Boundary Element Inverse Analysis for Stress Separation in Thermoelastic Stress Analysis*

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Thermoelastic stress analysis (TSA) is a technique for measuring stresses through temperature changes induced by the thermoelastic effect of elastic bodies. Unfortunately, TSA can only provide the sum of the principal stresses. In this paper, the stress separation problem to determine individual stress components from the sum of the principal stresses is considered. The problem is divided into two parts: (1) an inverse problem to estimate unknown boundary values from the sum of the principal stresses inside the body, and (2) a direct problem to determine the stress components inside the body based on the estimated boundary values. This two-step approach is formulated and solved by BEM. It is found that the combination of Tikhonov regularization with supplementary data and Hansen's L-curve method is effective for attaining an accurate result of stress separation.

Key Words: Thermoelastic Stress Analysis, Inverse Problem, Boundary Element Method, Tikhonov Regularization, L-Curve Method

1. Introduction

Thermoelastic stress analysis (TSA) is an experimental technique for measuring stresses through temperature changes induced by the thermoelastic effect of elastic bodies. Recently, it has become one of the most useful techniques because of the evolution of the infrared thermography equipment. Unfortunately, TSA can only provide the sum of the principal stresses which is the first invariant of the stress tensor. Therefore, a method of stress separation is needed to obtain individual stress components from the sum of the principal stresses.

In the experimental stress analysis, it is often the case that boundary values are unknown at some parts of the boundary of the specimen. In the TSA, the sum of the principal stresses can be obtained at many points on the surface of the specimen. Therefore, the

stress separation problem can be solved by dividing it into the following two parts:

(1) the inverse problem to estimate the unknown boundary values from TSA data,

(2) the direct problem to determine the individual stress components inside the body.

These two problems can be solved numerically by using FEM or BEM etc. Waldman et al.⁽¹⁾ and Murakami et al.⁽²⁾ have proposed methods based on this idea and demonstrated their effectiveness. Since the second problem is an ordinary boundary value problem, it can be solved rather easily. On the other hand, the inverse problem to estimate the boundary values is often ill-posed in the sense that the uniqueness or stability of the solution are violated⁽³⁾. Since temperature changes induced by the thermoelastic effect are very small, TSA data are usually very noisy. Therefore, a straightforward solution of the inverse problem often provides a significantly poor estimate of the boundary values and, as a result, the stress separation fails. Hence some sort of regularization is required to solve the inverse problem appropriately. Waldman et al.⁽¹⁾ introduced a set of complete orthogonal functions to describe the unknown load distribution. This technique stabilizes the solution by confining it inside a space spanned with the set of

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orthogonal functions. However, it may be difficult to find an appropriate set of orthogonal functions for various practical problems. Kishimoto et al.^{(4),(5)} formulated the stress separation problem by BEM and applied a regularization technique using the singular value decomposition and Hansen's L-curve method. Although this approach provides fairly good result of the stress separation, further improvement of the accuracy is required for practical applications.

In this study, to attain a better result of stress separation, we adopt Tikhonov regularization and, in addition, utilize supplementary data obtained by using strain gauges.

2. Thermoelastic Stress Analysis

The fundamental equation of the TSA is given by⁽⁶⁾

$$\Delta T = -K_m T \Delta s, \quad (1)$$

$$K_m = \frac{\alpha}{\rho C_p}, \quad (2)$$

where T is the absolute temperature, s the sum of the principal stresses, α the coefficient of thermal expansion, ρ the mass density, C_p the specific heat at constant pressure, and Δ denotes small change in physical quantity. The constant K_m is called the thermoelastic constant. Since Eq. (2) is valid under adiabatic condition, the specimen is usually subjected to cyclic loading with a constant amplitude. The change in the sum of the principal stresses, Δs , due to cyclic loading is obtained by measuring the change in the temperature, ΔT . If the infrared thermography is used for measuring the temperature change, two-dimensional distribution of the sum of the principal stresses is obtained.

In what follows, we suppose a reference state in which the specimen is stress free and the temperature is T_0 . In addition, the change in the sum of the principal stresses from the reference state is denoted by s instead of Δs for abbreviation.

