\end{equation}
the governing equations for the two half-spaces become indistinguishable and reduce to a single flat-punch problem specified by the boundary condition
\begin{equation}
\tilde{u}_3 = 1; \quad \tilde{x}_3 = 0, \quad \tilde{r} \leq \tilde{C}(\tilde{\theta}),
\end{equation}
which is identical to that discussed in Section 2.

3.2 Adhesive contact and Coulomb friction

Frictionless contact was assumed in the previous discussion. This can be extended to incorporate contacts with Coulomb friction, or with full adhesion. First in the case of adhesive contact (no slip at the interface), the pertinent condition becomes
\begin{equation}
\tilde{u}_1 = \tilde{u}_2 = 0, \quad \tilde{u}_3 = \tilde{h}; \quad \tilde{x}_3 = 0, \quad \tilde{r} \leq \tilde{C}(\tilde{\theta}).
\end{equation}
The same scaling in Eqs. (5) and (6) can be applied to this case, and the problem is reduced to the corresponding flat punch problem.

On the other hand, in the case of Coulomb friction with a constant friction coefficient, either adhesion or slip takes place at each point on the interface. By the same procedure, each condition can be reduced to a form independent of $\tilde{a}$. The history-dependence of friction turns into spatial dependence (non-locality) in the reduced problem, which is analogous to the situation discussed by Biwa and Storåkers\textsuperscript{(10)} for hardening plastic solids.

3.3 Miscellaneous

The foregoing discussion was based on linear elastic behavior. The similarity, which plays an essential role in the present method, holds not only for linear elasticity but also for more general cases where stresses, strains and strain-rates are interrelated by homogeneous functions. These include power-law (nonlinear) elasticity, power-law creep, and power-law hardening plasticity as well as viscoplasticity.

Fig. 2 Normal contact between two elastic bodies

For axisymmetric problems, similarity natures and their numerical applications have been studied for such nonlinear cases\textsuperscript{(8)-(11),(13)-(16)}.

4. Three-Dimensional Indentation by Punch with Elliptic Cross-Section

4.1 Formulation and reduced problem

As the first example of the application, the indentation of an isotropic elastic half-space by a parabolic punch ($p=2$) with elliptic cross-section is described, as shown in Fig. 3(a). A similar version of this problem for linear viscous half-space has been considered by Storåkers et al.\textsuperscript{(11)}. When the principal axes of curvature are taken to coincide with $x_1$- and $x_2$-axes, the punch shape is given by
\begin{equation}
\bar{f}(x_1, x_2) = x_1^2/D_1 + x_2^2/D_2.
\end{equation}
\begin{equation}
D = \beta = D_1, \quad F(\theta) = \cos^2 \theta + (D_1/D_2) \sin^2 \theta.
\end{equation}
It is assumed without loss of generality that the cross-section has the major axis in $x_1$-direction and $D_1 > D_2$.

The problem can be reduced to a flat-punch problem shown as Fig. 3(b) by the aforementioned procedure, though with the contact boundary of an unknown shape $\bar{C}(\theta)$. To proceed, it is assumed that the boundary is also an ellipse having the same principal directions as the punch cross-section, with a non-similar aspect ratio. That is, the major half-axis of the contact boundary is unity and the minor half-axis $\sqrt{1-e^2}$, where $e$ represents the unknown eccentricity of the contact ellipse.

4.2 Results of analysis

The solution to the above reduced problem is available\textsuperscript{(14)}, and the scaled field $\bar{u}_3$ on the free surface $\bar{x}_3 = 0, \bar{r} \leq \bar{C}(\bar{\theta})$ satisfying $\bar{u}_3 = 1$ for $\bar{r} \leq \bar{C}(\bar{\theta})$ is determined as
\begin{equation}
\bar{u}_3(\bar{x}_1, \bar{x}_2, 0) = \frac{1}{2K(e)} \int_0^\infty \frac{dw}{\sqrt{(1+w)(1+w-e^2)w^{1/2}}},
\end{equation}
where $K(e)$ is the complete elliptic integral of the first
kind, and \( \lambda \) a positive root of the following equation.

\[
\frac{\bar{x}_1}{1+\lambda} + \frac{\bar{x}_2}{1-\varepsilon^2+\lambda} = 1. \tag{23}
\]

Using this solution, the relation between the eccentricity \( E, E' = 1 - D_k/D_0 \), of the elliptic cross-section of the original punch, and the eccentricity \( e \) of the resulting elliptic contact boundary is established below. To this end, Eq. (11) is integrated partially as

\[
c^2(\theta) = -\int_C \frac{d\lambda_3}{d\bar{F}} \frac{1}{\bar{x}_1^2} d\bar{F}, \tag{24}
\]

and Eq. (22) is used to rewrite it as

\[
c^2(\theta) = \frac{\cos^2 \theta}{e^2 K(e)} (K(e) - E(e)) \frac{1}{1 - e^2 - K(e)} - \frac{\sin^2 \theta}{e^2 K(e)} \left( \frac{E(e)}{K(e) - E(e)} - 1 \right), \tag{25}
\]

where \( E(e) \) is the complete elliptic integral of the second kind.

