Finite Element Analysis for Large Deformation Frictional Contact Problems with Finite Sliding*

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In the analysis of frictional contact problems with large deformation, the use of a convected coordinate system is a natural approach, by which the frame indifference of friction law can be maintained. However, in the case of the finite element method, a problem arises due to the discontinuity of the local coordinates between elements when sliding extends over the element boundary. In this work, a method is proposed to solve this problem, i.e., although the formulation is essentially based on a convected coordinate system, the sliding term is redefined as a spatial vector and is calculated using the reference configuration. Thus, finite sliding due to large deformation can be treated, regardless of the limitations of element coordinate systems. Also, the corresponding consistent tangent stiffness is derived to obtain quadratic convergence. The effectiveness of the proposed method is verified in this study by two numerical examples, including an elastoplastic frictional contact problem with large deformation.

Key Words: Computational Mechanics, Finite Element Method, Nonlinear Problem, Contact Problem, Frictional Problem, Finite Sliding

1. Introduction

Designs in engineering are becoming increasingly sophisticated in recent years, and the trend necessitates further development of the finite element methods in order to solve advanced problems. Included among these advanced problems is the large deformation frictional contact one. This is known to be difficult because of the geometrical and material nonlinearities which arise in addition to the nonlinearity that comes from boundary changes. Since a purely analytical approach is not applicable to this type of problem, the finite element methods have been investigated⁽¹⁾. In this case, the contact model was

extended from a point-to-point contact with the infinitesimal deformation to a point-to-surface contact with large deformation. Also, the frictional model has been studied and a friction law, which is analogous to the elastoplastic constitutive law, is used.

In contact problems, the incremental approach is generally taken and sliding is evaluated in each time steps because the problem is essentially path-dependent. It should be noted that the effect of the finite deformation of the structure must be taken into account in this evaluation. From this viewpoint, Laursen et al.⁽²⁾ developed a finite element method based on a convected coordinate system for large deformation frictional contact problems. This method seems reasonable because the relative sliding velocity is well described by the convected coordinates, i.e., the objectivity of the friction law is preserved. On the other hand, it is not easy for this method to deal with the sliding that occurs over the element boundary where local coordinates are discontinuous. Parish et al.⁽³⁾ used a global coordinate system to overcome this problem, but the objective velocity was not introduced in their formulation and hence the

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objectivity of the friction law seems to be lost for large deformation problems.

In this paper we propose a method of solving this problem. Although the formulation is essentially based on a convected coordinate system, the sliding term is redefined as a spatial vector and is calculated using the reference configuration. The finite sliding can thus be treated, regardless of the limitations of element coordinate systems. Furthermore, the corresponding consistent tangent stiffness is derived to assure quadratic convergence. The effectiveness of the proposed method is verified by two numerical examples including an elastoplastic frictional contact problem with large deformation.

2. Finite Element Formulation for Frictional Contact Problems

2.1 Basic relations

Assume that two bodies, 1 and 2, contact each other on the surface ${}^t\gamma_c^1$ of body 1 and ${}^t\gamma_c^2$ of body 2 at time t . Denoting the virtual displacements as $\delta\mathbf{u}^1$ and $\delta\mathbf{u}^2$, the virtual work done by the contact forces ${}^t\mathbf{p}^1$ and ${}^t\mathbf{p}^2$ is given as follows.

$$\begin{aligned}\delta^t W_c &= \int_{{}^t\gamma_c^1} {}^t\mathbf{p}^1 \cdot \delta\mathbf{u}^1 ds + \int_{{}^t\gamma_c^2} {}^t\mathbf{p}^2 \cdot \delta\mathbf{u}^2 ds \\ &= \int_{{}^t\gamma_c^1} [{}^t\bar{p}_n {}^t\mathbf{n} \cdot (\delta\mathbf{u}^1 - \delta\mathbf{u}^2) + {}^t\bar{p}_t {}^t\mathbf{t} \cdot (\delta\mathbf{u}^1 - \delta\mathbf{u}^2)] ds\end{aligned}\quad (1)$$

where ${}^t\mathbf{n}^i$ and ${}^t\mathbf{t}^i$ are the outer normal vector and the tangential vector on the contact surface of body i . Also, the following relations are used in writing the above equation.

$${}^t\gamma_c \equiv {}^t\gamma_c^1 = {}^t\gamma_c^2 \quad (2)$$

$${}^t\mathbf{p}^1 = {}^t\bar{p}_n {}^t\mathbf{n}^1 + {}^t\bar{p}_t {}^t\mathbf{t}^1, {}^t\mathbf{p}^2 = {}^t\bar{p}_n {}^t\mathbf{n}^2 + {}^t\bar{p}_t {}^t\mathbf{t}^2 \quad (3)$$

$${}^t\mathbf{n} \equiv {}^t\mathbf{n}^1 = -{}^t\mathbf{n}^2, {}^t\mathbf{t} \equiv {}^t\mathbf{t}^1 = -{}^t\mathbf{t}^2 \quad (4)$$

$${}^t\bar{p}_n \equiv {}^t\bar{p}_n^1 = {}^t\bar{p}_n^2, {}^t\bar{p}_t \equiv {}^t\bar{p}_t^1 = {}^t\bar{p}_t^2 \quad (5)$$