3. Inverse Boundary Value Problem

3.1 Formulation

In this paper, as a basic problem relative to the TSA, the plane stress problem is considered. Suppose

Fig. 1 General description of the problem

a two-dimensional isotropic elastic body Ω with boundary $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ as shown in Fig. 1. Assume that the displacement and the traction are known on boundary Γ_1 and Γ_2 , respectively, and that the boundary condition is unknown on boundary Γ_0 . On the other hand, the sum of the principal stresses is given at many points inside the domain Ω . We consider an inverse problem to estimate the traction on Γ_0 from the sum of the principal stresses inside Ω .

In the absence of body forces, the displacement $u_i(p)$ at a point p inside the domain Ω is given by Somigliana's equation as

$$u_i(p) = \int_{\Gamma} U_{ij}(p, Q) t_j(Q) d\Gamma - \int_{\Gamma} T_{ij}(p, Q) u_j(Q) d\Gamma, \quad (3)$$

where $t_j(Q)$ and $u_j(Q)$ are respectively the traction and the displacement at a point Q on the boundary Γ , and $T_{ij}(p, Q)$ and $U_{ij}(p, Q)$ are the fundamental solutions. From the definition of strain and the constitutive equation, the sum of the principal stresses, $s(p)$, at the point p is given by

$$s(p) = 2 \left(\mu + \frac{2\lambda\mu}{\lambda + 2\mu} \right) \left[\int_{\Gamma} U_{ij,i}(p, Q) t_j(Q) d\Gamma - \int_{\Gamma} T_{ij,i}(p, Q) u_j(Q) d\Gamma \right], \quad (4)$$

where λ and μ are Lamé constants. Discretizing the boundary into elements in a similar manner as in BEM⁽⁷⁾, we obtain

$$\mathbf{s} = \mathbf{G}_s \mathbf{t} - \mathbf{H}_s \mathbf{u}, \quad (5)$$

where $\mathbf{s}$ is a vector composed of the sums of the principal stresses given at points inside the domain Ω , $\mathbf{t}$ and $\mathbf{u}$ are vectors composed of the tractions and the displacements, respectively, at the nodal points on the boundary Γ and, in addition, $\mathbf{G}_s$ and $\mathbf{H}_s$ are known matrices derived from the fundamental solutions.

Let us utilize the boundary integral equation as an additional requirement to be satisfied by the traction and displacement on the boundary. The boundary integral equation is given as

$$C_{ij}(P) u_j(P) = \int_{\Gamma} U_{ij}(P, Q) t_j(Q) d\Gamma - \int_{\Gamma} T_{ij}(P, Q) u_j(Q) d\Gamma, \quad (6)$$

where both P and Q are points on the boundary Γ , and C_{ij} is a constant determined by the geometry of the boundary. Discretizing Eq. (6) in the same manner as in Eq. (5), we obtain

$$\mathbf{G} \mathbf{t} - \mathbf{H} \mathbf{u} = \mathbf{0}, \quad (7)$$

where $\mathbf{G}$ and $\mathbf{H}$ are matrices derived from the fundamental solutions. Substituting Eq. (7) into Eq. (5), and applying the known boundary conditions on Γ_1 and Γ_2 , we obtain the following simultaneous equations for traction vector $\mathbf{t}_0$ composed of traction components at the nodal points on the boundary Γ_0 :

$$\mathbf{A} \mathbf{t}_0 = \mathbf{b}, \quad (8)$$

where $\mathbf{A}$ and $\mathbf{b}$ are derived from known quantities. If M denotes the number of points at which the sums of the principal stresses are given, and if N denotes the number of nodes on the boundary Γ_0 , then Eq. (8) has M equations and $2N$ unknowns. Therefore, if $M \geq 2N$, we may obtain the least squares solution of Eq. (8).

3.2 Regularization

As mentioned above, the inverse problem to estimate the boundary values is often ill-posed. In this study, we apply Tikhonov regularization and, in addition, utilize supplementary data obtained by using rosette gauges to stabilize the solution.

