Automated and error controlled model reduction for durability based shape optimization of mechanical system

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Shape optimization based on durability values is an important tool to design light weight structure without lowering the durability of the component. The reliability and the dynamic behavior of the mechanical systems is improved.

The tool chain involved in a topology optimization consists of a Finite Element (FE)-Program and an Elastic Multibody System (EMBS) simulation, see [1]. Usually, the mechanical component to optimize is part of a complex mechatronic system. EMBS simulations are therefore used to calculate the dynamical loads and dynamical stresses of the system. From the EMBS results the stress and the damage distribution is calculated. Based on these results the optimization routine changes the structural design of the whole component. A new FE model needs to be built for every new design. For the simulation of the whole mechatronic system in EMBS a pre-processing step is necessary to reduce the full elastic body (FE model) of the optimized component to a reduced elastic body, see [2]. The selection and the number of the reduced elastic degrees of freedom have a big influence on the quality and the calculation time of the whole optimization process.

In this work, we concentrate on the reduction process which is one integral part in the optimization loop. In recent years the authors developed a new pre-processor Morembs [2] based on advanced model reduction techniques like Krylov-subspaces, Gramian matrix-based reduction techniques. Advantages of modern reduction techniques are, the distribution of the loads are taken into account a priori and very accurate models can be obtained within a predefined frequency range. Furthermore, a priori error bounds or efficient error estimators are available. The goal of this work is the development of a totally automated Krylov-subspace and a frequency weighted Gramian matrix based reduction process for mechanical systems. Only the load distribution, the frequency range of interest and a measure for the desired accuracy have to be provided by the user. This makes the method especially attractive for optimization because during the optimization loop no additional input has to be provided.

For Krylov-subspace based model reduction two points are necessary for an automated reduction process. Firstly, because of the expansion point dependency an automated selection of expansion points needs to be used, two possible methods were proposed in [3]. Secondly, an appropriate error estimator as stopping criteria when the desired accuracy is reached needs to be developed.

For reduction with frequency weighted Gramian matrices the number and location of the snapshots, which are needed for the calculation of the Gramian matrices, have a big influence on the reduction results. In Reduced Basis methods, sophisticated snapshot-selection methods are used for basis construction [4]. In the current study we transfer some of these methods to Gramian matrix based model reduction. The method can be viewed as an automatic determination of optimal frequency weighting and allows a totally automated reduction process.

AUTOMATED AND ERROR CONTROLLED MODEL REDUCTION FOR DURABILITY BASED STRUCTURAL OPTIMIZATION OF MECHANICAL SYSTEMS

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ABSTRACT
Structural optimization based on durability values is an important tool for the design of light weight structures. One essential part of the optimization loop is the Elastic Multibody System (EMBS) simulation. An important issue for the simulation of the EMBS is the automated reduction of the flexible body's degrees of freedom. Therefore, totally automated Krylov-subspace based reduction processes for mechanical systems are introduced. Besides, an $H_2$-optimal SOIRKA algorithm is compared with a Second Order Adaptive Global Arnoldi (SOAGA) algorithm. For the automated reduction process error estimation is necessary. Three different error estimators are introduced and compared. In addition, automated error controlled reduction with snapshot-based approximation of frequency-weighted Gramian matrices is a suitable reduction scheme. Advanced basis construction schemes from Reduced Basis methods, e.g. the Greedy and the adaptive Greedy algorithm, improve and accelerate the reduction process. A numerical example demonstrates the approximation capability in the frequency domain.

1 INTRODUCTION
Structural optimization based on durability values is an important tool for the design of light weight structures. With its help, the reliability of components in mechanical systems can be improved. If structural optimization is applied to an industrial problem, its effects on the dynamic behavior of the entire mechanical system usually have to be taken into account. Therefore, the tool chain involved in a conventional structural optimization loop based on durability values consists of an Elastic Multibody System (EMBS) simulation, a Finite Element (FE) calculation, a durability analysis and the optimization algorithm, see Tobias and Eberhard (2009). The EMBS simulation is used to calculate the dynamic stresses in the investigated component. Stresses are input for the durability analysis from which a damage distribution is derived. Based on these results, the optimization routine changes the FE mesh of the investigated component. A new FE model is generated which needs to be pre-processed for the next EMBS simulation in the loop. The pre-processing comprises a reduction of the full elastic body (FE model) and is mainly done to improve computational performance, see Fehr and Eberhard (2009b). The selection and the number of the reduced elastic degrees of freedom have a big influence on the quality and the calculation time of the whole optimization process. They are of particular importance if dynamic stresses are directly derived from the EMBS simulation, an approach that raises the efficiency of a structural optimization loop.

