Shear correction of thin structural beam and plate elements using absolute nodal coordinates

Oleg Dmitrochenko
Department of Mechanical Engineering
Lappeenranta University of Technology
Lappeenranta, 53850, Finland
Oleg.Dmitrochenko@lut.fi

Marko Matikainen, Aki Mikkola
Department of Mechanical Engineering
Lappeenranta University of Technology
Lappeenranta, 53850, Finland

This study is an extension of a newly introduced approach to account transverse shear deformation in absolute nodal coordinate formulation. In the formulation, shear deformation is usually defined by employing slope vectors in the element transverse direction. This leads to the description of deformation modes that, in practical problems, may be associated with high frequencies. These high frequencies, in turn, could complicate the time integration procedure, burdening numerical performance of shear deformable elements. In a recent study of this paper’s authors, the description of transverse shear deformation is accounted for in a two-dimensional beam element, based on the absolute nodal coordinate formulation without the use of transverse slope vectors. In the introduced shear deformable beam element, slope vectors are replaced by vectors that describe the rotation of the beam cross-section. This procedure represents a simple enhancement that does not decrease the accuracy or numerical performance of elements based on the absolute nodal coordinate formulation.

In this study, the approach to account for shear deformation without using transverse slopes is implemented for a thin rectangular plate element. In fact, two new plate elements are introduced: one within conventional finite element and another using the absolute nodal coordinates. Numerical results are presented in order to demonstrate the accuracy of the introduced plate element. The numerical results obtained using the introduced element agree with the results obtained using previously proposed shear deformable plate elements.
SHEAR CORRECTION OF THIN STRUCTURAL BEAM AND PLATE ELEMENTS
USING ABSOLUTE NODAL COORDINATES

Oleg Dmitrochenko
Department of Mechanical Engineering
Lappeenranta University of Technology
Lappeenranta, 53850, Finland
Oleg.Dmitrochenko@lut.fi

Marko Matikainen, Aki Mikkola
Department of Mechanical Engineering
Lappeenranta University of Technology
Lappeenranta, 53850, Finland
Aki.Mikkola@lut.fi

ABSTRACT
This study is an extension of a newly introduced approach to account transverse shear deformation in absolute nodal coordinate formulation. In the formulation, shear deformation is usually defined by employing slope vectors in the element transverse direction. This leads to the description of deformation modes that, in practical problems, may be associated with high frequencies. These high frequencies, in turn, could complicate the time integration procedure, burdening numerical performance of shear deformable elements. In a recent study of this paper’s authors, the description of transverse shear deformation is accounted for in a two-dimensional beam element, based on the absolute nodal coordinate formulation without the use of transverse slope vectors. In the introduced shear deformable beam element, slope vectors are replaced by vectors that describe the rotation of the beam cross-section. This procedure represents a simple enhancement that does not decrease the accuracy or numerical performance of elements based on the absolute nodal coordinate formulation.

In this study, the approach to account for shear deformation without using transverse slopes is implemented for a thin rectangular plate element. In fact, two new plate elements are introduced: one within conventional finite element, based on the absolute nodal coordinate formulation without the use of transverse slope vectors. In the introduced shear deformable beam element, slope vectors are replaced by vectors that describe the rotation of the beam cross-section. This procedure represents a simple enhancement that does not decrease the accuracy or numerical performance of elements based on the absolute nodal coordinate formulation.

In the absolute nodal coordinate formulation, finite elements are defined without the use of any floating reference frame. Instead, elements employ absolute position coordinates together with independent global slopes that are partial derivatives of a position vector with respect to the element local coordinates [2]. The use of absolute slope coordinates allows for the description of an arbitrary rigid body motion without using any rotation matrix. An arbitrary rigid body motion is described using the matrix of global shape functions and the vector of global nodal coordinates, similar to the conventional solid finite element method. This feature is unique among other large-displacement approaches and leads to the linear representation of a position vector of a material point and, consequently, to a constant mass matrix. Additionally, the vector of inertia forces, which is quadratic in velocities, vanishes in the expression of the equations of motion [3], [4].

A structural finite element that uses small transverse displacements and slopes as generalized coordinates can be formally transformed to a corresponding absolute nodal coordinate element. After transition, some important geometric, static and dynamic properties of the elements are preserved as compared with the original element [9], [10].