Rosette gauge provides individual stress components given by

$$\sigma_{ij}(p) = \int_{\Gamma} D_{ijk}(p, Q) t_k(Q) d\Gamma - \int_{\Gamma} S_{ijk}(p, Q) u_k(Q) d\Gamma, \quad (9)$$

where $D_{ijk}(p, Q)$, $S_{ijk}(p, Q)$ are known functions derived from the fundamental solutions. Discretizing Eq. (9), substituting Eq. (7) and applying the known boundary conditions on Γ_1 and Γ_2 , we obtain

$$\mathbf{C} \mathbf{t}_0 = \mathbf{d}, \quad (10)$$

where $\mathbf{C}$ and $\mathbf{d}$ are derived from known quantities. If M_p denotes the number of rosette gauges, Eq. (10) has $3M_p$ equations and $2N$ unknowns. Note that $3M_p \ll 2N$ for practical applications.

To consider both Eq. (8) and Eq. (10) simultaneously, we propose the following functional to be minimized:

$$\Pi = \left\| \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} \tilde{\mathbf{t}}_0 - \begin{bmatrix} \mathbf{b} \\ \mathbf{d} \end{bmatrix} \right\|^2 + \lambda (\|\tilde{\mathbf{t}}_0\|^2 + \|\tilde{\mathbf{t}}_0'\|^2), \quad (11)$$

where λ is the regularization parameter. It is important to appropriately determine the regularization parameter λ . If λ is too small, regularization is ineffective and the solution remains unstable. On the other hand, if it is too large, regularization is excessive and the solution becomes inaccurate. In this study, Hansen's L-curve method^{(8),(9)} is adopted to determine λ . This method is based on the behavior of the L-curve which is a plot of the norm of the stabilizer $\sqrt{\|\tilde{\mathbf{t}}_0\|^2 + \|\tilde{\mathbf{t}}_0'\|^2}$ versus the norm of the residual $\left\| \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} \tilde{\mathbf{t}}_0 - \begin{bmatrix} \mathbf{b} \\ \mathbf{d} \end{bmatrix} \right\|$.

4. Numerical simulation

4.1 Method of numerical simulation

Numerical simulation on the tension of a strip with single edge crack was conducted to verify the effectiveness of the stress separation technique developed in this study. Figure 2 shows the simulation model. On the boundary DE, the model is subjected to tractions $q(x)$ and $p(x)$ in the x - and y -direction, respectively. Boundary conditions of this model are

summarized in Table 1. Young's modulus was assumed to be $E = 205$ GPa, Poisson's ratio $\nu = 0.3$, the thermoelastic constant $K_m = 2.99 \times 10^{-12}$ 1/Pa, and the reference temperature $T_0 = 300$ K.

Before conducting the numerical simulation of stress separation, direct problem was solved by BEM to generate TSA data to be used in the simulation. For the case when the boundary DE is subjected to stepwise tension as shown in Fig. 3, the distributions of the individual stress components were obtained as shown in Figs. 4(a)-(c), respectively. The temperature

Table 1 Boundary conditions of the model

Boundary	u_x	u_y	t_x	t_y
AB	*	*	0	0
BC	*	0	0	*
CD	*	*	0	0
DE	*	*	$q(x)$	$p(x)$
EA	*	*	0	0

* : unknown

Fig. 2 Simulation model

Fig. 3 Assumed traction

Fig. 4 Exact stress distribution (MPa)

Fig. 5 Distribution of temperature change (K)

change was derived from the sum of the principal stresses as shown in Fig. 5.

The stress separation was conducted on the basis of the temperature data shown in Fig. 5. The traction $p(x)$ and $q(x)$ on the boundary DE were assumed to be unknown and were estimated by solving the inverse problem. After that, individual stress components were determined by solving the direct problem on the basis of the traction estimated by the inverse analysis. Each boundary of the model was discretized into second order elements of constant length. The temperature data were given at grid points which equally divide the model both in the x - and y -direction. To verify the effectiveness of the regularization technique adopted in this study, random numbers with zero mean and 0.001 K amplitude were added to the temperature data.

The stress components σ_x , σ_y , τ_{xy} measured with rosette gauges were also generated by BEM. It can be considered that the supplementary data are effective if they are given near the boundary on which the traction is unknown. Therefore, the strain gauges were assumed to be attached on the line FG which is 5 mm distant from boundary DE as shown in Fig. 2. The stress components were given at M_p points which equally divide the line FG. Since the measurement error of the strain gauge is usually much less than that of the TSA data, we assumed that the strain gauge data are not contaminated with noise.