In this work, we concentrate on the reduction process which is one integral part in the optimization loop. In recent years, the authors developed a new pre-processor Morembs, see Fehr and Eberhard (2009b) based on advanced model reduction techniques like Krylov-subspaces and Gramian matrix-based reduction techniques. Not only the fact that the distribution of the loads is taken into account a priori but also that very accurate models can be obtained within a predefined frequency range have to be considered as advantages of modern reduction techniques. Furthermore, a priori error bounds or efficient error estimators are available. With these techniques, a totally automated pre-processing of the elastic bodies is possible. Only the load distribution, the frequency range of interest and a measure for the desired accuracy have to be provided by the user which makes the method especially attractive for optimization because during the optimization loop no additional input has to be provided.

In Section 2 the overall structural optimization loop is explained, then we recapitulate the idea of model reduction in flexible multibody systems. In Section 3 we give a short introduction to the Krylov-subspace based model reduction and explain the steps to be taken into consideration in order to achieve an automated reduction process. Afterwards, the automated reduction with frequency weighted Gramian matrices based on advanced basis construction from Reduced Basis methods Haasdonk and Ohlberger (2008) is explained in Section 4. The specific applicability of the different automated reduction techniques is then compared with a technical system
consisting of more than 5,000 degrees of freedom in Section 5. Our final conclusions constitute the last section.

2 Process of Durability-based Structural Optimization including EMBS-Simulation

The optimization loop presented in this contribution and its according workflow is summarized in Figure 1. The structural optimization method, which is used here, is based on the FE-mesh of the investigated structure, see Harzheim (2008). Design variables are FE-model nodes which are moved by the optimization algorithm. In many applications, the effects of these geometric changes (like an associated variation of the structure’s dynamic behavior) on an entire mechatronic system have to be considered by the structural optimization loop.

In order to satisfy this requirement, the entire mechatronic system can be modeled as an elastic multibody system (EMBS), containing the investigated structure as elastic bodies. The integration of an elastic body is not a trivial operation and is briefly to be abstracted now: The linear, displacement-based FEM approximates the structures linear elastic deformation $u \in \mathbb{R}^3$ by a superposition of nodal displacements $q \in \mathbb{R}^N$ and so-called global shape functions $\Phi$ yielding $u(R,t) = \Phi(R) \cdot q(t)$, with $R$ representing the position vector of an arbitrary material point inside the FE-model. Since the dimension $N$, which is equal to the number of degrees of freedom (dof) of the FE-model, is often very high, the entire FE-model usually cannot be handled in a non-linear dynamic EMBS-simulation. Therefore, it is necessary to reduce $N$ by projecting $q$ on a subspace $\text{span}(V) \subseteq \mathbb{R}^{N \times n}$ by $q = V \cdot q_{\text{red}}$, with $n = \dim(q_{\text{red}}) < N$. This step is done by special codes for model reduction. These tools have to provide data that can be used in an EMBS-model, e.g. the so-called standard input data (SID), see Wallrapp (1994). There are numerous possibilities of setting up $V$, see e.g. Lehner (2007). All of them require the system matrices of the FE-model as input. The derived SID-data allow an EMBS-simulation whereby the so-called generalized coordinates, $q_{\text{red}}(t)$ which describe the elastic behavior of the investigated structure, can be computed over long time intervals. Together with further information from the code for model reduction, the so-called stress modes $\Phi_{\sigma,\text{red}}(R)$, the recovery of stresses during EMBS-runtime is possible, $\sigma(R,t) \approx \Phi_{\sigma,\text{red}}(R) \cdot q_{\text{red}}(t)$, and no additional FE-solution is necessary. The derivation of the stress modes also requires the projection matrix $V$ and some stress information which can be delivered by the FE-model. The derivation process of the stress modes is explained precisely in Tobias and Eberhard (2010). A meaningful and appropriate method of dealing with the stress time series $\sigma(R,t)$ is the derivation of a damage distribution $D(R)$ in the investigated structure. The resulting damage values can be used as scalar performance criterion for the optimization problem after an appropriate definition of an utility function has been made. The minimization of damage $\min(D(R))$ at the automatically determined most critical positions by changes in the geometry is thus the final goal for the optimization.
As a special feature, the presented optimization loop integrates stress and durability analysis in the EMBS-simulation via user-routines. This leads to several advantageous properties of the presented loop compared to conventional approaches, see Häußler (2005) or Tobias and Eberhard (2009). A post-processing of EMBS- and FE-results is not required, the tool for structural optimization can directly be interfaced with the EMBS-tool. Accordingly, the structural optimization loop based on durability values is considerably simplified which implies an improvement of the computational performance.