1. INTRODUCTION
During the past decades, a considerable number of contributed works on large deformation formulations for multibody applications have been introduced. One of recent approaches, the absolute nodal coordinate formulation, has been established by A. Shabana [1]. The formulation is based on the finite element procedure and it is capable of correctly describing large rotations without employing incremental integration methods to satisfy the energy balance.
The vertical displacement of an arbitrary point of the centreline given by a horizontal coordinate \( x = 0 \) can be computed using the interpolation technique as follows:

\[
y(x) = \begin{bmatrix} s_1(x) & s_2(x) & s_3(x) & s_4(x) \end{bmatrix} \begin{bmatrix} y_0 \\ y_0' \\ y_1 \\ y_1' \end{bmatrix},
\]

where \( s_1, \ldots, s_4 \) are the shape functions for beams that play a role of interpolating functions:

\[
s_1(x) = 1 - 3\xi^2 + 2\xi^3, \quad s_2(x) = 1 (\xi - 2\xi^2 + \xi^3),
\]

\[
s_3(x) = 3\xi^2 - 2\xi^3, \quad s_4(x) = 1 (\xi^3 - \xi^2),
\]

\[\xi = \frac{x}{l}.\]

It is worth mentioning that this model assumes small displacements and small deformations. This means that all generalized coordinates, namely displacements \( u_0 \) and \( u_1 \) as well as slopes \( u_0' \) and \( u_1' \), are assumed to be small.

As shown in [9], in order to specify arbitrary displacements of the beam element, it is possible to parameterize the beam centerline using the arc parameter \( p = 0 \) and introduce two independent displacement fields \( x(p) \) and \( y(p) \), using the same interpolation polynomials. A new set of nodal parameters consists of the displacement and slope vectors at the end points of the beam:

\[
r_0 = [x_0 \ y_0]^T, \quad r_0' = [x_0' \ y_0']^T, \quad r_1 = [x_1 \ y_1]^T, \quad r_1' = [x_1' \ y_1']^T.
\]

as shown in Figure 2, where the primes ‘ denote derivatives with respect to arc parameter \( p \).
In Eqs. (5) and (6), symbols \( \otimes \) denote the Kronecker’s product, and \( \mathbb{R} \) denotes a real-number space of the same dimension as the identity matrix \( I \). This procedure can be called vectorization of finite element.

This transformation can be applied to any finite element having transverse displacements and slopes as nodal degrees of freedom. In the following sections, the transformation will be used for different types of elements such as beams/plates.

1.2. Transformation of Equations of Motion

The equations of motion of the conventional element can be modified in a certain regular way, which allows a forthcoming prediction of the static and dynamic properties of the new element based on the absolute nodal coordinate formulation, as shown in references [9], [10].

The transformation of the equations of motion can be briefly expressed in the following diagram:

\[
\begin{align*}
M_{\text{FEM}} \otimes \rho_{\mathbb{R}} + K_{\text{FEM}} \cdot u &= Q_{\text{FEM}}^{\text{raw}} \\
I \otimes \mathbb{R} &\downarrow \mathbb{R} \downarrow I \\
\downarrow &\downarrow \downarrow \downarrow \\
M_{\text{ANC}} \otimes \rho_{\mathbb{R}} + K_{\text{ANC}}(q) \cdot q &= Q_{\text{ANC}}^{\text{raw}}
\end{align*}
\]

The diagram shows that most terms of the equations of motion are transformed in a simple way in which their size is increased as a result of applying Kronecker’s product \( \otimes \). The only exception is the stiffness matrix \( K \), which is transformed in a non-trivial way using nonlinear mapping \( K_{\text{FEM}} \rightarrow K_{\text{ANC}}(q) \). It has been shown that some important static and dynamic properties are preserved after transformation. Particularly, inertia and stiffness properties near the undeformed configuration of the element are the same; this means that small deformations in linear static problems as well as eigenfrequencies and eigemodes remain unchanged at the undeformed configuration.

1.3. Structural and Fully Parameterized Elements

Research within the field of absolute nodal coordinate formulation can be subdivided into two large groups. In the first group, the formulation is used to describe conventional (or thin) beam and plate elements that cannot capture the transverse shear deformation. In this approach, an element is parameterized as the centerline of a beam or the mid-surface of a plate by employing global slope coordinates in the element longitudinal direction together with global position coordinates as described in Figure 3 [2], [5]. The beam element discussed in Sect. 2 falls into this category. In the second group of research, transverse shear deformable elements, based on the absolute nodal coordinate formulation, are developed and utilized in practical applications [8], [7]. In this approach, shear deformation can be accounted for by the parameterization of the element as a volume by introducing additional slopes in the element transverse direction, Figure 3 b. This makes it straightforward to define elastic forces using a continuum mechanics approach [4].