Fig. 6 Traction estimated without applying the regularization

4.2 Result of numerical simulation

Each boundary of the model was discretized into 16 second-order elements of constant length. The number of TSA data was $M=81$. Under these settings, number of equation M in Eq.(8) is 81 and number of unknowns $2N$ is 66.

First, inverse analysis was performed without applying regularization in order to examine the ill-posed nature of this inverse problem. Figure 6 shows the result. Comparing this result to the exact one (Fig. 3), it is understood that they are quite different. Namely, without applying regularization, the accuracy of the estimated traction is significantly poor. This indicates that the inverse problem is ill-conditioned. In addition, Figs. 7(a)-(c) show distributions of individual stress components derived from the estimated traction presented in Fig. 6. Comparing these results to the exact ones shown in Fig. 4, it is understood that these stress distributions are quite different from the exact ones. Therefore, the accuracy of estimation must be improved by applying regularization in order to attain a good result of stress separation.

To improve the accuracy of estimation, regularization was applied. At first, the singular values of the

Fig. 7 Contour plots of stress components derived from estimated traction shown in Fig. 6 (MPa)

Fig. 8 The L-curve in the case of $M_p=3$

matrix $\begin{bmatrix} A \\ C \end{bmatrix}$ were computed. It can be considered that the optimum λ exists between the square of the minimum singular value and the square of the maximum singular value. Therefore, the regularization parameter λ was changed from the square of the minimum singular value to the square of the maximum singular value. Figure 8 shows the L-curve in the case of $M_p=3$. The plot in Fig. 8 moves from the top left to the bottom right as the regularization parameter increases. In order to obtain an accurate and stable solution, it may be required that both norms of the residual $\left\| \begin{bmatrix} A \\ C \end{bmatrix} \tilde{t}_0 - \begin{bmatrix} b \\ d \end{bmatrix} \right\|$ and the stabilizer $\sqrt{\|\tilde{t}_0\|^2 + \|\tilde{t}'_0\|^2}$ are as small as possible. This requirement is met around the 'corner' of the L-curve. Therefore, the regularization parameter can be conveniently optimized by taking the value at which the curvature of the L-curve becomes maximum.

Figures 9(a)-(c) show estimated tractions in the cases of $M_p=0$, $M_p=3$ and $M_p=19$, respectively. In the case of $M_p=0$, the amplitude of the solution is controlled to be small, which means that the result of Fig. 9(a) is much better than the result of Fig. 6 obtained without regularization. But the estimate is still inaccurate around the discontinuous point of the

(a) $M_p=0$

(b) $M_p=3$

(c) $M_p=19$

Fig. 9 Traction estimated by applying the regularization technique

traction distribution. In particular, though $q(x)$ must be zero, it reaches 50 MPa. If the direct analysis is performed on the basis of this result, the distributions of individual stress components may be inaccurate near the boundary DE. In the case of $M_p=19$, the

Fig. 10 Contour plots of stress components derived from estimated traction shown in Fig. 9(b) (MPa)

Fig. 11 The estimation error versus the number of internal points

accuracy of the estimate is improved remarkably. However, from practical point of view, it is preferable that the number of strain gages is as small as possible.

In the case of $M_p=3$, the estimated traction is fairly accurate. The result of the stress separation in the case of $M_p=3$ is plotted in Fig. 10. This result is in good agreement with the exact one (Fig. 4).

Figure 11 shows the norm of the estimation error versus the number of internal points M in the case of $M_p=0$, $M_p=3$ and $M_p=19$. As M increases, the estimation error tends to decrease in all cases. However, the estimation error in the case of $M_p=19$ is almost constant. From practical point of view, $M_p=3$ seems sufficient to improve the accuracy of estimation.

5. Conclusions

In this paper, the problem of stress separation in the thermoelastic stress analysis (TSA) has been studied. The stress separation problem can be divided into two parts, namely, (1) the inverse problem to estimate the unknown boundary values from TSA data, and (2) the direct problem to determine the individual stress components on the basis of the estimated boundary values. It has been shown that these two problems can be efficiently formulated and suc-

cessfully solved by BEM. In addition, it has been shown that a regularization technique is necessary in order to stabilize the solution of the inverse problem and to improve the accuracy of the stress separation. Tikhonov's method with supplementary data is found to be effective for regularization of the inverse problem.

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