2.1 Elastic Multibody Simulation

Using e.g. Jourdain’s principles of dynamics, the equation of motion for a single body can be derived, as shown e.g. in Schwertassek et al. (1999), and reads

\[
\begin{bmatrix}
    M_r & M_{er}^T
    \\
    M_{er} & M_e
\end{bmatrix}
\begin{bmatrix}
a
    \\
    \ddot{q}
\end{bmatrix} =
\begin{bmatrix}
h_r
    \\
h_e
\end{bmatrix} +
\begin{bmatrix}
0
    \\
-\kappa_e \cdot \dot{q} - \gamma_e \cdot \dot{q}
\end{bmatrix},
\tag{1}
\]

where the sub matrix \( M_r \in \mathbb{R}^{6 \times 6} \) corresponds to the mass matrix known from rigid multibody dynamics, \( M_e \in \mathbb{R}^{N \times N} \), \( D_e \in \mathbb{R}^{N \times N} \) and \( K_e \in \mathbb{R}^{N \times N} \) are the flexible mass, damping and stiffness matrices, whereas \( M_{er} \in \mathbb{R}^{6 \times N} \) provides the coupling between the rigid body movement and the elastic deformation. The vector \( a \) contains the global accelerations of the body’s surface. This approach leads to \( 6 + N \) degrees of freedom per elastic body. In the following, all flexible bodies are considered simultaneously which yields to the same structure of the equation with just different dimensions. In the next step, the flexible degrees of freedom (dof) are reduced, e.g. by a Petrov-Galerkin projection of the flexible coordinates \( \dot{q} \) on an appropriate subspace \( \text{span}(V) \in \mathbb{R}^{N \times n} \) by

\[
q = V \cdot \ddot{q}_{\text{red}},
\tag{2}
\]

with \( n = \text{dim}(\ddot{q}_{\text{red}}) \ll \text{dim}(q) = N \) and requiring the residual to be orthogonal to the right projection space \( \text{span}(W) \in \mathbb{R}^{N \times n} \). A projection is called orthogonal if \( V = W \) and otherwise oblique. Here, only orthogonal projection is considered. This procedure leads to the reduced equations of motion

\[
\begin{bmatrix}
    M_r & M_{er}^T
    \\
    V^T \cdot M_e & V^T \cdot M_e \cdot V
\end{bmatrix}
\begin{bmatrix}
a
    \\
    \ddot{\ddot{q}}_{\text{red}}
\end{bmatrix} =
\begin{bmatrix}
h_r
    \\
V^T \cdot h_e
\end{bmatrix} +
\begin{bmatrix}
0
    \\
- V^T \cdot \kappa_e \cdot V \cdot \ddot{q}_{\text{red}} - V^T \cdot \gamma_e \cdot \dot{q}_{\text{red}}
\end{bmatrix}.
\tag{3}
\]

