Representation of Topology Using Homology Groups and Its Application to Structural Optimization*

(Fitness Value Based upon Topology for Unanalyzable Structures Generated in Genetic Algorithm and Its Effect on Performance of Optimization)

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Genetic algorithm (GA) is one of the most useful methods to optimize topology of structures. The performance of GA, however, deeply depends upon a rule of coding from a structure to a string (a chromosome), which must be decided before the execution of GA. An improper coding can cause a lot of unanalyzable structures to be generated in the process of optimization, which are separated into several pieces with no supporting or loading points. In this paper, a method to give such unanalyzable structures a fitness value based upon their topology using homology theory is proposed to raise the probability of obtaining the optimum structures. As numerical examples, topology of two-dimensional frames and three-dimensional structures consisting of triangular elements supported on a rigid wall and loaded vertically on a point distant from the wall are optimized under a constraint of constant weight. As a result, it was found that the proposed method increased the average fitness value and the number of optimum structures obtained in 100 trials of GA in comparison with other methods that considered unanalyzable structures to be useless.

Key Words: Optimum Design, Computational Mechanics, Numerical Analysis, Framed Structure, Homology Group, Genetic Algorithm, Topology, Torsion Coefficients

1. Introduction

Recently many new methods of optimizing structures using Genetic Algorithms (GAs) have been proposed, and they produced excellent results, especially in topology optimization problems^{(1),(2)}. When GA is applied to structural optimization problems, a method of encoding a structure to a string (a coding) must be decided prior to the execution of GA. In case the coding is improper, many structures which cannot be analyzed by finite element method (FEM) and so forth would be generated in the process of optimization. They deteriorate the performance of GA in most

cases. A structure divided into a few parts of which loading points are not connected to supporting points is an example of such unanalyzable ones.

In this paper, a new method of raising the probability of obtaining the optimum structures by GA is proposed; the above unanalyzable structures are given fitness values based upon their topology by homology theory (a field of topology in mathematics)^{(3),(4)}. A simplicial complex consisting of elements called simplexes is used in homology theory, and its topology is described as homology groups. Because the homology group is a kind of Abelian group, it can be written by canonical direct sum decomposition in the unique form, that is, direct sum of finite and infinite cyclic groups. The topology of any structures can be represented by the use of homology groups in a common form irrespective of their dimensions or shapes. In the previous paper⁽⁵⁾, the nature of the homology groups was studied in

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examples of an abstract combination problem of triangles, and it was proven that the homology groups can be treated by GA. In this paper, homology theory is applied to topology optimization of two-dimensional frames (in the second chapter) and three-dimensional structures consisting of planer triangular elements (in the third chapter) using GA as practical examples. Strain energy of a structure supporting a static load is minimized under a constraint condition of constant weight.

2. Topology Optimization Problem of Two-Dimensional Frames

2.1 Statement of problems

Strain energy of a two-dimensional frame loaded 9.8 N vertically at a point 1 m distant from a rigid wall as shown in Fig. 1 is minimized by GA under a condition of constant weight. All supporting points on the wall are not allowed any displacements and rotations. Structures (a) and (b) in Fig. 1 are ground structures. It is assumed that the upper and lower half parts of a structure obtained from the ground structure (b) are symmetrical. All members of a frame shall be made of homogeneous steel and have a circular cross-section in the same diameter. The prescribed weight of a frame is taken to be 39.1 kg, and this condition is satisfied by adjusting a diameter of members. Strain energy of a frame is calculated by FEM on the assumption that displacements are so small that the frame can be treated as an elastic body. The two-dimensional frame is replaced with a one-dimensional complex prior to a calculation of homology groups by regarding members and their nodes (joints) as 1- and 0-simplexes respectively. Zero- and one-dimensional homology groups of a one-dimensional complex are represented by direct sum of infinite cyclic groups (additive groups of integers). They do not have finite cyclic groups with torsion coefficients. Two or more-dimensional homology groups of a frame are always equal to zero.

