Analysis of 2-D Steady-State Thermoelasticity by a Simple Adaptive Boundary Element Method*

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This study is concerned with a new adaptive boundary element method based on sample-point error analysis for thermoelasticity under steady-state heat conduction. Mathematical formulations of the adaptive boundary element method are presented in detail for two-dimensional thermoelasticity. This scheme of adaptive meshing makes use of only the residual between the interpolated and calculated solutions as an error indicator. Adaptive boundary element analysis can be performed by simultaneously checking the 'displacement error indicator' and the 'temperature error indicator'. The h- and p-version mesh refinements based on this scheme are applied to some typical examples, and the usefulness of the proposed adaptive BEM is demonstrated through discussion of the results obtained.

Key Words: Boundary Element Method, Computational Mechanics, Structural Analysis, Thermal Stress, Elasticity, Thermoelasticity, Adaptive Method

1. Introduction

There are many objects from electronic equipment to aerospace structures which are used under a severe thermal environment. It is known that in these objects crack or damage is caused by change in temperature and unequal distribution of temperature. Therefore, it is important to accurately solve the thermoelastic problem for prediction and evaluation of such crack or damage.

The boundary element method is a numerical method, which can be an alternative to the finite element method and the difference method. The boundary element method has the advantage that only the boundary of the problem is divided into elements

and a highly accurate solution is obtained. Various techniques on the application of the boundary element method to the thermoelastic problem have been proposed, and it has been known that the problem can be analyzed with good accuracy⁽¹⁾⁻⁽¹²⁾.

However, even if we can obtain a highly accurate solution by the boundary element method, meshing greatly depends on experience and intuition of the analyst, and the numerical results are dependent on this meshing. In finite element analysis, the adaptive method⁽¹³⁾ has been proposed in order to solve this problem, and many kinds of such commercial software are available. Application of the adaptive method to the boundary element method has been recently developed⁽¹⁴⁾⁻⁽¹⁸⁾. An element with a large error is divided into smaller elements in the h-version adaptive method. In the p-version, the order of polynomials used as the interpolation function is increased.

In this paper, a simple adaptive boundary element method is proposed for 2-D steady-state thermoelasticity.

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Fig. 1 Solutions and residual on boundary element

2. Formulation

2.1 Integral equations and fundamental solutions

We shall consider two-dimensional thermoelasticity in steady-state heat conduction under the assumption of no body force and no heat source. The boundary integral equation is written as follows⁽¹²⁾:

$$cu_j(\mathbf{y}) = - \int_{\Gamma} u_I(\mathbf{x}) p_{IJ}^*(\mathbf{x}, \mathbf{y}) d\Gamma(\mathbf{x}) + \int_{\Gamma} p_I(\mathbf{x}) u_{IJ}^*(\mathbf{x}, \mathbf{y}) d\Gamma(\mathbf{x}) \quad (I, J=1, 2, 3) \quad (1)$$

In Eq. (1) the coefficient c is such that $c=1.0$ when the source point $\mathbf{y}$ is located inside the domain, and $c=0.5$ when the source point $\mathbf{y}$ is on a smooth boundary. In Eq. (1) the summation convention is implied for twice repeated indices. It is noted that $u_j(\mathbf{x})$ are the displacement components in x_1 and x_2 on two-dimensional space, and the temperature u_3 at a spatial point $\mathbf{x}$, respectively, whereas $p_j(\mathbf{x})$ are also the traction components in x_1 and x_2 , and the heat flux p_3 at point $\mathbf{x}$, respectively. The asterisked functions u_{IJ}^* are the fundamental solutions concerning displacement and temperature, while p_{IJ}^* are the fundamental solutions concerning traction and heat flux. It is interesting to note that isotropic materials and anisotropic materials can be equally expressed in the form of Eq. (1), although the detailed expressions are different⁽¹²⁾.

2.2 Adaptive boundary element method

There are many proposals for the adaptive BEM. Kawaguchi and Kamiya⁽¹⁷⁾ obtained the difference (residual) between the interpolated solution $u'(\mathbf{s})$ and the calculated one $u''(\mathbf{s})$ calculated from the boundary integral equation on the mid-point (sample point) of the linear boundary element. Solutions $u'(\mathbf{s})$, $u''(\mathbf{s})$ and the residual on boundary element is shown in Fig. 1. They called this concept 'sample point error analysis'. This concept entails that the residual is obtained by the effect from all boundary elements, and the

Fig. 2 L-shaped domain elastic problem for error distribution

effect of the error concerning the residual per element is calculated for all boundary elements. They carried out the adaptive analysis using an h-version mesh refinement.