As far as model reduction is concerned, the question of how to choose appropriate projection subspaces \( \text{span}(V) \) is of interest. In state of the art reduction methods like modal reduction, the projection space \( \text{span}(V) \) consists of the dominant eigen vectors of the system. Another approach to model reduction is the consideration of all the reaction and applied forces acting on the elastic body as inputs \( B_e \cdot u(t) \) and outputs \( y(t) = C_e \cdot q(t) \) to the elastic body, where \( B_e \in \mathbb{R}^{N \times p_e} \) and \( C_e \in \mathbb{R}^{N \times n} \). Then, the elastic part of the body can be considered as a linear time-invariant second order MIMO-system \( M_e \cdot \dot{\ddot{q}}(t) + D_e \cdot \dot{q}(t) + K_e \cdot q(t) = B_e \cdot u(t), \quad y(t) = C_e \cdot q(t) \). Using the Laplace transformation, the transfer matrix of the system \( H(s) = C_e \cdot (s^2 M_e + s D_e + K_e)^{-1} \cdot B_e \) is obtained. The projection space \( \text{span}(V) \) is then yielding the reduction space of the second order MIMO-system. The reduced order system reads

\[
M_e \cdot \dot{\ddot{q}}_{\text{red}}(t) + D_e \cdot \dot{\ddot{q}}_{\text{red}}(t) + K_e \cdot \ddot{\ddot{q}}_{\text{red}}(t) = B_e \cdot u(t), \quad y(t) = C_e \cdot \ddot{q}_{\text{red}}(t)
\tag{4}
\]

with the reduced order input matrix \( B_e = V^T \cdot B_e \) and output matrix \( C_e = C_e \cdot V \). Using the Laplace transformation, the transfer matrix of the reduced order system \( \overline{H}(s) = \overline{C}_e \cdot (s^2 \overline{M_e} + s \overline{D_e} + \overline{K_e})^{-1} \cdot \overline{B}_e \) is obtained.

3 Automated Krylov subspace based reduction

With Krylov subspace based reduction methods, certain values and derivatives (moments) of the original transfer matrix \( H(s) \) and the reduced transfer matrix \( \overline{H}(s) \) match. Overview articles about Krylov subspace reduction are e.g. Bai (2002) or Beattie and Gugercin (2005). For systems with many connection points, e.g. gear boxes, the Krylov reduction turns out to be challenging. In this work, we consider moment matching for second order systems while concentrating on systems characterized by few inputs. The second order structure of the system can be maintained thanks to projection with second order Krylov subspaces. The conservation of the second order structure is the enabling condition for simulating the elastic body in an EMBS simulation environment. In addition, if the system is reduced with an orthogonal reduction, meaning \( W = V \), and if the projection matrix \( V \) has full rank, the reduced mass, damping and stiffness matrices are also symmetric and positive definite, i.e. the structure of the system is preserved. In Lehner (2007) a second order Krylov subspace is defined as

\[
\mathcal{G}_d(A_1, A_2, G) = \text{colspan}\{P_0, P_1, \ldots, P_{r-1}\}
\tag{5}
\]

with

\[
\begin{align*}
P_0 &= G, \\
P_i &= A_1 \cdot P_{i-1} + A_2 \cdot P_{i-2}, \quad i = 2,3,\ldots,r-1.
\end{align*}
\tag{6}
\]

The second order Krylov subspace \( \mathcal{G}_d(-K_e^{-1} \cdot D_e, -K_e^{-1} \cdot M_e, -K_e^{-1} \cdot B_e) \) is called the input second order Krylov sub-

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The basic idea of this method is shown in Algorithm 1. This appears to be a very effective strategy in many circumstances. However, an appropriate error estimator is indispensable as stopping criterion when the desired accuracy is reached. In addition, an order adaptive selection of expansion points needs to be used.

3.1 Choice of expansion points

By using Laguerre functions it is possible to find one single optimal expansion point, see Eid et al. (2007). However, optimal Laguerre based model reduction is only possible for single input single output (SISO) systems which is a serious drawback for model reduction of elastic bodies.