Next, defining the penetration with the contact points ${}^t\mathbf{x}^1$ and ${}^t\mathbf{x}^2$ of each body, as

$${}^t g = ({}^t\mathbf{x}^2 - {}^t\mathbf{x}^1) \cdot {}^t\mathbf{n} \quad (6)$$

the mechanical and geometrical constraint conditions are given by the following equations.

$${}^t\bar{p}_n \leq 0, {}^t g \geq 0, {}^t\bar{p}_n \cdot {}^t g = 0 \quad (7)$$

Thus, the virtual work equation and the constraint condition are written in the weak form using the reference configuration:

$$\delta^t W_c = \int_{\Gamma_c} [{}^t\bar{p}_n {}^t\mathbf{n} \cdot (\delta\mathbf{u}^1 - \delta\mathbf{u}^2) + {}^t\bar{p}_t {}^t\mathbf{t} \cdot (\delta\mathbf{u}^1 - \delta\mathbf{u}^2)] dS \quad (8)$$

$$\int_{\Gamma_c} \delta^t \bar{p}_n \cdot {}^t g dS = 0 \quad (9)$$

where ${}^t\bar{p}_n {}^t\mathbf{n}$ and ${}^t\bar{p}_t {}^t\mathbf{t}$ are the normal and tangential part of the nominal contact force vector, and ${}^t\bar{p}_n$ corresponds to the Lagrange multiplier. Although “~” represents the nominal value in the above equation, it will be omitted for the sake of brevity in the

following sections. Likewise, the normal and tangential components of the contact force will simply be called contact force and friction force, respectively.

2.2 Virtual work equation in convected coordinate system

Objective friction law is derived using a convected coordinate system⁽²⁾. In this section, contact and friction forces are represented in a convected coordinate system, by which the virtual work equation is expressed. Figure 1 shows the surfaces of bodies 1 and 2. The hitting point $\mathbf{x}^h$ of body 2 is going to contact the target point $\mathbf{x}^t$ of body 1, where $\mathbf{x}^t$ is determined by drawing a line perpendicular to the surface of body 2 from $\mathbf{x}^h$. The left superscript t , denoting time, is omitted hereafter, and the right superscripts h and t represent the hitting and target points, respectively. In Fig. 1, ξ^1 and ξ^2 indicate the convected coordinates at $\mathbf{x}^t$, $\mathbf{t}_1$ and $\mathbf{t}_2$ indicate the covariant base vectors, and $\mathbf{n}$ indicates the outer normal vector with unit length. Namely,

$$\mathbf{t}_1 = \partial\mathbf{x}^t / \partial\xi^1, \mathbf{t}_2 = \partial\mathbf{x}^t / \partial\xi^2 \quad (10)$$

$$\mathbf{r} \equiv \mathbf{t}_1 \times \mathbf{t}_2, \mathbf{n} = \mathbf{r} / |\mathbf{r}| \quad (11)$$

It should be noted, from the definition of the target point, that the kinematic movement of $\mathbf{x}^t$ depends not only on the deformation of body 1 but also on the motion of $\mathbf{x}^h$, i.e., the slip of $\mathbf{x}^h$ relative to the surface of body 1. Thus, using the convected coordinate system and summation convention, the differentiation of $\mathbf{x}^t$ can be given as

$$d\mathbf{x}^t(\xi^1, \xi^2, t) = d\mathbf{u}^t + \mathbf{t}_i d\xi^i \quad (12)$$

For the material point of body 1, with a convected coordinate where $\mathbf{x}^t$ has been determined, $d\mathbf{u}^t$ represents the infinitesimal displacement (material time derivative) of this point due to the deformation of body 1. Also, $\mathbf{t}_i d\xi^i$ indicates the movement (slip) relative to body 2. The introduction of vector

$$\mathbf{g} \equiv (\mathbf{x}^t - \mathbf{x}^h) \quad (13)$$

and the use of the definitions of penetration and contact point (Fig. 1) yield

$$\mathbf{g} \equiv \mathbf{g} \cdot \mathbf{n} \quad (14)$$

$$\mathbf{g} = g\mathbf{n} \quad (15)$$

$$\mathbf{t}_i \cdot \mathbf{g} = 0, (i=1, 2) \quad (16)$$

Fig. 1 Contact surfaces

The variation of $\mathbf{g}$ is obtained as

$$\delta \mathbf{g} = \mathbf{t}_i \delta \xi^i + \delta \mathbf{u}^t - \delta \mathbf{u}^h \quad (17)$$

by considering the dependence of $\mathbf{x}^t$ on $\mathbf{x}^h$. Since the base vector of the convected coordinate system is perpendicular to the normal vector, it follows that

$$\mathbf{n} \cdot (\delta \mathbf{u}^t - \delta \mathbf{u}^h) = \mathbf{n} \cdot \delta \mathbf{g} \quad (18)$$