Each of the two groups of elements that can be correspondingly named as ‘thin’ and ‘thick’ elements has its own advantages and drawbacks.

The thin elements are, in general, simpler in geometry and do not have as many degrees of freedom as the thick elements. For this reason, the numerical performance of a thin element is enhanced as compared to a corresponding thick element. The main drawback of the thin elements is due to their simplified nature: the elements are only capable of representing the beam centerline or the plate mid-surface, and the treatment of shear deformation in the volume of the
elements is questioned.

The thick elements can be used to describe cross-section deformations in case of beams and, correspondingly, fiber deformations in case of plates and shells. However, as it has been recently pointed out, the volumetric parameterization of the thick elements can lead to elements with low numerical performance [11]. This is due to inefficient bending strain description as well as shear and curvature lockings. These problems can be overcome in several different manners, such as introducing additional nodes or slope coordinates [12], [13]. The performance of thick elements can be improved using an implicit integration scheme [14]. Nevertheless, cross-section or fiber deformation usually plays an insignificant role in multibody applications. For this reason, shear deformable elements may be excessively accurate for practical usage.

This paper is devoted to revealing the shortcomings of the thin elements towards accounting for the shear deformation in conventional models of beams and plates as one and two-dimensional manifolds. In the following sections, this approach is briefly described for a thin beam element, and after that, the generalization is made for a thin plate element.

2. Shear Correction for a thin Beam

In paper [15], shear deformation is accounted for in the thin beam elements by using rotations directly as nodal coordinates. In order to shed some light on this approach, the geometric configuration of the thin beam is depicted in Figure 4. In the figure, the centerline of the beam element is shown as a dashed line.

![Figure 4. CENTERLINE AND THE CROSS SECTION OF BEAM](image)

The geometry of the centerline can be represented by a cubic polynomial as follows:

\[ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3, \]

where \( a_0, ..., a_3 \) are unknown coefficients to be identified. Consequently, the derivative of this expression, called slope \( y' = dy/dx \), identifies the normal to the centerline. The cross section of the beam is assumed to be shifted with respect to the normal line in such a way that the absolute rotation of the cross-section is given by the expression:

\[ \varphi = y' + \gamma, \]

where value \( \gamma \) is called shear, or shear angle. The shear is a quantity that is absent from the Euler–Bernoulli thin beam theory, where \( \gamma = 0 \), and \( \varphi = y' \), as shown in Figure 5. Euler–Bernoulli beam kinematics can be represented in the following matrix expression:

\[
\begin{bmatrix}
\gamma \\
\varphi
\end{bmatrix} =
\begin{bmatrix}
s_1 & s_2 & s_3 & s_4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y_0 \\
y_1
\end{bmatrix},
\]

In the approach proposed by Narayanaswami and Adelman [15], the shear is not zero, \( \gamma \neq 0 \), and its value is found using the conditions of static equilibrium of the cross section, which for this 1-D case can be written as:

\[
\frac{dM}{dx} = Q,
\]

where \( M \) is the bending moment and \( Q \) is the transverse shear force. The bending moment and shear force can be written as:

\[
M = EIy', \quad Q = kGA\gamma.
\]

The substitution of forces and moments from Eq. (11) together with Eqs. (7) and (8) into the equilibrium equation (10) leads to solution for shear and rotation angle as follows:
\[ \gamma = \frac{1}{2}l^2a_0, \]
\[ \varphi = a_1 + 2a_2x + 3a_3x^2 + \gamma, \]

where \( \theta \) is a dimensionless constant called slenderness ratio:

\[ \theta = \frac{12EI}{kGAI^2}. \]

In Eq. (13), \( E, \nu \) and \( G = E/(2(1+\nu)) \) are elastic constants of the material; \( k = 5/6 \) is the shear factor. For example, for rectangular cross-section \( A = bh \), \( I = bh^3/12 \), and the ratio is

\[ \theta = \frac{2(1+\nu)}{k} \left( \frac{h}{l} \right)^2, \quad \theta \rightarrow 0, \]

i.e. it rapidly approaches zero when the thickness is much smaller than its length. In such a case, as seen from Eq. (12), shear \( \gamma \rightarrow 0 \) and the beam model agrees with the Euler–Bernoulli theory.