In this paper, this concept of sample point error analysis is adopted but the error of the element is evaluated only by the residual, and adaptive analysis by the h- and p-versions is carried out.

Residual $r(\mathbf{s})$ for sample point $\mathbf{s}$ is given in the following equation:

$$r(\mathbf{s}) \equiv u'(\mathbf{s}) - u''(\mathbf{s}) \quad (2)$$

when $u''(\mathbf{s})$ is obtained, it is necessary to evaluate the integral equation in the sense of Cauchy's principal value. Here, it is evaluated by using the sub-element method near the sample point.

Error $\epsilon(\mathbf{s})$ on a sample point is calculated as follows:

$$\epsilon(\mathbf{s}) = r(\mathbf{s}) / |u|_{\max} \times 100(\%) \quad (3)$$

In Eq. (3) $|u|_{\max}$ is the maximum of absolute value of the solution for the boundary integral equation. $|u|_{\max}$ can be uniquely determined, and hence the residual $r(\mathbf{s})$ is proportional to the error $\epsilon(\mathbf{s})$. In this paper, the thermoelastic problem is treated. The governing equations of displacement field and temperature field are solved. Therefore, the residuals and errors for displacement and temperature are obtained as follows:

$$r_i = u'_i - u''_i, \quad \epsilon_i = r_i / |u_i|_{\max} \times 100(\%) \quad (i=1, 2, 3) \quad (4)$$

where the residual $r_i(\mathbf{s})$ is for displacement and temperature, and the error $\epsilon_i(\mathbf{s})$ is also for displacement and temperature.

2.3 Error distribution on boundary element

We do not know how the error is distributed on boundary element. Therefore, the error distribution is examined on second- and third-degree polynomials of

Fig. 3 Error ε_1 distribution for an L-shaped domain elastic problem

the interpolation function, which is later described, in the p-version referring to L-shaped domain problem (plate thickness is assumed as 1 mm)⁽¹⁹⁾ shown in Fig. 2. In this case, Young's modulus is 206 GPa, Poisson's ratio is 0.3 and the plane strain state is assumed. A uniformly distributed load of $p_2=1$ N/mm (force per unit plate thickness) acts on the edge CD to x_2 direction. The boundary of the problem is divided into 12 elements of a quadratic-shape function. It is confirmed that the results obtained using a second degree interpolation function are identical with those by a standard quadratic element. The error distribution of the displacement in the x_1 direction is shown in Fig. 3. The error is calculated at a position which has an equal distance between 10 collocation points. The abscissa in Fig. 3 is the distance from point A along the tangential coordinate, and the ordinate is the displacement error in the x_1 direction. It is seen that the error at an arbitrary point on the boundary changes almost continuously and becomes a peak near the center between the nodal point and the collocation point.

2.4 Algorithm of adaptive analysis

We shall consider mesh refinement by the h- and p-versions. In both versions, we use a quadratic expression such as Eq. (5) for the shape function and the interpolation functions as shown in Eq. (6)⁽²⁰⁾.

$$\begin{aligned}\phi_0 &= \frac{1}{2}\xi(1-\xi), & \phi_1 &= \frac{1}{2}\xi(1+\xi), \\ \phi_2 &= \frac{1}{2}(1-\xi)(1+\xi)\end{aligned}\quad (5)$$

$$\begin{aligned}\phi_0 &= \frac{1}{2}(1-\xi), & \phi_1 &= \frac{1}{2}(1+\xi), \\ \phi_p &= \begin{cases} \frac{1}{p!}(\xi^p - 1) & \text{if } p \text{ is even} \\ \frac{1}{p!}(\xi^p - \xi) & \text{if } p \text{ is odd} \end{cases} & p \geq 2\end{aligned}\quad (6)$$

where ξ denotes local coordinate that is $-1 \leq \xi \leq 1$. In the h-version, the degree of the interpolation function is 2. In the p-version, it changes from 2 to 10, and it

Fig. 4 Collocation and sample points on boundary element

can be increased corresponding to the error value.