Also, from Eqs.(13) and (17), the variation of Eq.(16) is given by

$$\delta \mathbf{t}_i \cdot (\mathbf{x}^t - \mathbf{x}^h) + (\mathbf{t}_i \cdot \mathbf{t}_j) \delta \xi^j + \mathbf{t}_i \cdot (\delta \mathbf{u}^t - \delta \mathbf{u}^h) = 0 \quad (19)$$

Since in the equilibrium state,

$$\mathbf{g} = (\mathbf{x}^t - \mathbf{x}^h) = \mathbf{0} \quad (20)$$

must be satisfied, we have

$$\mathbf{t}_i \cdot (\delta \mathbf{u}^t - \delta \mathbf{u}^h) = -(\mathbf{t}_i \cdot \mathbf{t}_j) \delta \xi^j \quad (21)$$

Thus, by replacing the superscripts 1 and 2 in Eq.(8) with t and h and using Eqs.(18) and (21), the virtual work due to contact and friction forces is rewritten as

$$\delta W_c = \int_{\Gamma_c} [p_n \mathbf{n} \cdot \delta \mathbf{g} - p_{ij} \delta \xi^j] dS \quad (22)$$

where p_{ij} is the covariant component of the friction force vector $\mathbf{p}_t$ which is decomposed with the covariant base vector as

$$\mathbf{p}_t \equiv p_i \mathbf{t}^i = p'_i \mathbf{t}_i \quad (23)$$

Additionally, the relationship between the covariant and contravariant components is used, namely,

$$p_{ij} = (\mathbf{t}_i \cdot \mathbf{t}_j) p'_i \quad (24)$$

2.3 Friction law

To avoid the discontinuity of mechanical response, friction laws similar to the constitutive relationship of elastoplasticity have been proposed (for example, Ref.(4)), i.e., small slip is allowed, even in the stick state. Such procedures essentially introduce a penalty constraint for sticking. For the material points of bodies 1 and 2, located at $\mathbf{x}^t$ and $\mathbf{x}^h$, the difference between their material time derivatives gives the relative displacement velocity, which can then be expressed with the normal component $\dot{\mathbf{u}}_n$ and tangential component $\dot{\mathbf{u}}_t$ as

$$\dot{\mathbf{u}} = \dot{\mathbf{u}}^t - \dot{\mathbf{u}}^h = \dot{\mathbf{u}}_n + \dot{\mathbf{u}}_t \quad (25)$$

Moreover, the use of the additive decomposition of the tangential component into the elastic part (stick) and plastic part (slip) leads to

$$\dot{\mathbf{u}}_t = \dot{\mathbf{u}}_t^e + \dot{\mathbf{u}}_t^p \quad (26)$$

On the other hand, in the equilibrium state, the normal component becomes

$$\dot{\mathbf{u}}_n = \dot{\mathbf{u}}_n^e = \dot{\mathbf{u}}_n^p = \mathbf{0} \quad (27)$$

due to the geometrical constraint condition. The slip condition which corresponds to the yield condition takes the form

$$\phi = |\mathbf{p}_t| - \mu |\mathbf{p}_n| = 0 \quad (28)$$

where μ , $\mathbf{p}_t$ and $\mathbf{p}_n$ are the friction coefficient, friction force and contact force, respectively. Although the yield function is taken as the plastic potential in the flow rule of elastoplasticity theory, the frictional potential is defined for frictional problems as

$$\Psi = |\mathbf{p}_t| \quad (29)$$

when Eq.(27) is considered. Thus, the flow rule may be written as follows:

$$\dot{\mathbf{u}}_t^p = -\lambda \frac{\partial \Psi}{\partial \mathbf{p}_t} = -\lambda \frac{\mathbf{p}_t}{|\mathbf{p}_t|} \quad (30)$$

$$\dot{\lambda} \geq 0 \quad (31)$$

Next, let the velocity of friction force be decomposed with the contravariant base vectors of the convected coordinate as

$$\dot{\mathbf{p}}_t = \dot{p}_{ti} \mathbf{t}^i \quad (32)$$

Due to the fact that the elastic part of the relative velocity must be small enough, the following form can be obtained:

$$\dot{\mathbf{u}}_t^e = -\frac{\dot{\mathbf{p}}_t}{\varepsilon} = -\frac{\dot{p}_{ti} \mathbf{t}^i}{\varepsilon} \quad (33)$$

where ε indicates the penalty parameter. Then the use of Eqs.(26), (30) and (33) leads to

$$-\frac{\dot{p}_{ti} \mathbf{t}^i}{\varepsilon} = \dot{\mathbf{u}}_t + \lambda \frac{\mathbf{p}_t}{|\mathbf{p}_t|} \quad (34)$$