2.2 Method of coding and fitness functions in GA

A value of a feature (a gene) shall be 0 or 1 in this chapter. Each feature in a string corresponds to a

Fig. 1 Ground structures for optimization of planar frames

certain member of a ground structure. When its value is 0, a member corresponding to it shall be eliminated from the ground structure. Therefore the length of a string is 7 when the structure (a) is used as a ground structure, and that of the structure (b) is 14. This coding is one of typical methods that generate many unanalyzable structures in the process of optimization using GA.

A fitness function to evaluate frames which can be analyzed by FEM is different from that for unanalyzable ones in this paper. The following function F_1 is employed as a fitness function for the analyzable frames.

$$F_1 = 1 / (\{u\}^T \{f\}) \quad (1)$$

where $\{u\}$ and $\{f\}$ are displacement and load vectors in FEM, and $\{u\}^T$ denotes a transposed vector of $\{u\}$. The value of this function increases with a decrease in strain energy of the frame. A fitness value F_2 is given to unanalyzable frames according to the following procedure. A temporary fitness value F_t based upon homology groups is calculated first.

$$F_t = 1 / \{rank(H_0(K)) - 1 - rank(H_1(K)) / M\} \quad (2)$$

where $rank(H_0(K))$ and $rank(H_1(K))$ denote ranks of zero- and one-dimensional homology groups of a frame (a complex K), and M is a rank of a one-dimensional homology group of the ground structure. $rank(H_0(K))$ and $rank(H_1(K))$ correspond to the numbers of connected components and loops consisting of members, respectively. The value of F_t increases with a decrease in $rank(H_0(K))$ or an increase in $rank(H_1(K))$. The (true) fitness value F_2 for unanalyzable frames is obtained as follows after all frames in the generation of GA are given a value F_1 in Eq. (1) or F_t in Eq. (2).

$$F_2 = C F_t F_{max1} / F_{maxt} \quad (3)$$

where C is a coefficient (a real number between 0 and 1). F_{max1} denotes the maximum value of F_1 in the generation, and F_{maxt} is that of F_t . The coefficient C is employed to heighten F_{max1} than any value of unanalyzable frames in the generation. Although some frames which have two or more connected components can be analyzed by FEM, any frame of which $rank(H_0(K))$ is more than 1 shall be treated as an unanalyzable frame in this paper. Frames which do not connect a loading point with the rigid wall are also regarded as unanalyzable frames, even if their $rank(H_0(K))$ are equal to 1.

2.3 Numerical examples

Strain energy of two-dimensional frames is minimized using the ground structure (a) or (b). Young's modulus, modulus of rigidity and the mass density of members are taken as 202 GPa, 81.4 GPa and 7.83×10^3 kg/m³, respectively. Simple crossover and an elitist model are employed as processes of

Fig. 2 Optimum frames

Fig. 3 Relation between ratio of average fitness values $\bar{F}/\bar{F}_e$ and coefficient C (Ground structure (a))

crossover and selection in GA. When the ground structure (a) is used, the probability of mutation to each string position shall be 1.43% in order to adjust the number of strings including mutant genes to about 10% of population size p , where p is taken to be one of three values 10, 20 and 30. Those of the ground structure (b) shall be 0.71%, and one of five values 10, 20, 30, 60 and 100. When a fitness value of the elitist (best) string has not changed for generations of the prescribed number g , the search for the optimum frame using GA shall be completed, where g is taken to be one of two values 10 and 50. The coefficient C shall be one of 10 values 0, 0.1, 0.2, ..., 0.9. The numbers of combinations of the prescribed generation number g , the population size p and the coefficient C to the ground structure (a) and (b) amount to 60 and 100 respectively. GA is executed 100 times in each case, and all features are changed at random in the beginning of each trial.