In discretization of the boundary integral equation (1) by Eq. (6), the number of discretized equations is to be identical with the number of unknown quantities. Therefore, it is necessary to consider collocation points on boundary elements, so that they are located at positions which equally divide the element. In this case, we can calculate the temperature and displacement directly at nodes, but we cannot obtain these physical quantities directly at collocation points. Here, we can transform these quantities to their physical quantities using Eq. (6). The position where the error of element is evaluated is the center of two adjacent collocation points in Fig. 3 for the error distribution on the boundary. Figure 4 shows collocation points and sample points on the boundary element. In the p-version the number of sample points on one element is dependent on the degree of the interpolation function. Hence, it is necessary to employ a common measure for evaluating the error indicator of one element by the h- and p-versions. Here, we calculate the error of one element using Eq. (7) for the sake of simplification.

$$\varepsilon^i = \sum_{j=1}^{p_i} |\varepsilon^{i,j}| \quad (7)$$

where $\varepsilon^{i,j}$ denotes the error of the j th sample point on the i th element, and p_i denotes the degree of interpolation function of the i th element.

New elements are generated when the error

Fig. 5 Mesh refinement for h-version

Fig. 6 Analysis model 1 (Square plate)

ε_k^i ($k=1, 2$) of displacement or the error ε_3^i of temperature is greater than the reference value. The criteria of the generation in h- and p-versions are shown as follows:

$$\text{Mesh refinement} \begin{cases} \text{generate new element;} \\ h\text{-version} \\ p_i = p_i + 1, 2 \leq p_i \leq 10; \\ p\text{-version} \end{cases}$$

$$\text{if } \varepsilon_1^i \geq \varepsilon_1^{\text{ref}} \text{ or } \varepsilon_2^i \geq \varepsilon_2^{\text{ref}} \text{ or } \varepsilon_3^i \geq \varepsilon_3^{\text{ref}} \quad (8)$$

where $\varepsilon_1^{\text{ref}}$, $\varepsilon_2^{\text{ref}}$ and $\varepsilon_3^{\text{ref}}$ denote the reference value of error for displacement and temperature, respectively. New elements by the h-version are generated by dividing the old element into two equal elements, as shown in Fig. 5.

3. Numerical Results

(1) Analysis case 1

Figure 6 shows the analysis model of the square plate (plate thickness is assumed as 1 mm)⁽²¹⁾. The upper edge is divided into 2 elements, and the other edges are divided into 1 element, respectively. The

Fig. 7(a) Final boundary element meshes for h-version

Fig. 7(b) Temperature u_3 on boundary for h-version Fig. 7(c) Displacement u_1 on boundary for h-version

adaptive analysis is carried out from this mesh. The plane strain state and isotropic material are assumed. Material properties are defined as in Eq. (9) and the reference values of errors are defined as in Eq. (10). E is Young's modulus, ν Poisson's ratio and α thermal expansion coefficient.

$$E = 2.0594 \times 10^{11} (\text{Pa}), \quad \nu = 0.3, \quad \alpha = 0.00001 (/^{\circ}\text{C}) \quad (9)$$

$$\varepsilon_1^{\text{ref}} = 0.5(\%), \quad \varepsilon_2^{\text{ref}} = 0.5(\%), \quad \varepsilon_3^{\text{ref}} = 0.5(\%) \quad (10)$$

Fig. 8(a) Final boundary element meshes for p-version

Fig. 8(b) Temperature u_3 on boundary for p-version Fig. 8(c) Displacement u_1 on boundary for p-version

In this analysis, the initial mesh is counted as the first mesh. In the h-version, the refinement was finished after 10 trials under 58 nodes. In the p-version, the refinement was completed after 9 trials under 35 nodes (collocation points were translated to nodes). Figure 7 shows the results of the h-version, while Fig. 8 shows those of the p-version. Here, Fig. (a) shows the final mesh refinement, (b) shows the temperature u_3 on the boundary and (c) shows the displacement u_1 in the x_1 direction. The numbers on the element in Fig. 8(a) mean the degrees of the interpolation func-

Fig. 9 Temperature u_3 at inner points Fig. 10 Displacement u_1 at inner points

tion in the p-version. The abscissas in (b) and (c) show the distance from point A along the tangential coordinate of the boundary. Figure 9 shows the temperature u_3 at $x_1=5$ mm of the initial mesh in the h- and p-versions compared with FEM results. Figure 10 shows similar results on the displacement u_1 . The FEM results are obtained by finite element analysis software I-DEAS Model Solution⁽²²⁾. The results were calculated using a square quadratic thin shell element with the side length of 0.25 mm. The relative energy norm error⁽²³⁾ for this solution was 1.02%. This FEM solution converged sufficiently. Figure 11 shows errors of temperature and displacement at sample points for the h-version. Figure 12 shows errors of temperature and displacement at sample points for the p-version. The ordinates of Figs. 11 and 12 are absolute values of the errors. In the p-version, the error of temperature in the element, which corresponded to the edges AB and BC of Fig. 6, did not achieve the reference value ϵ_3^{ref} . The mesh refinements were stopped before convergence. It is seen that the mesh in this position was very small in the h-version. At point B, the heat flux is discontinuous and remarkably large. Therefore, this effect influences the error of temperature. This technique which evaluates the error only using the residual is