Referring to Eq.(17), in the equilibrium state, the time derivative of Eq.(20) becomes

$$\mathbf{t}_i \dot{\xi}^i + \dot{\mathbf{u}}^t - \dot{\mathbf{u}}^h = \mathbf{0} \quad (35)$$

From this equation, as well as Eqs.(25) and (27) and the relationship between the covariant and contravariant base vectors, namely,

$$\mathbf{t}_i = (\mathbf{t}_i \cdot \mathbf{t}_j) \mathbf{t}^j \quad (36)$$

it can be obtained that

$$\dot{\mathbf{u}}_t = -\mathbf{t}_i \dot{\xi}^i = -(\mathbf{t}_i \cdot \mathbf{t}_j) \dot{\xi}^j \mathbf{t}^j \quad (37)$$

Finally, by substituting this equation into Eq.(34) and combining the result with Eqs.(28) and (31), the friction law is derived as follows:

$$\phi = |\mathbf{p}_t| - \mu |\mathbf{p}_n| \leq 0 \quad (38.a)$$

$$\dot{p}_{ti} \mathbf{t}^i = \varepsilon \left[(\mathbf{t}_i \cdot \mathbf{t}_j) \dot{\xi}^j - \lambda \frac{p_{ti}}{|\mathbf{p}_t|} \right] \mathbf{t}^i \quad (38.b)$$

$$\dot{\lambda} \geq 0 \quad (38.c)$$

$$\phi \cdot \dot{\lambda} = 0 \quad (38.d)$$

Furthermore, the left-hand side of Eq.(38.b) and the first term on the right-hand side are in the form of the Oldroyd rate of a vector. As well, the second term on the right-hand side includes a scalar and a product of the friction force vector with its norm. Thus, it is clear that the friction equations satisfy the objectivity requirement.

3. Analysis of Frictional Contact Problems with Large Sliding

3.1 Incremental decomposition of friction law

Due to the deformation path dependence of frictional problems, it is necessary to carry out an incremental analysis. Using the formulas of Section 2, it is, in principle, possible to compute the increments of friction force and sliding by the incremental decomposition. Using the backward Euler integration for incremental analysis, the incremental form of Eq.(38)

becomes

$$\phi_{(i+1)} = |\mathbf{p}_{t(i+1)}| - \mu |\mathbf{p}_{n(i+1)}| \leq 0 \quad (39.a)$$

$$\begin{aligned} p_{tk(i+1)} \mathbf{t}_{(i+1)}^k &= p_{tk(i)} \mathbf{t}_{(i+1)}^k + \varepsilon \left[(\mathbf{t}_k \cdot \mathbf{t}_j)_{(i+1)} (\xi_{(i+1)}^j - \xi_{(i)}^j) \right. \\ &\quad \left. - \Delta \lambda \frac{p_{tk(i+1)}}{|\mathbf{p}_{t(i+1)}|} \right] \mathbf{t}_{(i+1)}^k \end{aligned} \quad (39.b)$$

$$\Delta \lambda \geq 0 \quad (39.c)$$

$$\phi_{(i+1)} \cdot \Delta \lambda = 0 \quad (39.d)$$

where subscripts (i) and $(i+1)$ indicate the incremental steps. Considering Eq.(39.b), the increment of the convected coordinate appears as the result of incremental decomposition. In the finite element method, the contact surface is divided into elements and the local coordinate of every element is used practically as a convected coordinate. Thus, the increment of a convected coordinate cannot be computed if the contact point slides over the boundary of element. Because of the difficulty of defining a general convected coordinate on the whole contact surface for an arbitrary body, in Ref.(2) the element local coordinate was extended as an expedient. However, since this treatment requires a control of the load increment in order to limit sliding over elements, it is considered impractical for the *a priori* unknown contact state of a deforming body.

In this work, the following approach is presented to overcome this problem. Defining

$$\Delta \mathbf{r} \equiv (\mathbf{t}_k \cdot \mathbf{t}_j)_{(i+1)} (\xi_{(i+1)}^j - \xi_{(i)}^j) \mathbf{t}_{(i+1)}^k \quad (40)$$

for the term in Eq.(39.b), including the increment of the convected coordinate, it is clear from Eq.(37) that $\Delta \mathbf{r}$ represents the increment of relative displacement for the contact points. By referring to the discretized contact surfaces shown in Fig. 2, $\Delta \mathbf{r}$ can be approximated with

$$\Delta \mathbf{r} \doteq \mathbf{x}_{(i+1)}^t - \mathbf{x}_{(i)}^t \quad (41)$$