By fixing the displacements \( y \) and total cross-section rotations \( \varphi \) at both ends of the beam, as Figure 4, the boundary conditions can be written as follows:

\[ y(0) = y_0, \quad \varphi(0) = \varphi_0, \]
\[ y(l) = y_l, \quad \varphi(l) = \varphi_l. \]

These four equations are solved for unknown coefficients:

\[ a_0 = y_0, \]
\[ a_1 = \frac{1}{2l(1+\nu)}(2l \varphi_0 + \theta(-2y_0 + 1 \varphi_0 + 2y_l - l \varphi_l)), \]
\[ a_2 = \frac{1}{2l(1+\nu)}(-6y_0 - 4l \varphi_0 + 6y_l - 2l \varphi_l - 10(\varphi_0 - \varphi_l)), \]
\[ a_3 = \frac{l}{l(1+\nu)}(2y_0 + l \varphi_0 - 2y_l + l \varphi_l). \]

These coefficients are to be substituted into Eqs. (7) and (12) to compute geometrical parameters of the beam:

\[
\begin{bmatrix}
  y \\
  \gamma \\
  \varphi
\end{bmatrix} =
\begin{bmatrix}
  s_1^* & s_2^* & s_3^* & s_4^* \\
  g_1 & g_2 & g_3 & g_4 \\
  f_1 & f_2 & f_3 & f_4
\end{bmatrix}
\begin{bmatrix}
  y_0 \\
  \varphi_0 \\
  y_l \\
  \varphi_l
\end{bmatrix},
\]

where the following shape functions are contained:

\[ s_1^* = \frac{1}{12}(1 - 3\xi^2 + 2\xi^3 + \theta(1 - \xi)), \quad g_1 = \frac{h}{12(1+\nu)}, \quad f_1 = s_1^* + g_1, \]
\[ s_2^* = \frac{1}{12}(\xi - 2\xi^2 + \xi^3 + \theta(\xi - \xi^2)), \quad g_2 = \frac{h}{12(1+\nu)}, \quad f_2 = s_2^* + g_2, \]
\[ s_3^* = \frac{1}{12}(\xi^2 - 2\xi^3 + \theta\xi), \quad g_3 = \frac{h}{12(1+\nu)}, \quad f_3 = s_3^* + g_3, \]
\[ s_4^* = (-\xi + 2\xi^2 - \xi^3 + \theta\xi^2), \quad g_4 = \frac{h}{12(1+\nu)}, \quad f_4 = s_4^* + g_4. \]

The graphic representation of the shear corrected beam model is shown in Figure 6, together with nodal coordinates.

![Figure 6. SHEAR CORRECTED BEAM ELEMENT (BY NARAYANASWAMI & ADELMAN)](image)

It can easily be seen that for a slender beam, i.e. when \( h << l \) and \( \theta \rightarrow 0 \), all terms from Eq. (15) converge with the corresponding terms in Eq. (9), and, more general, the shear deformable model depicted in Figure 6 converges with the thin beam model shown in Figure 5.

### 2.1. Stiffness Matrix and Test for Shear Locking

The stiffness matrix of the element is composed of two terms, one due to bending of the centerline, and the second one due to shear in the cross-section, as follows:

\[
K_\theta = EI \int_0^l \begin{bmatrix}
  s_1^* & s_2^* & s_3^* & s_4^* \\
  g_1 & g_2 & g_3 & g_4 \\
  f_1 & f_2 & f_3 & f_4
\end{bmatrix}
\begin{bmatrix}
  g_j \\
  g_j \\
  f_j
\end{bmatrix} \mathrm{d}x + kGA \int_0^l \begin{bmatrix}
  12 & 6l & -12 & 6l \\
  6l & (4+0)l^2 & -6l & (2-0)l^2 \\
  -12 & -6l & 12 & -6l \\
  6l & (2-0)l^2 & -6l & (4+0)l^2
\end{bmatrix}. \quad \text{(16)}
\]

It can be seen that in the limiting case \( \theta \rightarrow 0 \) the stiffness matrix is identical to that for a standard Euler-Bernoulli beam.