The optimum frames shown in Fig. 2 were obtained from the ground structures (a) and (b) by GA under the above conditions. Figure 3 shows the relation between the coefficient C in Eq. (3) and average fitness values of frames obtained from the ground structure (a). The abscissa and the ordinate in Fig. 3 are the coefficient C and a ratio of average fitness values $\bar{F}/\bar{F}_e$ respectively, where $\bar{F}_e$ denotes an average of 100 fitness values (the maximum value in each run of GA, which is not always the optimum one) obtained by 100 runs of the existing GA ignoring unanalyzable frames by regarding their fitness values

Fig. 4 Relation between ratio of average fitness values $\bar{F}/\bar{F}_e$ and coefficient C (Ground structure (b))

as zero, and $\bar{F}$ is that of the present GA utilizing homology groups. Figure 4 shows a ratio of average fitness values obtained by the ground structure (b). $\bar{F}/\bar{F}_e$ in Figs. 3 and 4 are values averaged on the population size (averages of 3 and 5 ratios of average fitness values respectively). The solid lines in these figures show results obtained when the prescribed generation number g is taken to be 10, and the broken ones are those of $g=50$. When the value of the ordinate is higher than 1, it can be said that the performance of the existing GA on fitness values is improved by applying homology theory. Most values of the ordinates exceeded 1 except only one case of $C=0.3$ in Fig. 3. We think these results are evidence of the validity of the present GA. It was also found that a value between 0.4 and 0.8 is proper for the coefficient C , and the present GA is more effective when g is small. The results obtained by the ground structure (b) are deeply influenced by conditions, such as the value of the coefficient C and the prescribed generation number g , in comparison with those of the ground structure (a). For example, an index of improvement $\bar{F}/\bar{F}_e - 1$ in Fig. 4 is outstanding at $C=0.5$. Because the ground structure (a) has much less members than (b), many frames with a high fitness value can be easily obtained even though unanalyzable frames are ignored. The present GA cannot display its ability in such a case. However it can work effectively when the ground structure (b) is used, because a large number of unanalyzable frames frequently appear. We also found that unanalyzable frames survive till a later generation with an increase in the coefficient C , though data about this are not printed here for want of space.

Figure 5 shows the relation between the population size p and a ratio of average fitness values $\bar{F}/\bar{F}_e$ obtained by the ground structure (b). $\bar{F}/\bar{F}_e$ is a value averaged on the coefficient C (an average of 10 results of $C=0, 0.1, 0.2, \dots, 0.9$). It can be said that the effects of the present GA are prominent when both of the

Fig. 5 Ratio of average fitness values $\bar{F}/\bar{F}_e$ in relation to population p (Ground structure (b))

Fig. 6 Ratio of standard deviations σ/σ_e in relation to coefficient C (Ground structure (b))

population size p and the prescribed generation number g are taken to be small values, although the present GA can hardly work when p or g is a high value.

The relation between σ/σ_e and the coefficient C is shown in Fig. 6, where σ_e denotes a standard deviation of fitness values of structures obtained from the ground structure (b) by the existing GA ignoring unanalyzable frames, and σ is that of the present GA. The solid and the broken lines are results obtained when g equals 10 and 50 respectively. Figures 5 and 6 show that the ratio σ/σ_e decreases with an increase in $\bar{F}/\bar{F}_e$, and they have a close correlation. It was also found that the value of σ/σ_e does not generally depend on the prescribed generation number g .

Figure 7 shows the relation between the coefficient C and the probability of obtaining the optimum frames by the ground structure (a). The ordinate is a ratio of P to P_e , where P_e and P denote the probabilities of obtaining the optimum frames using the existing and the present GA respectively. Figures 3 and 7 show that the ratio of average fitness values $\bar{F}/\bar{F}_e$ and P/P_e closely correlate to each other. The value of P/P_e obtained by the ground structure

Fig. 7 Ratio of probabilities of obtaining optimum structures P/P_e vs. coefficient C (Ground structure (a))

Fig. 8 Ratio of probabilities of obtaining optimum structures P/P_e vs. coefficient C (Ground structure (b))

(b) is shown in Fig. 8. Although $\bar{F}/\bar{F}_e$ and P/P_e of the ground structure (b) also have a close correlation as shown in Figs. 4 and 8, the number of the optimum frames falls short of our expectation based on the value of $\bar{F}$ in some cases such as $C=0.6$.