where $\mathbf{x}_{(i+1)}^t$ is the position vector of the contact point at increment $i+1$, whereas $\mathbf{x}_{(i)}^t$ is the position vector, at increment $i+1$, of the material point where the contact point has been located at increment i . However, because of movement due to deformation, $\mathbf{x}_{(i)}^t$ is unknown at increment $i+1$. Because this movement affects the tangent stiffness due to friction, it is not convenient to use Eq.(41) directly. Now a mapping of the contact point to the reference configuration, as shown in Fig. 3, is considered. Let $\mathbf{F}$ be the deformation gradient tensor, $\mathbf{t}_{j(i+1)}$ the covariant base vector at the contact point $\mathbf{x}_{(i+1)}^t$ in the current configuration (increment $i+1$), and ${}^0\mathbf{t}_{j(i+1)}$ the covariant base vector at the mapped point $\mathbf{X}_{(i+1)}^t$ in the reference configuration. Use of the relationship⁽⁵⁾

$$\mathbf{F} = \delta_k^i (\mathbf{t}_{j(i+1)} \otimes {}^0\mathbf{t}_{k(i+1)}^j) \quad (42)$$

gives

$$\Delta \mathbf{r} = (\mathbf{t}_{j(i+1)} \cdot \mathbf{t}_{k(i+1)}) ({}^0\mathbf{t}_{(i+1)}^k \cdot \Delta^0 \mathbf{r}) \mathbf{t}_{(i+1)}^j \quad (43)$$

where

Fig. 2 Discretized contact surface

Fig. 3 Motion of contact surface

$$\Delta^0 \mathbf{r} \equiv (\xi_{(i+1)}^j - \xi_{(i)}^j) {}^0\mathbf{t}_{j(i+1)} \quad (44)$$

is the relative displacement increment mapped to the reference configuration and can be calculated as

$$\Delta^0 \mathbf{r} \doteq \mathbf{X}_{(i+1)}^t - \mathbf{X}_{(i)}^t \quad (45)$$

Thus, because $\mathbf{X}_{(i)}^t$ in Eq.(45) is fixed to the reference configuration, its influence on the frictional tangent stiffness vanishes and only the change of $\mathbf{X}_{(i+1)}^t$ has to be considered.

Finally, the use of Eqs.(40) and (43) with Eq.(39) results in

$$\begin{aligned} p_{tk(i+1)} \mathbf{t}_{(i+1)}^k &= p_{tk(i)} \mathbf{t}_{(i+1)}^k + \varepsilon \left[(\mathbf{t}_k \cdot \mathbf{t}_j)_{(i+1)} ({}^0\mathbf{t}_{(i+1)}^j \cdot \Delta^0 \mathbf{r}) \right. \\ &\quad \left. - \Delta \lambda \frac{p_{tk(i+1)}}{|\mathbf{p}_{t(i+1)}|} \right] \mathbf{t}_{(i+1)}^k \end{aligned} \quad (46)$$

The mapped contact point can be obtained from its element local coordinate in the current configuration and the node coordinate before deformation.

The algorithm adopted to update friction force is analogous to the radial return approach in elastoplasticity analysis. First, assuming the stick state, the trial friction force is found from

$$p_{tk(i+1)}^{(T)} = p_{tk(i)} + \varepsilon (\mathbf{t}_k \cdot \mathbf{t}_j)_{(i+1)} ({}^0\mathbf{t}_{(i+1)}^j \cdot \Delta^0 \mathbf{r}) \quad (47)$$

If the result satisfies

$$|p_{tk(i+1)}^{(T)}| - \mu |\mathbf{p}_{n(i+1)}| \leq 0 \quad (48)$$

then the final friction force is determined as

$$p_{tk(i+1)}^{(F)} = p_{tk(i+1)}^{(T)} \quad (49)$$

Otherwise, if the trial friction force results in

$$|p_{tk(i+1)}^{(T)}| - \mu |\mathbf{p}_{n(i+1)}| > 0 \quad (50)$$

then, in order to pull the state point back onto the yield surface, it has to be modified as follows:

$$T_{k(i+1)} \equiv \frac{p_{tk(i+1)}^{(T)}}{|\mathbf{p}_{t(i+1)}^{(T)}|} \quad (51)$$

$$p_{tk(i+1)}^{(F)} = \mu |\mathbf{p}_{n(i+1)}| T_{k(i+1)} \quad (52)$$

3.2 Tangent stiffness due to frictional contact

For nonlinear problems, the consistent tangent

stiffness is required to ensure a quadratic rate of convergence of the Newton-Raphson iteration scheme. The objective here is to obtain a tangent stiffness which is consistent with the algorithm proposed in Section 3.1. Differentiation of the virtual work equation leads to

$$d(\delta W_c) = \int_{\Gamma_c} [dp_n(\mathbf{n} \cdot \delta \mathbf{g}) + p_n d\mathbf{n} \cdot \delta \mathbf{g} + p_n \mathbf{n} \cdot d(\delta \mathbf{g})] dS - \int_{\Gamma_c} [dp_{ij} \delta \xi^j + p_{ij} d(\delta \xi^j)] dS \quad (53)$$

where dp_n is treated as the independent variable, in the case of using the Lagrange multiplier constraint. Let the differentiation of friction force be derived later. By referring to Section 2.2 and considering the geometrical relationship on the contact surface and the fact that $\mathbf{x}^t$ is affected by $\mathbf{x}^h$, the differentiation of the other terms can be obtained in the following way. The definition of

$$\mathbf{a}_{,i} \equiv \frac{\partial \mathbf{a}}{\partial \xi^i} \quad (54)$$

$$\mathbf{b}_{,ij} \equiv \frac{\partial \mathbf{b}}{\partial \xi^i \partial \xi^j} \quad (55)$$

will be used hereafter.