To check the performance of the element, a simple cantilever beam loaded by a vertical free-end force \( F \) is considered. After applying boundary conditions, the bottom-right quarter of the matrix (16) is employed in a linear system of equations, which is used to find free-end
nodal coordinates $y_i$ and $\phi_i$:

$$\frac{EI}{l^3(1+\theta)} \begin{bmatrix} 12 & -6l & -10l \end{bmatrix} \begin{bmatrix} y_1 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}.$$  

The solution for cross-section rotation is $\phi_i = \frac{q_i^2}{2EI}$, which is correct value. The solution for the free-end displacement is:

$$y_1 = \frac{(4+\theta)Fl^3}{12EI} - \frac{Fl}{3EI} + \frac{Fl}{kGA}, \quad (17)$$

which includes both a term due to pure bending and a term due to pure shear. This solution agrees to both cases of thin and thick beams. Indeed, when beam is thick, the last term in Eq. (17) vanishes, and the solution tends to the value predicted by Euler–Bernoulli theory: $y_1 \to Fl^3/3EI$ if $\theta \to 0$. In case when $h$ is comparable to $l$, values $\theta$ and 4 are comparable, and terms in (17) are of same magnitude. Relative difference between Timoshenko and Euler–Bernoulli solutions for this test problem is estimated by

$$\frac{\varphi_i}{\varphi} = \frac{(1+\nu)h^2}{12},$$

and given $\nu = 0.3$, the latter formula can be illustrated by Figure 8.

Figure 7. RELATIVE ERROR (IN %) BETWEEN TIMOSHENKO AND EULER–BERNOULLI SOLUTIONS VERSUS SLENDERNESS VALUE $S=H/l$

2.2. Shear Correction for Absolute Nodal Coordinate Formulation

In paper [6], the procedure of shear correction described in the previous section has been implemented for a beam element, which employs absolute nodal coordinates. In this paper, the derivation is not described in detail. Instead, the final form is obtained by using the procedure of updating the finite element to the absolute nodal coordinate formulation element as defined in Sect. 1.1. The algorithm of transformation, which is given by a series of equations (4) – (6), is applied to Eq. (15), which describes the shear corrected thin beam element shown in Figure 6. The kinematics of the new element in ANCF is depicted in Figure 8 and can be written as:

$$\begin{bmatrix} r \\ \gamma \\ \phi \end{bmatrix} = \begin{bmatrix} s_1 I & s_2 I & s_3 I & s_4 I \\ g_1 I & g_2 I & g_3 I & g_4 I \\ f_1 I & f_2 I & f_3 I & f_4 I \end{bmatrix} \begin{bmatrix} r_0 \\ \varphi_0 \end{bmatrix}. \quad (18)$$

Figure 8. SHEAR CORRECTED BEAM ELEMENT BASED ON FINITE CROSS-SECTION ROTATION VECTORS IN ANCF

In other words, all scalar values in Eq. (15) and in Figure 6 are vectorized as a result of Kronecker’s product $\otimes$, which forms the procedure of transition described in Eqs. (5) and (6). As shown in [9] and [10], after such transition, stiffness properties of the element in small-deflection problems are conserved, and the discussion about absence of shear locking in Sect. 2.1 remains correct for the ANCF beam element.

In the consequent section, the procedures introduced for a thin beam element will be applied for a thin plate element.

3. Extension of the shear correction for a thin plate

In this section, the transformations of a Kirchhoff thin plate element to a shear deformable plate element are performed in a similar manner to the one described in Sect. 2 for a Euler–Bernoulli thin beam element.

3.1. Shear Correction for a Thin Plate Element

The rectangular thin plate element is usually formulated using twelve nodal degrees of freedom. Correspondingly, the representation of its mid-surface as an incomplete quartic polynomial can be written as:

$$0_x = \frac{(1+\nu)h^2}{12},$$

and given $\nu = 0.3$, the latter formula can be illustrated by Figure 8.
\[ w = c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2 + c_7 x^3 + c_8 x^2 y + c_9 xy^2 + c_{10} y^3 + c_{11} x^3 y + c_{12} xy^3. \] (19)

In Eq. (21), relations between fiber rotations, tangent slopes of the mid-surface and shears (cf. Eq. (8)) as follows:

\[ \begin{align*}
\varphi_x &= w_x' + \gamma_x, \\
\varphi_y &= w_y' + \gamma_y.
\end{align*} \] (20)