3. Topology Optimization Problem of Three-Dimensional Structures

Topology of two-dimensional frames was optimized by applying homology theory to GA in a previous chapter. In this chapter, our proposed GA is utilized for topology optimization of three-dimensional structures consisting of planer triangular elements.

3.1 Statement of problems

Strain energy of a three-dimensional structure is minimized under a constraint condition that its total mass equals 78.3 kg. The structure shall be sustained at several points on a rigid wall and loaded 98 N vertically at a point 2 m distant from the wall. Four structures (c), (d), (e) and (f) in Fig. 9 are used as ground structures. All supporting points marked with gray triangles are not allowed any displacements and rotations. While boundary lines showing the regions where triangular elements exist are drawn in Fig. 9, the elements are not described because these structures have too many elements to draw. The ground structure (c) has six nodes (black dots in Fig.

Fig. 9 Ground structures for optimization of three-dimensional structures (Triangular elements are not drawn.)

9), and three vertexes of each triangular element are chosen from among these nodes. Elements of which all vertexes are supporting points shall be excluded. Therefore, the number of elements in the ground structure (c) amounts to 19. The ground structure (c) is also utilized to construct other ones. The ground structures (d), (e) and (f) consist of two, four and eight parts corresponding to (c), and they have 38, 64 and 128 triangular elements. Elements with three supporting points and duplicated extra ones in the interfaces between the parts shall be excluded. It is assumed that all elements are made of homogeneous steel and have the same thickness. The thickness is adjusted to a proper value based on the gross area of elements to satisfy the constraint condition of constant weight. Structures obtained from these ground structures (c) - (f) are assumed to be the plane symmetry. Strain energy of a structure is calculated by FEM on the assumption that displacements are small enough to consider it an elastic body. Planer shell elements (non-conforming triangular elements) in which bending and 'in plane' forces are presumed to cause independent and small deformations are employed. When topology of a three-dimensional structure is calculated, it should be replaced with a two-dimensional simplicial complex first by regarding triangular elements, their sides and vertexes as 2-, 1- and 0-simplexes respectively. Zero-, one- and two-dimensional homology groups consist of infinite cyclic groups. Only one-dimensional homology group of a two-dimensional complex can include finite cyclic groups with torsion coefficients besides infinite ones. Homology groups of other dimensions are always equal to zero.

3.2 Method of coding and fitness functions in GA

When a structure is coded to a string, a number

Fig. 10 Complex L generated from complex K

assigned to each triangular element shall be reduced by one and encoded to a binary number in this chapter. The binary numbers are arranged side by side in a string. For example, a string (101 | 111 | 010) corresponds to a structure consisting of three elements 6, 8 and 3. This coding method generates relatively short strings. For example, a string of a structure consisting of fewer than 18 elements is shorter than that coded by the method in a previous chapter when the ground structure (f) is used. Structures shall be allowed to have separate multiple elements. In other words, the same binary numbers can exist in the identical string.

The fitness function F_1 in Eq.(1) is employed again in this chapter to give a fitness value to a structure which can be analyzed by FEM. The following function is utilized as a temporary fitness function F_t for an unanalyzable structure K . This function corresponds to Eq.(2).

$$F_t = 1 / \left\{ \sum_{i=1}^t \gamma_i - t + \text{rank}(H_0(K)) / N_0 + \text{rank}(H_0(L)) / (N_0 N_1) - \text{rank}(H_1(K)) / (2 N_0 N_1^2) - \text{rank}(H_2(K)) / (2 N_0 N_1 N_2) \right\} \quad (4)$$

where γ_i denotes a value of a torsion coefficient in a one-dimensional homology group of the structure K , and t is the number of torsion coefficients. N_0 , N_1 and N_2 are the numbers of 0-, 1- and 2-simplexes in a ground structure, respectively. The structure L in Eq. (4) is an imaginary structure (a one-dimensional simplicial complex) generated by exchanging each 2-simplex in the structure K for two 1-simplices as shown in Fig. 10. $\text{rank}(H_0(L))$ and $\text{rank}(H_0(K))$ correspond to the numbers of connected components of the structure L and K respectively. $H_0(L)$ is not always equal to $H_0(K)$. In the other words, the number of connected components of the structure L is different from that of the structure K in general. For example, the structure L in Fig. 10(b) has more connected components than the structure K , although they have the same one-dimensional homology group in Fig. 10(a). The value of the temporary fitness