Noting that $\delta \mathbf{n} \cdot \mathbf{n} = 0$, the variation of Eq.(15) becomes

$$p_n d\mathbf{n} \cdot \delta \mathbf{g} = p_n g d\mathbf{n} \cdot \delta \mathbf{n} \quad (56)$$

The differentiation of Eq.(17) leads to

$$p_n \mathbf{n} \cdot d(\delta \mathbf{g}) = p_n [\delta \xi^i (\mathbf{n} \cdot \mathbf{t}_{i,j}) d\xi^j + \mathbf{n} \cdot (d\mathbf{u}_{,i}^t \delta \xi^i + d\mathbf{u}_{,j}^t d\xi^j)] \quad (57)$$

The calculation of $d(\delta \xi^j)$ is a little cumbersome, yet by taking the differentiation of Eq.(19), the final result can be found by solving the following equations:

$$\begin{aligned} A_{ij} d(\delta \xi^j) = & -\mathbf{t}_i \cdot d\mathbf{u}_{,j}^t \delta \xi^j - \mathbf{t}_i \cdot d\mathbf{u}_{,j}^t \delta \xi^j \\ & -(\mathbf{t}_i \cdot \mathbf{t}_{j,l} + g \mathbf{n} \cdot \mathbf{t}_{i,j,l}) \delta \xi^j d\xi^l - \delta \xi^j \mathbf{t}_j \cdot (d\mathbf{u}_{,i}^t + \mathbf{t}_{i,l} d\xi^l) \\ & -d\xi^j \mathbf{t}_j \cdot (d\mathbf{u}_{,i}^t + \mathbf{t}_{i,l} d\xi^l) \\ & -(\mathbf{x}^t - \mathbf{x}^h) \cdot (d\mathbf{u}_{,ij}^t \delta \xi^j + d\mathbf{u}_{,ij}^t \delta \xi^j) \\ & -(\delta \mathbf{u}^t - \delta \mathbf{u}^h) \cdot (d\mathbf{u}_{,i}^t + \mathbf{t}_{i,l} d\xi^l) \\ & -(d\mathbf{u}^t - d\mathbf{u}^h) \cdot (d\mathbf{u}_{,i}^t + \mathbf{t}_{i,l} d\xi^l) \end{aligned} \quad (58)$$

$$A_{ij} = (\mathbf{t}_i \cdot \mathbf{t}_j) + (\mathbf{x}^t - \mathbf{x}^h) \cdot \mathbf{t}_{i,j} \quad (59)$$

It is seen that the tangent stiffness contributed by the kinematic action of contact point is symmetrical, due to the exchangeability between the differentiation and variation.

Next, based on the algorithm presented in Section 3.1, the differentiation of the friction force will be obtained by starting from the differentiation of Eqs. (47) and (52).

(1) Stick state:

Differentiation of Eq.(47) becomes

$$\begin{aligned} dp_{tk(i+1)} = & \varepsilon [d(\mathbf{t}_k \cdot \mathbf{t}_j)_{(i+1)} ({}^0 \mathbf{t}_{(i+1)}^t \cdot \Delta^0 \mathbf{r}) \\ & + (\mathbf{t}_k \cdot \mathbf{t}_j)_{(i+1)} (d^0 \mathbf{t}_{(i+1)}^t \cdot \Delta^0 \mathbf{r}) \\ & + (\mathbf{t}_k \cdot \mathbf{t}_j)_{(i+1)} ({}^0 \mathbf{t}_{(i+1)}^t \cdot d\Delta^0 \mathbf{r})] \end{aligned} \quad (60)$$

Applying

$$d\mathbf{t}_{k(i+1)} = \mathbf{t}_{k(i+1),l} d\xi^l + d\mathbf{u}_{,k}^t \quad (61)$$

leads to

$$\begin{aligned} d(\mathbf{t}_k \cdot \mathbf{t}_j)_{(i+1)} = & (\mathbf{t}_{k(i+1),l} \cdot \mathbf{t}_{j(i+1)} + \mathbf{t}_{j(i+1),l} \cdot \mathbf{t}_{k(i+1)}) d\xi^l \\ & + \mathbf{t}_{k(i+1)} \cdot d\mathbf{u}_{,j}^t + \mathbf{t}_{j(i+1)} \cdot d\mathbf{u}_{,k}^t \end{aligned} \quad (62)$$