For a fiber of the plate, it is possible to formulate combinations of the values of indices \( i, j \) independently take values \( x, y \). In Eq. (21), \( D \) is the flexural rigidity of the plate, \( D = E h^3/(12(1-v^2)) \); \( E, \nu \) and \( G = E/(2(1 + \nu)) \) are elastic constants of the material; \( h \) is the thickness of the plate; \( k = 5 \) is the shear factor; symbol \( \delta_{ij} \) denotes Kronecker's delta. Other symbols in Eq. (23) are the components of the curvature tensor

\[ \kappa_{ij} = \frac{1}{2} \left( \frac{\partial \varphi_i}{\partial y} - \frac{\partial \varphi_j}{\partial x} \right). \] (24)

After substitution of relations (19) and (20) into (23) and (24), the equations of equilibrium (22) can be solved for shears as linear functions:

\[ \begin{align*}
\gamma_x &= (\alpha_0 + \alpha_1 x) \theta, \\
\gamma_y &= (\beta_0 + \beta_1 x) \theta.
\end{align*} \] (25)

where \( \alpha_0, \alpha_1, \beta_0, \beta_1 \) are coefficients defined as combinations of the values \( c_1, \ldots, c_{12} \), while \( \theta \) is the slenderness ratio:

\[ \theta = \frac{2}{k(1-v) ab} = \frac{12D}{kGhab}. \] (26)

The slenderness ratio approaches zero when the thickness of the plate decreases.

In order to find unknown coefficients \( c_1, \ldots, c_{12} \) from Eq. (19), the boundary conditions can be defined by employing values of nodal displacements \( w \) and total rotations \( \varphi \) at the four nodes of the plate, see FIGURE 10, as follows:

\[ \begin{align*}
w(0,0) &= w_1, & \varphi_x(0,0) &= \varphi_{x1}, & \varphi_y(0,0) &= \varphi_{y1}, \\
w(a,0) &= w_2, & \varphi_x(a,0) &= \varphi_{x2}, & \varphi_y(a,0) &= \varphi_{y2}, \\
w(a,b) &= w_3, & \varphi_x(a,b) &= \varphi_{x3}, & \varphi_y(a,b) &= \varphi_{y3}, \\
w(0,b) &= w_4, & \varphi_x(0,b) &= \varphi_{x4}, & \varphi_y(0,b) &= \varphi_{y4}.
\end{align*} \] (27)

Solving linear equations (27) for coefficients \( c_1, \ldots, c_{12} \) and substituting the coefficients into (19), (20) and (25), fiber displacements, shears and rotations as a linear combination of shape functions and nodal displacements can be written as shown below:

\[ M_{ij} = D((1-\nu)\kappa_{ij} + \nu(\kappa_{xx} + \kappa_{yy})\delta_{ij}), \]
\[ Q_i = kGh\gamma_i, \] (23)
The geometrical representation of the nodal displacements is described in FIGURE 10.

![Figure 10. Shear Corrected Thin Plate Element](image)

Values of shape functions that appear in (28) are too long to be represented in this paper. For example, the shape function $s_1^\ast$ reads as

$$
1 - \frac{(12 \theta^2 b^2 + 12 \theta^2 a^2 b + 12 \theta a b^2 + 6 \theta a^2)}{12 (a^2 \theta + a b + \theta b^2) (a + \theta b)} \\
+ \frac{(6 \theta b - 12 \theta^2 a^2 b - 12 \theta a b^2 a - 12 \theta^2 a^3)}{12 (a b^2 + \theta b^3 + \theta^2 b^3 a + 2 \theta a b^3 + \theta^3 a^3)} \\
\frac{(a - 6 a^2 \theta - 12 \theta b^2 - 12 a b) \xi}{4 (a^2 \theta + a b + \theta b^2) (a + \theta b)} \\
+ \frac{(a - 3 a^2 \theta + 12 a b) \xi}{4 (a b^2 + \theta b^3 + \theta^2 b^3 a + 2 \theta a b^3 + \theta^3 a^3)} \\
\frac{3 a^2 \xi b}{2 a^2 \theta + a b + \theta b^2} \\
\frac{3 b a^2 \xi n}{2 a^2 \theta + a b + \theta b^2} \\
\frac{3 b a^2 \xi n}{2 a^2 \theta + a b + \theta b^2}
$$

and one can see that in case $\theta \rightarrow 0$ this shape function tends to the known function $s_1$ for the Kirchhoff plate element. In general, for a very thin plate, when the thickness and slenderness ratio approaches zero, $\delta << ab$ and $\theta \rightarrow 0$, all terms of Eq. (28) approach the corresponding terms of Eq. (21). Accordingly, the plate model shown in FIGURE 10 converges with the Kirchhoff model depicted in Figure 9.