The indentation depth \( h \) is related to the reference contact size \( a \) by Eq. (12), and Eq. (1) requires

\[
\cos^2 \theta + \frac{D_i}{D_k} \sin^2 \theta = \cos^2 \theta + \frac{E(e)}{1 - e^2 - K(e)} \sin^2 \theta. \tag{26}
\]

This relation is satisfied by setting \( e \) as

\[
D_i \left( \frac{1}{1 - E(e)} \right) = \frac{E(e)}{1 - e^2 - K(e)} \tag{27}
\]

which validates the assumption made on the ellipticity of the resulting contact region. The obtained relation between the punch cross-sectional eccentricity \( E \) and the contact zone eccentricity \( e \) is delineated in Fig. 4. It is observed that the two eccentricities differ in general, the contact ellipse being slightly more slender than the punch cross-sectional ellipse.

The displacement, strain, and stress fields for the original curved-punch problem can be obtained by cumulative superposition of the corresponding reduced fields. In particular, when the punch cross-section is circular \((D_i/D_k = 1)\), the eccentricity \( e \) vanishes and the results shown here become identical to the classical axisymmetric ones, i.e. \( c^2 = 1/2 \) and \( h = 2a^2/D \).

5. Plane-Strain Indentation by Asymmetric Punch

5.1 Formulation and reduced problem

A similar problem to the one discussed above is the two-dimensional plane-strain indentation problem in which the punch shape is given by

\[
f(x_i) = \begin{cases} \beta_+ (x_i/D)^p, & x_i > 0, \\ \beta_- (|x_i|/D)^p, & x_i < 0, \end{cases} \tag{28}
\]

which is asymmetric with respect to the vertical axis, and \( 0 < \beta_- < \beta_+ \) with no loss of generality. The punch is pressed normally onto the half-plane with constraint on horizontal movement and rotation, Fig. 5 (a).

Using the procedure in Section 2, the problem is reduced to that for a half-plane of Fig. 5(b) for which the boundary condition is given by \( \bar{a}_3 = 1 \) in the interval \( \eta \leq \bar{x}_3 \leq 1 \) and \( \bar{x}_3 = 0 \), where an unknown parameter \( \eta \) relates to the contact edge for the negative direction of \( \bar{x}_3 \)-axis.

5.2 Results of analysis

The solution to the reduced problem is readily obtained from a known elastic solution for a flattened punch \((\psi)\) by appropriately changing the coordinate frame, i.e.

\[
\begin{align*}
\psi(\xi) & : \bar{x}_1 > 1, \\
\psi(-\xi) & : \bar{x}_1 < -\eta,
\end{align*} \tag{29}
\]

\[
\xi = \frac{\bar{x}_1 - (1 - \eta)}{2 (1 + \eta)} / 2, \tag{30}
\]

Substitution of the result into the plane-strain version of Eqs. (11) and (12) establishes the connection between the unknown parameter \( \eta \) and the punch shape parameters \( \beta_+ \) and \( \beta_- \), which yields

\[
\frac{\beta_+}{\beta_-} = \frac{c^p_+}{c^p_-}, \tag{31}
\]

Fig. 4 Relation between the eccentricities of the punch cross-section and the resulting contact region

Fig. 5 Plane-strain indentation by an asymmetric punch

\begin{center}
(a) Original problem \hspace{1cm} \hspace{1cm} (b) Reduced problem
\end{center}
\[ c_{\alpha} = 1 - \rho \left( \frac{2}{1 + \eta} \right)^p \int_1^\infty \frac{\Psi(\xi)}{[\xi + (1 - \eta)/(1 + \eta)]^{p+1}} d\xi, \]  
(32.a)

\[ c_{\beta} = \left( \frac{1}{\eta} \right)^p - \rho \left( \frac{2}{1 + \eta} \right)^p \times \int_1^\infty \frac{\Psi(\xi)}{[\xi - (1 - \eta)/(1 + \eta)]^{p+1}} d\xi. \]  
(32.b)

The relation between the punch asymmetry parameter \( 0 \leq \beta_\alpha, \beta_\beta \leq 1 \) and the contact edge parameter \( 0 \leq \eta \leq 1 \) is computed numerically and shown in Fig. 6.