Fig. 11(a) Error ε_3 on sample points for h-version Fig. 11(b) Error ε_1 on sample points for h-version Fig. 12(a) Error ε_3 on sample points for p-version Fig. 12(b) Error ε_1 on sample points for p-version

Fig. 13 Analysis model 2 (L-shaped domain)

sensitive, if heat flux changes discontinuously, and also when the mesh is remarkably small. In this example, the accuracy for the p-version is rather worse than for the h-version. However, a remarkable improvement of accuracy can be achieved in both

versions.

(2) Analysis case 2

Figure 13 shows the analysis model of an L-shaped domain (plate thickness is assumed as 1 mm). The whole boundary is equally divided into 8 elements. Adaptive analysis is carried out by using it as an initial mesh. The plane stress state and anisotropic material are assumed. Material properties are defined as in Eq.(11) and the reference value of error is defined as in Eq.(12). C_{ij} is compliance coefficient in anisotropic material, k_{ij} thermal conductivity and β_{ij} thermal expansion coefficient. That is,

$$\begin{aligned} C_{11} &= C_{22} = 5.4750130 \times 10^{-9}, \\ C_{12} &= -4.5249870 \times 10^{-9}, \\ C_{13} &= C_{23} = 1.4997400 \times 10^{-9}, \\ C_{33} &= 3.9000520 \times 10^{-9} (\text{Pa}^{-1}), \\ k_{11} &= k_{22} = 1.50, \quad k_{12} = -0.50 (\text{W}/(\text{mm} \cdot ^\circ\text{C})), \\ \beta_{11} &= \beta_{22} = 0.00125, \quad \beta_{12} = -0.00050 (/\text{ } ^\circ\text{C}) \end{aligned} \quad (11)$$

$$\varepsilon_1^{\text{ref}} = 0.5(\%), \quad \varepsilon_2^{\text{ref}} = 0.5(\%), \quad \varepsilon_3^{\text{ref}} = 0.5(\%) \quad (12)$$

In this numerical example, the refinement was finished in the h-version after 5 trials under 52 nodes. For the p-version, it took 8 trials under 39 nodes (collocation points were translated to nodes). Figures 14 and 15 show the final results on mesh refinements. Figure 16

Fig. 14 Final boundary element meshes for h-version

Fig. 15 Final boundary element meshes for p-version

Fig. 16 Temperature u_3 at inner points

shows the temperature u_3 at $x_1=37.5$ mm of the initial mesh in the h- and p-versions compared with FEM results. Figure 17 shows the displacement u_1 . The results were calculated using a square quadratic thin shell element with the side length of 0.5 mm. The relative energy norm error for this solution was 0.86%. This FEM solution converged sufficiently. The errors of temperature and displacement on sample points are shown in Figs. 18 and 19, respectively. The errors decrease by this adaptive analysis in both versions. The errors of the final mesh in both versions are almost equal. Both versions provide good con-

Fig. 17 Displacement u_1 at inner points Fig. 18(a) Error ϵ_3 on sample points for h-version Fig. 18(b) Error ϵ_1 on sample points for h-version

verged results. The accuracy for the h-version is higher than the p-version. This is remarkable in Fig. 17 which shows displacement at inner points. In the h-version the element can be locally divided, whereas in the p-version the degree of the interpolation functions of elements with large errors is increased. It is pointed out that since the technique used for each version is different, a difference in accuracy may occur.

4. Conclusion

A simple adaptive boundary element method has

Fig. 19(a) Error ε_3 on sample points for p-version Fig. 19(b) Error ε_1 on sample points for p-version

been proposed for 2-D steady thermoelasticity. It is shown that good results can be obtained by applying the h- and p-version mesh refinements. Through computation of some examples, it is concluded that the accuracy of the h-version is higher than that of the p-version in this technique. The present technique, which evaluates the error using only the residual, is likely to result in generating remarkably small elements for the problems in which the heat flux changes discontinuously. This technique is simple, and good results can be obtained. As future work, it is recommended to extend this technique to 3-D problems.