3.1.1 Iterative Rational Krylov Algorithm

Recently, in Gugercin et al. (2008) a method called Iterative Rational Krylov Algorithm (IRKA) was proposed where the $\mathcal{H}_2$-norm of the error $H_e(s) = H(s) - \bar{H}(s)$ is minimized if the moment matching conditions are chosen in an optimal way. An $\mathcal{H}_2$ optimal placement of the expansion points is achieved if the moments are matched at the mirror image of poles. This method is improved for MIMO systems in Bunse-Gerstner et al. (2007) and Geschwinder (2009) extended those methods for model reduction of second order mechanical systems. The optimal moment matching conditions are achieved by solving the following minimization problem

$$\min_{s_k} \mathcal{F} = \|H_e(s) - \bar{H}(s)\|^2_{\mathcal{H}_2(s)}.$$  \hspace{1cm} (7)

The necessary optimality condition for minimizing the error system $H_e(s)$ is $\nabla \mathcal{F} = 0$, meaning $H(-\lambda_i) = H'(\lambda_i) = H'(-\lambda_i)$. The Newton direction search method is used. This leads to a successive substitution framework. At the $(i+1)$th step moment matching is ensured by a biorthogonal Arnoldi algorithm at the mirror images of the Ritz values from the $i$-th step. Despite its simplicity, this appears to be a very effective strategy in many circumstances. The basic idea of this method is shown in Algorithm 1. This method has some drawbacks. First, in every calculation step new expansion points are generated. This leads to long calculation because in every iteration for the $k$ new expansion points the matrices $\hat{K}_k$ need to be decomposed with an LU-decomposition to approximate $K^{-1}$, which is the main computational task in Krylov subspace based model reduction. Moreover, applied to mechanical systems this algorithm tends to converge against a local optimum, not against the global optimum, see Fehr et al. (2009a).

Algorithm 1 Second Order Iterative Rational Krylov Algorithm (SOIRKA)

1: Input: $M_e, D_e, K_e, B_e, C_e$, the sets of expansion points: $\hat{S}$ and scalar quantity: $\text{contol}$ (convergence tolerance)
2: Output: reduced system matrices like $\hat{M}_e, \hat{D}_e, \hat{K}_e, \hat{B}_e, \hat{C}_e$
3: /*Initialize*/ $i = 1$ and $\hat{S}_{i0} = \hat{S}$
4: repeat
5: $[V, W] = \text{BiOrthArnoldi}(\hat{M}_e, \hat{D}_e, \hat{K}_e, \hat{B}_e, \hat{C}_e, \hat{S}_{si-1}, \ldots)$
6: $\hat{M}_e = W^T \cdot M_e \cdot V; \ldots$ /* build reduced system*/
7: $\hat{S}_{si} = -\text{polyeig}(\hat{K}_e, \hat{D}_e, \hat{M}_e)$;
8: $\text{rel} \Delta \{ \hat{S} \} = \ldots$ /* calc. rel. change of $\hat{S}*/$
9: $i = i + 1$
10: until $\text{max} \{ \text{rel} \Delta \{ \hat{S} \} \} < \text{contol}$

3.1.2 Adaptive-order based algorithm

Another expansion point selection strategy was shown by the authors in Fehr and Eberhard (2009a). In this method called Second Order Adaptive Global Arnoldi (SOAGA) the error estimator and automated selection of appropriate expansion points is based on the global Arnoldi process, see Jbilou et al. (1999). The Second Order Global Arnoldi (SOGA) is the Second Order Adaptive Global Arnoldi (SOAR) algorithm by Bai and Su (2005) applied to the following matrix pair $((I_p \otimes A_1),(I_p \otimes A_2),vec(G_1))$ leading to the SOGA Algorithm, see Fehr and Eberhard (2009a). Global Arnoldi algorithms are similar to rational Arnoldi algorithms. The fact that all matrices are treated as stacked vectors marks their uniqueness. The inner product of two stacked vectors is equivalent to $\text{vec}(A) \cdot \text{vec}(B) = \text{trace}(A^T \cdot B)$. Therefore, the global Arnoldi algorithms will recursively generate a Frobenius orthonormal basis of the Krylov-subspace whereas standard Arnoldi algorithms will produce an orthonormal basis. In addition, an order adaptive expansion point selection scheme exists. The calculation starts with a given set $\hat{S} = \{ s_1, s_2, \ldots, s_i \}$ of expansion points. The error is minimized by building a new basis at the respective expansion point with the maximum error as defined in Bai and Su (2005) $\varepsilon_3G(s_i) = ||h_2(s_i)C_e \cdot E_{\hat{S}}(s_i)||_F$. Here $E_{\hat{S}}(s_i)$ is the residual error for a second order system and is defined as

$$E_{\hat{S}}(s) = (\bar{s}^2M_e + sD_e + K_e) \cdot E_{\hat{S}}(s),$$  \hspace{1cm} (8)