The derivation of this model of the plate element is completely analogous to that for the beam element above. Consequently, in a way similar to Sect. 2.1, it can be demonstrated that the shear corrected thin plate element does not suffer from shear locking and can be applied for both thin and thick plates.

### 3.2. Shear Corrected Plate Element based on the Absolute nodal Coordinate Formulation

The thin plate element with shear correction obtained above was developed under the assumption that all nodal displacements are small. To overcome this limitation and allow large displacements, the procedure of transition explained in Sect. 1.1 can be implemented for the plate element. This means that all terms in Eq. (28) and in FIGURE 10 are vectorized and as a result of the transition procedure, the plate element kinematics, shown in FIGURE 11, can be expressed as:

![Figure 11. Shear Corrected Thin Plate Element Based on the Absolute Nodal Coordinate Formulation](image)

$$
\begin{pmatrix}
\mathbf{r} \\
\mathbf{w}
\end{pmatrix} =
\begin{bmatrix}
s_{1x} & K & s_{1x}^\ast \\
g_{x1} & K & g_{x1}^\ast \\
g_{y2} & K & g_{y2}^\ast \\
f_{x1} & K & f_{x1}^\ast \\
f_{y1} & K & f_{y1}^\ast
\end{bmatrix}
\begin{pmatrix}
\varphi_x \\
\varphi_y \\
\varphi_{y1} \\
\varphi_{y2} \\
\varphi_{y3}
\end{pmatrix}
\begin{pmatrix}
r_1 \\
r_2 \\
r_3 \\
r_4 \\
r_5
\end{pmatrix}
$$

### 3.3. Equations of Motion of the Shear Deformable Plate Element

Over the course of this section, it is convenient to split Eq. (29) into components, introducing partial shape function matrices and the vector of nodal coordinates as shown below:
\[ \mathbf{r} = \mathbf{S} \cdot \mathbf{q}, \]

\[ \gamma_\alpha = \Gamma_\alpha \cdot \mathbf{q}, \quad \varphi_{s1}, \quad \mathbf{q} = \begin{bmatrix} r_1 \\ \varphi_{s2} \end{bmatrix}. \] (30)

Equations of element motion have the following matrix representation:

\[ \mathbf{M} \cdot \ddot{\mathbf{q}} + \mathbf{Q}^e(\mathbf{q}) = \mathbf{Q}^s \] (31)

where \( \mathbf{M} \) is the mass matrix, \( \mathbf{Q}^s \) is the vector of generalized gravity forces, and \( \mathbf{Q}^e(\mathbf{q}) \) is the vector of generalized elastic forces. These components of the equations of motion can be written as follows:

\[ \mathbf{M} = \rho h \int_{\Pi} \mathbf{S}^T \cdot \mathbf{S} \, d\Pi, \quad \mathbf{Q}^s = \rho h \int_{\Pi} \mathbf{S}^T \cdot \mathbf{g} \, d\Pi, \quad \mathbf{Q}^e(\mathbf{q}) = \frac{\partial U}{\partial \mathbf{q}} = \mathbf{K}(\mathbf{q}) \cdot \mathbf{q}. \]

where \( \rho \) is the mass density of the plate material. In equations above, the integration is assumed within the plate mid-surface \( \Pi \) while \( d\Pi = dx dy \).

In equations of motion (31), the most complicated term is \( \mathbf{Q}^e(\mathbf{q}) \), the gradient of the strain energy \( U(\mathbf{q}) \), which is highly non-linear in respect to \( \mathbf{q} \). For the shear deformable plate, the strain energy can be decomposed onto three parts:

\[ U = U_e + U_\kappa + U_\gamma, \] (32)

where component \( U_e \) is responsible for deformations in mid-surface:

\[ U_e = \frac{1}{2} \frac{Eh}{1 - \nu^2} \int_{\Pi} \left( (\epsilon_{xx} + \epsilon_{yy})^2 - 2(1 - \nu) (\epsilon_{xx} \epsilon_{yy} - \epsilon_{xy}^2) \right) \, d\Pi, \]

and component \( U_\kappa \) is responsible for transverse bending deformations of the plate:

\[ U_\kappa = \frac{1}{2} \frac{Eh^3}{12(1 - \nu^2)} \int_{\Pi} \left( (\kappa_{xx} + \kappa_{yy})^2 - 2(1 - \nu) (\kappa_{xx} \kappa_{yy} - \kappa_{xy}^2) \right) \, d\Pi, \]

and component \( U_\gamma \) deals with shear deformations:

\[ U_\gamma = \frac{1}{2} kGh \int_{\Pi} \left( \gamma_{\xi}^2 + \gamma_{\eta}^2 \right) \, d\Pi. \]