function F_t in Eq. (4) increases with decreases in the values of torsion coefficients γ_i , $rank(H_0(K))$ and $rank(H_0(L))$ or increases in $rank(H_1(K))$ and $rank(H_2(K))$. $rank(H_1(K))$ is equal to the number of holes made in faces of the structure K , and $rank(H_2(K))$ corresponds to the number of three-dimensional spaces enclosed with 2-simplexes. The (true) fitness values F_2 of unanalyzable structures are given by Eq. (3) as mentioned in a previous chapter.

3.3 Numerical examples

Strain energy of a three-dimensional structure consisting of triangular elements is minimized under a constraint condition of constant weight using the ground structures (c) - (f). The elements are assumed to be made of the same material with that in a previous chapter. A prescribed upper limit of the number of elements constituting a structure obtained from the ground structure (c) is taken to be 2, and those of the ground structures (d), (e) and (f) shall be 6, 6 and 10 respectively. Simple crossover and an elitist model are employed again in GA. The probability of mutation to each string position shall be 2.0% when the ground structure (c) is used, and those of the ground structures (d), (e) and (f) are taken to be 0.56%, 0.56% and 0.29% respectively so that the number of strings including mutant genes can account for about 10% of population size p in any case. The population size p is taken to be one of the following three values; 10, 20 and 30 when the ground structure (c) is utilized, 200, 400 and 600 to the ground structures (d) and (e), 300, 600 and 900 to the ground structure (f). When the best structure in each generation has kept a constant fitness value for 50 generations, the execution of GA shall be completed. The coefficient C is taken as one of ten values 0, 0.1, 0.2, ..., 0.9 as stated in chapter 2. GA is executed 100 times in each combination of the population size p and the coefficient C . Therefore, the total number of trials of GA amounts to 3 000 to each ground structure. All features in strings are changed at random before the first generation of each run.

Figure 11 shows the optimum structures obtained under the above conditions. The triangles colored deep gray in Fig. 11 represent separate double elements. The optimum structures (e) and (f) consist of elements 2 and 3 fewer than the prescribed upper limit numbers given to them. Figure 12 shows the relation between a ratio of average fitness values $\bar{F}/\bar{F}_e$ and the coefficient C . $\bar{F}_e$ denotes an average fitness value of structures obtained by the existing GA ignoring unanalyzable structures, and $\bar{F}$ is that of the present GA. $\bar{F}/\bar{F}_e$ in Fig. 12 is an average of three ratios of average fitness values obtained under three different conditions of the population size. The struc-

Fig. 11 Optimum three-dimensional structures

Fig. 12 Relation between ratio of average fitness values $\bar{F}/\bar{F}_e$ and coefficient C (Ground structures (c) - (f))

Fig. 13 Ratio of average fitness values $\bar{F}/\bar{F}_e$ in relation to population p (Ground structures (c) - (f))

ture (f) obtained from the ground structure of the greatest number of elements has the highest $\bar{F}/\bar{F}_e$ in most cases. The relation between a ratio of average fitness values $\bar{F}/\bar{F}_e$ and the population size p is shown in Fig. 13. $\bar{F}/\bar{F}_e$ in Fig. 13 is a value averaged on the coefficient C (10 values; 0 - 0.9). The ratio of average

Fig. 14 Ratio of standard deviations σ/σ_e in relation to coefficient C (Ground structures (c)-(f))