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and $E_S(s)$ is the Laplace-transform of the time domain error $e_S(t) = q(t) - V \cdot q_{\text{red}}(t)$. Figure 2 graphically illustrates the SOAGA process.

It was shown in Fehr et al. (2009a) that $E_S(s)$ decays fast and stays constant after some iterations. Once the error is constant the iteration should be stopped with the actual set of expansion points $\bar{S}$. Usually, the previously decayed error can still be improved by adding a new expansion point $s_{\text{new}}$. Those ideas can be used to develop a totally automated Krylov subspace based reduction method. The idea is shown in Figure 3. The algorithm starts with a set of minimal expansion points $\bar{S}_1$ to save computationally expensive LU-decompositions. After the first stopping criterion $\epsilon_l - \epsilon_{l-1} < \text{tol}_1$, where $l$ is the iteration number, is satisfied, (no further improvement can be achieved), a new expansion point is added $s_{\text{new}}$ until the second stopping criterion $\epsilon_k - \epsilon_{k-1} < \text{tol}_2$ is fulfilled and the reduction process is finished.

### 3.2 Error estimators

A widely used error expression for model reduction is the relative error $\epsilon(\omega) = \frac{\|H(\omega) - \bar{H}(\omega)\|_F}{\|H(\omega)\|_F}$ measured in the Frobenius norm. However, the calculation of the relative error involves an evaluation of the original system $H$. For each $\omega$ a set of linear equations of size $N$ and $n$ needs to be solved. Error estimators are applied in order to reduce the time needed to calculate $\|H(\omega)\|_F$. Within a given frequency range $[\omega_{\text{min}}, \omega_{\text{max}}]$ this can be achieved e.g. achieved by replacing the original system with another reduced system $\bar{H}$. The approximation error is then estimated by the relative error between the two reduced order systems

$$\bar{\epsilon}(\omega) = \frac{\|\bar{H}(\omega) - \bar{\bar{H}}(\omega)\|_F}{\|\bar{H}(\omega)\|_F}. \quad (9)$$

For every $\omega$ the time needed to solve the system of size $N$ is reduced to the solution time of a system of size $\bar{n}$, where $\bar{n}$ represents the size of the second reduced system $\bar{H}$. However, viable and meaningful results require that the second reduced system meets certain conditions. One possibility is the error estimator proposed in Grimme (1997). The reduced order systems $\bar{H}$ and $\bar{\bar{H}}$ are calculated by the SOAGA reduction method where two different sets of expansion points $\bar{S} = \{s_1, s_2, \ldots, s_i\}$ and $\bar{\bar{S}} = \{\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_i\}$ are used. The moments of $H$ match at $\bar{S}$ with the moments of the original system $H$ and the moments of $\bar{H}$ match at $\bar{\bar{S}}$. Good results are achieved if the expansion points are alternating $s_i < \bar{s}_i < s_{i+1}$ and equidistantly spaced in the given frequency range $[\omega_{\text{min}}, \omega_{\text{max}}]$ and $s_{i+1} - \bar{s}_i = s_i + 1$, see Figure 4. The error estimator by Grimme

Figure 3. Process flow of the automated SOAGA algorithm

Figure 4. Distribution of expansion points

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(1997) \( \varepsilon_G = \| \hat{H}(\omega) - \bar{H}(\omega) \|_F / \| \bar{H}(\omega) \|_F \) usually achieves good results results but twice the numerical effort is needed in comparison to the error estimator proposed by Bechtold et al. (2005). Instead of building a second reduced system \( \hat{H} \) at different reduction points, the reduced order model from the iteration before \( \hat{H}_{i-1} \) is used as a second reduced system. The error estimator then reads \( \varepsilon_{BBK} = \| \hat{H}(\omega) - \hat{H}_{i-1}(\omega) \|_F / \| \hat{H}(\omega) \|_F \).