The expression for mid-surface strains has the standard form of the Green–Lagrange tensor as follows:

\[ \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \varphi_j}{\partial x_i} - \delta_{ij} \right), \]

while the vectorized fiber rotations and shears should be transformed back to scalars by multiplying by the normal vector \( \mathbf{n} \) to the plate as follows:

\[ \kappa_{ij} = \frac{1}{2} \left( \frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \varphi_j}{\partial x_i} \right) \cdot \mathbf{n} = \frac{1}{2} \mathbf{q}^T \cdot \frac{\partial \mathbf{F}^T}{\partial x_i} + \frac{\partial \mathbf{F}^T}{\partial x_j} \cdot \mathbf{n}, \]

\[ \gamma_i = \mathbf{q}_i \cdot \mathbf{n}, \quad \mathbf{n} = \frac{\partial \mathbf{r}}{\partial x_1} \times \frac{\partial \mathbf{r}}{\partial x_2}. \]

To evaluate integrals in Eq. (32), numerical integration should be used.

### 3.4. Example of simulation

In the simulation example being considered in this section the accuracy of the newly developed thin plate element with shear correction is studied as compared to other (thick) elements that use fully parametrized representation of the plate. This example was first considered in paper [16]. The rectangular plate shown in FIGURE 12 is clamped while the structure is subjected to a force of 50 N at point \( A \). The length, width and thickness of the plate are 1.0, 1.0 and 0.01 m, respectively. The material is assumed isotropic, Young’s modulus of the material is 2.07·10^11 N/m^2 and Poisson’s ratio is equal to 0. The shear correction factor \( k \) is assumed to be equal to 1.

In Table 1, the vertical displacement of point \( A \) is presented, using different discretizations containing \( nxn \)
elements with different numbers $n$. The types of elements used in the table are as follows:

0 – solution according to Kirchhoff thin plate theory;

I – continuum based thick ANCF element, which is known to be suffering from shear locking;

II – thick ANCF element developed by authors of paper [16] with shear locking removed;

III – commercial ANSYS plate element (type SHELL63);

IV – the thin plate element with the shear correction developed in the current paper.

Table 1. Vertical displacement of point A in nm, obtained by using different types of finite elements

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1^2</td>
<td>1.2467</td>
<td>0.7248</td>
<td>1.2464</td>
<td>1.2446</td>
<td>1.2468</td>
</tr>
<tr>
<td>2^2</td>
<td>1.2847</td>
<td>0.9267</td>
<td>1.2843</td>
<td>1.2753</td>
<td>1.2845</td>
</tr>
<tr>
<td>3^2</td>
<td>1.2966</td>
<td>1.0338</td>
<td>1.2960</td>
<td>1.2901</td>
<td>1.2965</td>
</tr>
</tbody>
</table>

As one can see from the table, thick element I explicitly shows its locking behavior in this case of slender plate since it gives significantly smaller displacement than the others that are much closer to the real displacement in this case. Thick element II with improved stiffness parameters and ANSYS element III give similar results. So does the new thin element IV and shows its ability to compete with thick elements that have 1.5 times more degrees of freedom. It can be also shown that for moderate-slower and thin plates the solution obtained by the new plate element is in a good accordance with results obtained for beam, see Figure 7.

CONCLUDING REMARKS

In absolute nodal coordinate formulation, the use of transverse slope vectors in the definition of shear deformation may lead to problems that burden the numerical simulation and lower the performance of the finite elements. In this study, the shear deformation in a thin plate model is accounted for by using vectors that describe the orientation of the cross-section fiber. No degrees of freedom in the element transverse direction are introduced. The orientation of the cross-section is coupled with the displacement field of the element with the help of the equilibrium equations describing forces and moments.

Acknowledgement

The research is supported by the Academy of Finland, project 122899.

References