References

- (1) Tanaka, M., Togoh, H. and Kikuta, M., Fracture Mechanics Application in Thermoelastic States, Topics in Boundary Element Research, ed. by Brebbia, C.A., Vol. 1 (1984), p. 59-77, Springer-Verlag.
- (2) Tosaka, N., Boundary Integral Equation Formulations for Linear Coupled Thermoelasticity, Proc. 3rd JASCOME Symposium on BEM, (1986), p. 207-212.
- (3) Suh, I. and Tosaka, N., Boundary Element Analysis of Linear Coupled Thermoelasticity Problem, Proc. 4th JASCOME Symposium on BEM, (in Japanese), (1987), p. 99-104.
- (4) Chaudouet, A., Three-dimensional Transient Thermoelastic Analysis by the BIE Method, Int. J. Num. Meth. Eng., Vol. 24 (1987), p. 25-45.
- (5) Dargush, G.F. and Banerjee, P.K., Boundary Element Methods for Poroelastic and Thermoelastic Analyses, Developments in Boundary Element Methods-5, (1989), p. 119-156, Elsevier Appl. Sci.
- (6) Ishiguro, S. and Tanaka, M., Analysis of Two-Dimensional Transient Thermoelasticity by the Time-Stepping Boundary Element Method, Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 57, No. 544, A (1991), p. 2999-3004.
- (7) Deb, A., Henry, Jr.D.P. and Wilson, R.B., Alternate Bem Formulations for 2- and 3-D Anisotropic Thermoelasticity, Int. J. Solids Struct., Vol. 27, No. 13 (1991), p. 1721-1738.
- (8) Sekiya, T., Ochiai, Y. and Ishida, R., Thermal Stress Analysis by Boundary Element Method, (in Japanese), (1992), p. 33-56, Morikita-Syuppan.
- (9) Ishiguro, S. and Tanaka, M., Analysis of Two-Dimensional Orthotropic Thermoelasticity by Boundary Element Method, Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 60, No. 569, A (1994), p. 205-211.
- (10) Matsumoto, T, Tanaka, M. and Kosasayama, H., A Study on Boundary Integral Approach for Coupled Thermoelastic Problems, Proc. of the 1994 Annual Meeting of JSME/MM, (in Japanese), A (1994), p. 80-81.
- (11) Ishiguro, S. and Tanaka, M., Analysis of Two-Dimensional Orthotropic Thermoelasticity by Boundary Element Method (2nd Report), Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 62, No. 595, A (1996), p. 713-719.
- (12) Ishiguro, S. and Tanaka, M., Analysis of Two-Dimensional Anisotropic Thermoelasticity by Boundary Element Method, Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 63, No. 613, A (1994), p. 1963-1970.
- (13) Zienkiewicz, O.C. and Taylor, R.L., The Finite Element Method Fourth Edition, (in Japanese), Vol. 1 (1996), p. 401-440, Scitech Press.
- (14) Alacon, E. and Reverter, A., p-Adaptive Boundary Elements, Int. J. Num. Meth. Eng., Vol. 23, No. 5 (1986), p. 801-829.
- (15) Rencis, J.J. and Mullen, R.L., Solution of Elasticity Problems by a Self-Adaptive Mesh Refinement Technique for Boundary Element Computation, Int. J. Num. Meth. Eng., Vol. 23, No. 8 (1986), p. 1509-1527.
- (16) Rencis, J.J. and Jong, K.Y., A Self-Adaptive h-Refinement Technique for the Boundary Element Method, Comp. Meth. Appl. Mech. Eng., Vol. 73, No. 3 (1989), p. 295-316.
- (17) Kawaguchi, K. and Kamiya, N., An Adaptive BEM by Sample Point Error Analysis, Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 56, No. 529, A (1990), p. 2032-2039.
- (18) Yuuki, R., Cao, G. and Ueda, S., Development of

- Expert System for BEM Analysis (2nd Report: Error Estimation and Adaptive Meshing), Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 57, No. 533, A (1991), p. 202-208.
- (19) Yuuki, R., Tamaki, M. and Noda, Y., Comparison of Adaptive Meshes Obtained by BEM-h Method and FEM-p Method, Proc. 9th JASCOME Symposium on BEM, (in Japanese), (1992), p. 109-114.
- (20) Zienkiewicz, O.C. and Taylor, R.L., The Finite Element Method Fourth Edition, (in Japanese), Vol. 1, (1996), p. 139-141.
- (21) Kamiya, N., Aikawa, Y. and Kawaguchi, K., Adaptive Boundary Elements for Thermoelastic Analysis, Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 60, No. 574, A (1994), p. 1416-1424.
- (22) I-DEAS Model Solution and I-DEAS Optimization User's Guide, (1990), SDRC.
- (23) Zienkiewicz, O.C. and Taylor, R.L., The Finite Element Method Fourth Edition, (in Japanese), Vol. 1 (1996), p. 405.
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