Both error estimators show in practice satisfactory results but lack a provable error bound.

**Error estimator by Konkel et al. (2008)** The development of general a-priori error bounds independent from the choice of extension points is still an active field of research. Recently, in Konkel et al. (2008) a new provable error estimator was proposed. This estimator is based on the fact that frequently in mechanical systems a certain frequency range \( \mathcal{K} = \{ \omega_{\text{min}} \leq \omega \leq \omega_{\text{max}} \} \) is of special interest. In addition, it is considered that the eigen modes \( \Phi \) of the system, see e.g. Müller and Schiehlen (1985) are partitioned into the subset \( S \) with the known modes in the interesting range \( \Phi_{[p]} \) and into the subset \( S_r \) with the usually unknown modes beyond the interesting frequency range \( \Phi_{[n]} \). The output error \( E_{r,s}(s) = C_r \cdot E_S(s) \) is partitioned into two complementary subspaces \( S_r \) and \( S_s \). The subspace \( S_r \) is spanned by the colspace of the mass orthogonal eigen modes \( \Phi_{(p)} \). Consequently, the projector on subspace \( S_s \) is

\[
P_r = (M_r \cdot \Phi_{(p)}) \cdot [(M_r \cdot \Phi_{(p)})^T \cdot (M_r \cdot \Phi_{(p)})]^{-1} (M_r \cdot \Phi_{(p)})^T.
\]

The complementary projector on subspace \( S_r \) is \( P_r = I - P_r \). After the residuum \( E_S \) and the output matrix \( C_r \) are partitioned into the two previously defined subspaces, as shown by Konkel et al. (2008), the output error can be simplified to

\[
E_{r,s}(s) = C_r \cdot P_r \cdot \Phi_{(n)} \cdot L^{-1} \cdot \Phi_{(n)}^T \cdot P_r \cdot E_S.
\]

4 **Gramian matrix based model reduction**

Frequently, in mechanical systems a certain frequency range is of special interest. The information about the interesting frequencies can be included in the reduction process by projecting the system with the dominant frequency-weighted version of the Gramian matrix \( P^w_r \). In Lehner (2007); Fehr and Eberhard (2009a) it is shown, that if the projection space \( \text{span}(V_r) \) consists of the dominant eigen vectors \( \text{span}(V_r) \) of \( P^w_r \), which is associated with the largest eigen values \( \sigma_{(r)} \) of the frequency-weighted position Gramian matrix then the frequency-weighted \( \mathcal{H}_2 \) error of the reduced system can be written as

\[
\| H_{E}(s) \cdot W(s) \|_{\mathcal{H}_2} \leq \frac{1}{\pi} \sum_{j=n+1}^{N} \sigma_{(r)}^j.
\]

The sum of neglected eigen values can be calculated as soon as the eigen decomposition of the Gramian matrices is known, causes the upper bound of the error to be known before the reduced system is obtained. The sum of neglected eigen values \( \sum_j \sigma_{(r)}^j \) can be used to determine the size \( m \) of the reduced order model because the error is bounded below the sum of the neglected eigen values. Usually, the eigen values of the Gramian matrix decay rapidly in mechanical systems. As a consequence, the first neglected eigen value is the dominant share in the sum of neglected eigen values and allows a fully automated reduction process, see Fehr et al. (2008).

For small asymptotically stable systems, the Gramian matrices \( P^w_r \) can be calculated by evaluating a matrix logarithm in addition to the solution of a suitable Lyapunov equation, see Antoulas (2005). Direct solution of the Lyapunov equation is only possible for small- to medium-scale models because the solution requires \( O(n^3) \) operations and the storage requirement is \( O(n^2) \). As to large-scale models there are several possibilities to approximate the subspace of dominant eigen vectors of the Gramian matrix.
One approach for approximating the Gramian matrix is introduced in Lehner (2007). According to this method, the Gramian matrix is numerically approximated by a POD based procedure. The so-called frequency-weighted position controllability Gramian matrix is