Use of the relationship between the covariant and contravariant base vectors at the mapped contact point in the reference configuration, namely,

$${}^0 \mathbf{t}_{(i+1)}^t \cdot {}^0 \mathbf{t}_{l(i+1)} = \delta_l^i \quad (63)$$

gives

$$d^0 \mathbf{t}_{l(i+1)}^t \cdot {}^0 \mathbf{t}_{l(i+1)} = -{}^0 \mathbf{t}_{(i+1)}^t \cdot d^0 \mathbf{t}_{l(i+1)} \quad (64)$$

By decomposing the mapping of the incremental relative displacement vector to be the form as

$$\Delta^0 \mathbf{r} = \Delta^0 \mathbf{r}^{i0} \mathbf{t}_{l(i+1)} \quad (65)$$

and considering

$$d^0 \mathbf{t}_{l(i+1)}^t = {}^0 \mathbf{t}_{l(i+1),m} d\xi^m \quad (66)$$

and Eq.(64), it is given that

$$d^0 \mathbf{t}_{(i+1)}^t \cdot \Delta^0 \mathbf{r} = -\Delta^0 \mathbf{r}^{i0} \mathbf{t}_{(i+1)}^t \cdot {}^0 \mathbf{t}_{l(i+1),m} d\xi^m \quad (67)$$

On the other hand, by means of the definition of the covariant base vector at the mapped contact point in the reference configuration,

$$d\Delta^0 \mathbf{r} = {}^0 \mathbf{t}_{l(i+1)} d\xi^l \quad (68)$$

is obtained from Eq.(45). Then use of Eq.(63) yields

$${}^0 \mathbf{t}_{(i+1)}^t \cdot d\Delta^0 \mathbf{r} = d\xi^j \quad (69)$$

Rearranging the above results, the differentiation of friction force is given as

$$\begin{aligned} dp_{tk(i+1)} = & \varepsilon \{ ({}^0 \mathbf{t}_{(i+1)}^t \cdot \Delta^0 \mathbf{r}) [(\mathbf{t}_{k(i+1),l} \cdot \mathbf{t}_{j(i+1)} \\ & + \mathbf{t}_{j(i+1),l} \cdot \mathbf{t}_{k(i+1)}) d\xi^l + \mathbf{t}_{k(i+1)} \cdot d\mathbf{u}_{,j}^t + \mathbf{t}_{j(i+1)} \cdot d\mathbf{u}_{,k}^t] \\ & - (\mathbf{t}_k \cdot \mathbf{t}_j)_{(i+1)} \Delta^0 \mathbf{r}^{i0} \mathbf{t}_{(i+1)}^t \cdot {}^0 \mathbf{t}_{l(i+1),m} d\xi^m \\ & + (\mathbf{t}_k \cdot \mathbf{t}_j)_{(i+1)} d\xi^j \} \end{aligned} \quad (70)$$

(2) Slide state:

In this case, the differentiation of friction force can be derived from the differentiation of Eq.(52), namely,

$$dp_{tk(i+1)}^{(F)} = \mu d|\mathbf{p}_{n(i+1)}| T_{k(i+1)} + \mu |\mathbf{p}_{n(i+1)}| dT_{k(i+1)} \quad (71)$$

For $dT_{k(i+1)}$, note that the covariant base vector is perpendicular to the contravariant one, Eq.(51) is differentiated to obtain

$$\begin{aligned} dT_{k(i+1)} = & \frac{dp_{tk(i+1)}^{(T)}}{|\mathbf{p}_{t(i+1)}^{(T)}|} - \frac{dp_{ij(i+1)}^{(T)}}{|\mathbf{p}_{t(i+1)}^{(T)}|} T_k T^j \\ & + T_k T^j \mathbf{T} \cdot d\mathbf{t}_{j(i+1)} \end{aligned} \quad (72)$$

where

$$T^j = \frac{p_{t(i+1)}^{(T)}}{|\mathbf{p}_{t(i+1)}^{(T)}|} \quad (73)$$

represents the normalized trial friction force. Since $p_{n(i+1)} < 0$ in the contact state, the differentiation of friction force can be obtained in the following form:

$$\begin{aligned} dp_{tk(i+1)}^{(F)} = & -\mu dp_{n(i+1)} T_{k(i+1)} - \mu p_{n(i+1)} \frac{dp_{ij(i+1)}^{(T)}}{|\mathbf{p}_{t(i+1)}^{(T)}|} (\delta_k^j \\ & - T_k T^j) - \mu p_{n(i+1)} T_k T^j \mathbf{T} \cdot d\mathbf{t}_{j(i+1)} \end{aligned} \quad (74)$$

It is observed from Eq.(70) or (74) that the tangent stiffness due to friction is no longer symmetrical.

4. Numerical Examples

If the element local coordinates and values of

differentiation are computed, the equivalent nodal contact force, friction force and tangent stiffness due to frictional contact can be constructed based on the formulation performed in Section 3. These are implemented into a finite element analysis code⁽⁶⁾ for large deformation elastoplastic problems developed by the authors using a hexahedral type of element with 8 nodes. Numerical examples are carried out to assess the presented algorithm.