Fig. 15 Ratio of probabilities of obtaining optimum structures P/P_e vs. coefficient C (Ground structures (c)-(f))

fitness values $\bar{F}/\bar{F}_e$ tends to decrease with an increase in the population size p like that in Fig. 5. Figure 14 shows the relation between a ratio of standard deviations σ/σ_e and the coefficient C . σ_e denotes a standard deviation of fitness values of structures obtained by the existing GA, and σ is that of the present GA. The ratio of standard deviations σ/σ_e generally decreases at faster rate than that in Fig. 6 with an increase in the ratio of average fitness values $\bar{F}/\bar{F}_e$ or the coefficient C . The relation between P/P_e and the coefficient C is shown in Fig. 15. P_e is the probability of obtaining the optimum structures by the existing GA, and P denotes that of the present GA. It can be said that P/P_e increases with an increase in the number of elements of the ground structure. Figures 12 and 15 show that P/P_e and the ratio of average fitness values $\bar{F}/\bar{F}_e$ have a close correlation except a case of the ground structure (c).

Although there is no qualitative differences between results in chapters 2 and 3, the quantities, such as $\bar{F}/\bar{F}_e$ and P/P_e , in the topology optimization problem of three-dimensional structures are much higher than those of two-dimensional frames. It can

Fig. 16 Relation between ratio of average fitness values $\bar{F}/\bar{F}_e$ and coefficient C (Ground structure (g))

be said that the present GA can show its ability in the more difficult problem of the two.

4. Influences of a Coding Method on the Performance of the Present GA

We think that we could reach almost our goal mentioned in the first chapter, because the numerical examples in chapters 2 and 3 showed the validity of the present GA. A coding method, however, deeply influences the performance of GA in general, and needless to say, our proposed GA is also affected by it. Figure 16 shows a ratio of average fitness values $\bar{F}/\bar{F}_e$ obtained by the ground structure (g) shown in an upper right corner. The same coding as that in chapter 3 was employed; the numbers assigned to members of a structure were encoded to binary ones to make a string. It was assumed that the upper and lower parts of the structure are symmetric. The prescribed upper limit of the number of members, the population size p and the prescribed generation number g to complete GA were taken as 34, 50 and 50, respectively. The optimum structure consists of 26 members. Although strings are relatively long (the number of features is 85), the effect of the present GA is obvious as shown in Fig. 16 except a case of $C=0.7$. An index of improvement $\bar{F}/\bar{F}_e - 1$ in Fig. 16 is larger than those in other Figs. 3, 4 and 12. However the present GA was not effective when another coding method in chapter 2 was adopted; the existing and eliminated members were represented by 1 and 0 respectively. Therefore we must say that the performance of the present GA is easily influenced by the method of coding, though it can show some effects irrespective of methods of coding. Further improvement of GA is necessary to reduce interference of coding methods.

5. Conclusions

In the process of topology optimization using genetic algorithms (GAs), useless structures, which

cannot be analyzed by finite element method and so forth, often appear because of an improper coding. We proposed new method of improving the performance of GA, such as the probability of obtaining the optimum structures and an average fitness value, by giving such unanalyzable structures fitness values based upon their topology using homology theory. Strain energy of a two-dimensional frame and a three-dimensional structure consisting of triangular elements was minimized as numerical examples to verify the validity of our proposed GA. As a result, the following were found. (1) When the population size and the prescribed generation number to complete GA are taken to be small values, the present GA can work effectively and improve an average fitness value and the probability of obtaining the optimum structures. (2) A value between 0.4 and 0.8 is proper for the coefficient C adjusting the magnitude of fitness values of unanalyzable structures. (3) When an average fitness value is increased by the effects of the present GA, the probability of obtaining the optimum structures also increases, whereas the standard deviation of fitness values decreases; these three values have a close correlation. (4) The above tendencies (1)-(3) are qualitatively common to two topology optimization problems of two-dimensional frames and three-dimensional structures. Indices of the performance of the present GA such as an average fitness value in the optimization problem of three-dimensional structures are quantitatively superior to those of two-dimensional frames in general. (5) The proposed method utilizing homology theory rarely changes the performance of GA for the worse. Its adverse effects are, however, so slight as to be ignored in comparison with much benefit from the present GA.

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