\[
P_p^{\omega} = \frac{1}{2\pi} \int_{-\omega_{\text{max}}}^{\omega_{\text{max}}} \mathbf{Q}(i\omega) \cdot \mathbf{Q}^T(i\omega) d\omega
\]

with \( \mathbf{Q}(i\omega) = [\text{Re} \{ \mathbf{Q}(i\omega) \}] + i [\text{Im} \{ \mathbf{Q}(i\omega) \}] \) and \( \mathbf{Q}(i\omega) = (-\omega^2 \mathbf{M} + i\omega \mathbf{D} + \mathbf{K})^{-1} \cdot \mathbf{B} \). This matrix can be approximated as

\[
P_p^{\omega} = c \sum_{i=1}^{I} \mathbf{Q}(i\omega) \cdot \mathbf{Q}^T(i\omega)
\]

by quadratures using integral kernel snapshots \( \mathbf{U}'(i\omega_0) = \mathbf{Q}'(i\omega_0) \), where \( \alpha \) is a real constant following from the approximation scheme. This method can be viewed as an extension of the Poor Man’s TBR Phillips and Silveira (2005) scheme for second order systems. The number and location of these snapshots have a big influence on the reduction results Fehr et al. (2009b).

In a first step, a greedy search algorithm is used for the calculation of the next snapshot at the maximum of an error measure calculated for the finite training set \( \mathcal{S}_n \in \mathcal{F}_{\text{training}} \) of training frequencies \( \omega_0 \), see Figure 5. Better results are obtained if the absolute error \( \epsilon(\omega_0) = ||\mathbf{H}(\omega_0) - \tilde{\mathbf{H}}(\omega_0)||_F \) is used as an error measure. The results of the greedy search algorithm are better in comparison with uniformly or randomly chosen basis construction strategies. However, the calculation of the absolute error \( \epsilon(\omega_0) \) is expensive. As mentioned the calculation of the absolute error involves an evaluation of the original system \( \mathbf{H} \). For each \( \omega_0 \) a set of linear equations of size \( N \) and \( n \) needs to be solved. Further improvement of the snapshot selection is achieved by an adaptive basis construction scheme, derived from Haasdonk and Ohlberger (2008). The idea is to start with a small training set \( \mathcal{S}_n \) and adaptively increase the training set until the error is converged. The risk of overfitting as explained in Fehr et al. (2009b) is overcome by the use of a second validation set \( \mathcal{S}_v \). The process flow of the adaptive greedy algorithm is shown in Figure 6. For a model with 30 000 dofs the computational speedup of the adaptive Greedy versus Greedy was more than four. The method can be viewed as an automatic determination of optimal frequency weighting and as an adaptive learning of quadrature rules.

5 Example and Results

The model used in this context is a simulated three body pendulum consisting of two rigid bodies connected via interface nodes to the elastic carrier. The elastic carrier has about 5700 degrees of freedom, see Figure 7. In addition, the elastic carrier is coupled via one additional interface node to the ground. The interface nodes are connected by light and stiff beams, as suggested in Fehr and Eberhard (2009b), to the corner nodes of the carrier, see Figure 7. Forces and torques are acting on all three additional interface nodes in all three directions. Accordingly, the carrier is considered as a MIMO system with 18 inputs and outputs, \( \mathbf{B} = \mathbf{C}' \). The interesting frequency range is from 1 to 100 Hz.

In Figure 8 the relative error is plotted for the different automated reduction methods. With the automated SOAGA algorithm a medium-scaled systems of order \( n = 108 \) was derived. Due to the low error of the SOAGA algorithm, it is admissi-
FIGURE 7. Simulated three body pendulum

FIGURE 8. Relative reduction error using different approaches

FIGURE 9. Comparison of the different error estimators

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The goal of this investigation is the development of a totally automatic reduction process for mechanical systems. Only the load distribution, the frequency range of interest and a measure for the desired accuracy have to be provided by the user. This makes the methods especially attractive for structural optimization or error-controlled computations. In an example the automated reduction methods are used to find the best elastic ansatz function for the carrier in the structural optimization loop.

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