4.1 Indentation of a rigid body into elastic foundation

The distribution of contact traction is computed for the finite element model shown in Fig. 4. The degree of freedom in the direction out of the plane is constrained to simulate a 2-D problem. The fine mesh, as shown in Fig. 4, is made to get a smooth distribution of traction, but a rough mesh, with dimension of 500 mm × 250 mm, is connected to the fine part to approximate a half-infinite body. The penalty parameter for friction is set to be $\epsilon=10^6$. Displacement is specified to be $u_x=3.33$ mm and $u_y=-0.01$ mm, with respect to the width of 1.67 mm for an element such that finite sliding is produced. The results of contact traction are shown in Fig. 5. The gradient of traction is sharp at two sides and the larger gradient is on the left side. This result is considered to be due to the reaction force, which resists the moment resulting from friction, of the rigid

Fig. 4 Indentation of a rigid body into elastic body

Fig. 5 Contact pressure distribution on contact surface

body whose top is constrained. This trend becomes considerable when the friction coefficient is increased. For comparison, the results of the general code MARC are shown in Fig. 5 and agree well with the present work.

4.2 An elastoplastic frictional contact problem of two beams with large deformation

The effectiveness of the presented algorithm is verified for the elastoplastic frictional contact problem with large deformation shown in Fig. 6. The depth of two beams is set at 2 mm and the degree of freedom in the depth direction is constrained. The left side of the upper beam and the right side of the lower beam are clamped to rigid walls. The beams are then deformed by applying downward displacement on the left rigid wall. The penalty parameter used for the stick state is $\epsilon=10^4$. In this example, the contact constraint is given such that the upper beam cannot penetrate the lower one. Von Mises yield function and the associated flow rule with isotropic hardening are assumed for elastoplastic analysis. Also, the total Lagrangian formulation is adopted for treating large deformation. The prescribed displacement $U=30$ mm is divided into equally 60 steps, but as a result of computation, two beams separated before 30 mm was reached.

The results for the coefficient of friction $\mu=0.5$ are first shown. The nodal displacements on the contact surfaces of the upper and lower beams are

Fig. 6 Two-beam frictional contact system

Fig. 7 Displacement of beam surface

depicted in Fig. 7 for typical increments. Although it is illegible in the figure, the results show that the nodes on the free side released from the contact surface in early deformation due to the deflection of the beam. Along with deformation, the contact nodes changed and the two beams separated after the nodes of the free side made contact. Thus, the large amount of sliding of the contact node over the element boundary was successfully simulated. Figure 8 shows the deformation of the prescribed $U=15\text{ mm}$ and the yielding area is shaded black.

The analysis of frictionless contact problems for elastic beams is also performed for comparison. Figure 9 shows the relationship between the prescribed displacement and the reaction force of the rigid wall. As expected, the reaction force in the case of an elastic beam was larger than that of the elastoplastic one. Also, a larger reaction force due to friction was observed. In these analyses, smooth results were obtained despite the repetition of the contact and release of the contact nodes and sliding over the element boundary. Moreover, since the relative displacement at the contact point is defined in the current deformed configuration, it is confirmed that the direction of friction force always coincides with the tangential direction at contact point. In order to judge the convergence of iterative computa-

Fig. 8 Deformation of two beams

Fig. 9 Reaction force vs prescribed displacement

tion, tolerance for the normalized energy residual is set to be $1.0e-10$. In the first load step, all nodes on the contact surface are designated as candidates of contact and nine iterations are needed. After this the iterations do not exceed six to get convergence even though the contact node is changed, and in general, convergence is obtained with four iterations. This is aided by using consistent tangent stiffness. For comparison, the same problem is analyzed by the general code MARC. Although the results depicted in Fig. 9 show good agreement, only 60 increments are required in the present work while, for the friction coefficient 0.5, MARC takes over 500 increments for elastic beams and 1200 increments for elastoplastic beams in order to avoid oscillation in the load-displacement curve and the dependence of the results on the number of increments. In other words, a quite fine division of increments is necessary to get stable results, and even so, oscillation in the load-displacement curve is observed. It should be noted that although the best efforts were made in the computations, there is no conclusive evidence that the performance has been perfectly executed, since treatment of the general code requires experience.

5. Conclusions

A finite element approach that enables the analysis of frictional contact problems including finite sliding was proposed. The method utilizes the convected coordinate system so that the objectivity of the friction law was preserved. Conventional methods have not effectively dealt with the sliding that extends over the element boundary, whereas the proposed method solves this problem by introducing the reference configuration. In this formulation, the consistent tangent stiffness was derived to obtain quadratic convergence. The effectiveness of the proposed method was verified by numerical examples including an elastoplastic frictional contact problem with large deformation. Using this method, superior convergence and accuracy were confirmed.

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