As a special case, when \( \beta_\alpha = \beta_\beta \), it implies \( \eta = 1 \) and the integration in Eq.(32) can be carried out analytically, which leads to the known solutions for punch shapes for a wedge \( (p=1) \) and a parabola \( (p=2) \).

6. Hybrid Numerical Analysis with Boundary Elements

6.1 Scope of the analysis

The flat-punch problem introduced in the present method is certainly easier to analyze numerically than its original one, since the boundary conditions are given for fixed regions. In this section, it is examined for plane-strain problems how further efficiency of the analysis can be achieved by employing the boundary element method to solve the reduced problem. To check its accuracy as well, frictionless contact problems for symmetric punches are considered for which exact solutions are available.

The shape of the punch is specified as

\[ f(x_1) = \beta(|x_1|/D)^p. \]  
(33)

The problem is symmetric with respect to \( x_1 \)-axis, and the corresponding reduced problem becomes the indentation by a flat punch with unit half-width, which is a special case of the problem discussed in Section 5.

The reduced problem is now solved by the boundary element method. Consequently, based on the BEM flat-punch solution, Eqs.(11), (13) - (15) are evaluated numerically to obtain the desired original solutions. Below, these are compared with the exact solutions to demonstrate the effectiveness of the method. Of special concern is the accuracy of the numerical evaluation of the cumulative superposition in Eqs.(13) - (15).

6.2 Numerical analysis

The boundary element analysis employed is a standard one with second-order elements. The use of Melan-Mindlin fundamental solution\(^{16}\) satisfying the traction-free condition on the plane boundary especially facilitates the analysis as only the interval \(-1 \leq \tilde{x}_1 \leq 1\) of the boundary is to be discretized. The solution is obtained with the condition \( \tilde{u}_3 = 1 \) for this interval and used to obtain the original curved-punch solutions. Young's modulus was selected as the stress-normalizing factor \( \Sigma \), and Poisson's ratio was put as \( \nu = 0.3 \) when necessary.

Figure 7 shows the so obtained distributions for the free-boundary displacement \( \tilde{u}_3 \) and the contact pressure \( \tilde{p} = -\sigma_{33} \), represented by circles, together with the corresponding exact distributions\(^{34,41}\) represented by solid curves;

\[ \tilde{u}_3 = 1 - \frac{1}{\ln 2} \ln \left( |\tilde{x}_1| + \sqrt{\tilde{x}_1^2 - 1} \right) ; |\tilde{x}_1| \geq 1, \]  
(34)

\[ \tilde{p} = \frac{1}{\ln \left( 4(1 - \nu^2)/\sqrt{1 - \nu^2} \right)} ; -1 < \tilde{x}_1 < 1. \]  
(35)

In the flat-punch problem, the stress distribution exhibits singularity at the ends of the pressed region \( \tilde{x}_1 \to \pm 1 \). Although a special choice of elements or interpolation functions may be reasonable to treat this singularity\(^{19}\), presently it was found convenient to simply extrapolate the numerical results by a curve with known elastic singular behavior. A preliminary analysis showed that even without such consideration the error to the obtained outcome was insignificant, which demonstrated robustness of the present procedure. No special technique was used for the computa-

![Fig. 6](image)

**Fig. 6** Relation between the punch asymmetry parameter and the contact width ratio

![Fig. 7](image)

**Fig. 7** Surface displacement and contact pressure by a flat-ended punch (reduced problem)


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tion of cumulative superposition formulae except that
fine division of evaluation points was provided near
the contact edges $x_1 = \pm 1$, roughly corresponding to
circle intervals shown in Fig. 7.

It is a feature of two-dimensional elasticity that
the displacement diverges at infinity for the present
loading condition (119). However, this does not make
significant influences on the integration of Eq.(13)
because the integrand contains a factor $\tilde{\beta}^{-(p+1)}$, and
the integration was cut off at certain finite number, say
$\tilde{\beta} = 500$. Furthermore, due to the nature of the problem
there exists uncertainty of rigid-body translation
in the relation between the load and the displacement.
Presently this was circumvented by adequately
specifying the displacement beneath the flat punch (13)(17).
The uniqueness of the displacement so
determined is succeeded by the curved-punch results
appearing below via cumulative superposition.

In the following subsections, some representative
results for wedge-shaped ($p=1$) and parabolic ($p=2$)
punches are illustrated. The parameter $c^p$, which makes
a role similar to $c^0(0)$ in Section 2, is simply
denoted by $c^p$ hereafter. In contrast to axisymmetric
problems where the parameter $c^p$ serves as a quantita-
tive indicator of surface sinking (9), in this plane-
strain situation $c^p$ is merely a numerical constant and
depends on the particular choice of the rigid-body
displacement aforementioned.

### 6.3 Results for wedge-shaped punch

When $p=1$, Eq.(33) represents a wedge with inclination
angle $\delta$ specified by $\tan \delta = \beta/D$. According to
the exact solution to this original problem,

$$ c^1 = \tan \delta \frac{a}{h}, $$

(36)


which compares favorably with the present numerical
solution $c^1 \approx a \tan \delta / h \approx 2.2397$. It is remarked here
that the results in Eq.(36), and Eqs.(37) and (40) in
the sequel have been obtained by analytically evaluating
the derived formulae analogous to Eqs.(11) and

(13) by aid of a symbolic manipulation software
Mathematica (Ver. 3.0, Wolfram Research).

Figure 8 compares the surface displacement and
the contact pressure obtained by the present method
(open circles) to those given by classical elasticity (21)
(solid line);

$$ \frac{t_{sl}}{a \tan \delta} = \begin{cases} 
\frac{1}{c_1} - \frac{x_1}{a}, & x_1 \leq a, \\
\frac{1}{c_1} - \frac{2}{\pi} \frac{x_1}{a} \arctan \left( \frac{1}{\sqrt{(x_1/a)^2 - 1}} \right) + \ln X \\
; x_1 > a,
\end{cases} 
$$

(37)

$$ X = \frac{x_1}{a} + \sqrt{\frac{x_1}{a}} - 1, 
$$

(38)

$$ \frac{p}{E \tan \delta} = \frac{1}{1 - \nu^2} \cosh^{-1} \left( \frac{a}{x_1} \right), \quad 0 < x_1/a < 1. 
$$

(39)

### 6.4 Results for parabolic punch

The case of $p=2$ and $\beta = D$ in Eq.(33) implies a
parabolic punch, which also approximates a circular
cylinder of diameter $D$ in the vicinity of the resulting
contact area. The exact solution in this case yields

$$ c^2 = a^2 / h D = \frac{1}{\ln 2} \approx 1.4427, 
$$

(40)

while the present analysis gives $c^2 = a^2 / (h D) \approx 1.4432$.
Figure 9 illustrates the surface displacement and the
contact pressure, which analytical expressions are

$$ \frac{u_{sl}}{a \tan \delta} = \begin{cases} 
\frac{1}{c_2} - \frac{x_1}{a}, & 0 \leq x_1 \leq a, \\
\frac{1}{c_2} - \frac{1}{2} - \frac{1}{2} \ln X, & x_1 > a,
\end{cases} 
$$

(41)

$$ \frac{p}{E \cdot a / D} = \frac{1}{1 - \nu^2} \sqrt{1 - \frac{x_1}{a}^2}, \quad -a \leq x_1 \leq a, 
$$

(42)

where $X$ is given by Eq.(38). In all cases illustrated
here, the computed results are found almost indistin-
guishable from the corresponding exact solutions.
6.5 Remarks

From the above results, it is seen that the original problem with curved punches can be efficiently and accurately solved by the boundary element analysis of the reduced flat-punch problem and the numerical evaluation of the cumulative superposition. It is also remarked that the results for arbitrary values of \( p \) can be readily constructed systematically from the flat-punch solution in Eqs. (34) and (35). The analogous procedure can be applied to adhesive/frictional contact problems with asymmetric punches without essential difficulty, as partially demonstrated by Ogaki[23].

In contrast to direct numerical methods, the present procedure has an advantage that the contact area size can be quite accurately determined for prescribed loading or displacement data. Namely, the relation \( \delta = a^0 \) is satisfied exactly, and there only remains to obtain the numerical value of the constant \( c^0 \). Furthermore, while conventional numerical methods would demand sophisticated incremental or iterative procedures for the contact area, the present hybrid method only requires discretization of a fixed portion of the boundary and a single-step solution of a boundary value problem.

7. Conclusion

Some features and advantages of the cumulative superposition procedure have been examined for elastic indentation problems from analytical as well as numerical points of view. The applicability of the principle has been illustrated for examples of three-dimensional as well as plane-strain indentation problems. Although the results demonstrated herein can be obtained by existing theories, the present method is believed to have its own virtue in principle. Numerical aspects of this method have been examined by comparison of some plane-strain indentation results with the known exact solutions. The use of boundary element analysis for the reduced flat-punch problem has proven to provide satisfactory accuracy and to facilitate the analysis.

True virtue of this method would be enjoyed for inherently three-dimensional problems, for which no exact solutions exist, and direct numerical methods would meet severe difficulties in determining the contact area in an accurate and effective manner. The present study is thought to provide basic and useful hints toward this end.

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