A Study on Three-Dimensional Disturbances in a Compressible Shear Layer*

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Early stages of transition to turbulence of compressible shear layers were investigated in the viewpoint of a linear stability analysis. The primary stage of the transition is by which instability of a laminar flow (*primary instability*), in which disturbances develop into spanwise vortices and form a secondary flow. The secondary stage of the transition is by which instability of the secondary flow (*secondary instability*), in which disturbances develop into streamwise vortices. In the present study, the primary and secondary instability problems were formulated as eigenvalue problems and solved numerically using spectral methods. At first, the primary instability problems were solved. From the eigenvectors of the primary instability, we obtain density, velocity, temperature and pressure fields of the secondary flow. Using these quantities, the instability of the secondary flow was calculated. For the secondary instability, it is shown that (1) growth rates of disturbances are larger than those in the primary instability, (2) growth rates of disturbances have a peak when the ratio of wavelength of the primary and secondary instability is approximately 0.8 and (3) growth rates of disturbances are less affected by flow compressibility than those in the primary instability.

Key Words: Compressible flow, Transition to turbulence, Secondary instability, Mixing enhancement, Scramjet engine

1. Introduction

In development of scramjet engines for propulsion of hypersonic aircrafts, mixing enhancement between supersonic air stream and gaseous fuel stream is one of the most important technologies. Many experimental and computational works have shown that growth rates of shear layers decrease rapidly with the increase in convective Mach number, M_c [1-5]. This gain in stability leads to increase in combustor length. Therefore there has been great interest in understanding the stability characteristics of compressible shear layers.

Orszag et al. [10] and Pierrehunbert et al. [11] investigated the early stages of transition to turbulence in incompressible flows, using a linear stability analysis.

Early stages of the transition can be described as bellow, (1) instability of a laminar flow (*primary instability*) and formation of a periodic secondary flow; (2) instability of the secondary flow (*secondary instability*) and formation of a more complex flow. Further instability may appear one after another, and the flow gains complexity rapidly. Finally, the flow leads to chaotic behavior, i.e. turbulence.

For *incompressible* shear layers, these stages of the transition have investigated experimentally and analytically. Bernal et al. [12] have investigated vortex formations in incompressible shear layers experimentally. It is shown that, at first, spanwise vortices are formed in a shear layer, and then superimposed on the spanwise vortices, streamwise vortices are formed. Formation of the streamwise vortices is observed as onset of *streamwise streaks*. It is also shown that these streamwise vortices play important roles in mixing increase.

In the viewpoint of a linear stability analysis,

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Pierrehunbert et al. [11] investigated the formation of streamwise vortices. By superimposing small disturbances on Stuart's vortex street, which have similar configuration to the spanwise vortices in a shear layer, stability characteristics are examined. It is shown that Stuart's vortex street is unstable to small disturbances which have spanwise orientation. It means that the formation of the secondary streamwise vortices observed experimentally is due to the secondary instability. It is also shown that growth rates of the disturbances are considerable large. Therefore the secondary streamwise vortices are possible to affect drastically on mixing in shear layers.

For *compressible* shear layers, there seems few works on the secondary instability. Considering that it may have large growth rates even in compressible shear layers, it is necessary to investigate its effects on mixing in compressible shear layers. It is considered that understanding characteristics of the secondary instability of compressible shear layers is useful to optimize mixing-enhancement devices which use streamwise vortices [6-9]. In these experiments, streamwise vortices are induced into compressible shear layers by using three-dimensional configuration of trailing edges or nozzles. Increment in growth rates of the shear layers ranges approximately from 40 to 110%. It is considered that these large effects on mixing enhancement in these experiments are partly due to the secondary instability. In these experiments, the induced streamwise vortices would excite the secondary instability of the shear layers. The aim of the present work is to investigate characteristics of the secondary instability in compressible shear layers, in the viewpoint of a linear stability analysis.

2. Mathematical formulation of the stability problem

2.1. Governing equations of the stability problem

We take a coordinate system with x pointing in the streamwise direction, y in the transverse direction and z in the spanwise direction, respectively. The coordinate systems are depicted in Fig. 1. In the present study, we adopt two kinds of coordinate systems owing to the stage of instability, which is discussed in following two sections.

The evolution of the flow is governed by the equations bellow.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{1}{\gamma M^2} \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \quad (2)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{1}{\gamma M^2} \frac{1}{\rho} \frac{\partial p}{\partial y} = 0 \quad (3)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{1}{\gamma M^2} \frac{1}{\rho} \frac{\partial p}{\partial z} = 0 \quad (4)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} + (\gamma - 1) \frac{p}{\rho} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \quad (5)$$

These are continuity, momentum and energy equations. ρ , u , v , w , T and p are density, velocity, temperature and pressure, respectively. M and γ are Mach number of the faster stream and ratio of specific heats, respectively. We also used an equation of state,

$$p = \rho T. \quad (6)$$

Using this equation, pressure is omitted from the equation system.

Instability of a basic flow is examined by superimposing small disturbances into the basic flow quantities. In a linear stability analysis, flow quantities are expressed as bellow,

$$\Phi = \Phi_{\text{basic flow}} + \varepsilon \Phi_{\text{disturbance}} \quad (\varepsilon \ll 1) \quad (7)$$

where $\Phi = (\rho, u, v, w, T)$, i.e. Φ corresponds to flow quantities such as density, velocity and temperature. Basic flows which will be examined the instability are not necessarily laminar. Spatially periodic flows are also allowed [11]. ε denotes that amplitudes of these disturbances are small. Flow quantities expressed above are submitted into Eqs. (1)-(5), and linearized with respect to ε . Finally we will obtain a linear equation system governing evolution of the disturbances.

2.2. Primary instability

First we will investigate the evolution of small disturbances in *laminar* basic flows. Flow quantities are assumed to be of the form,

$$\Phi(x, y, z, t) = \overline{\Phi}(y) + \varepsilon [\hat{\Phi}_1(y) \exp\{i(\alpha_1 x - \omega_1 t)\}] \quad (\varepsilon \ll 1) \quad (8)$$

The first term of the right hand side, with an over bar, represents laminar flow quantities. The second term of the right hand side represents the primary

Fig. 1 Schematics of flow fields and coordinate systems; (a) a basic flow for the primary instability problem (laboratory coordinate system) and (b) a basic flow for the secondary instability problem (convective frame of reference) and (c) resulting flow by the secondary instability.

instability quantities. A quantity with a hat denotes complex amplitude of the disturbance. Subscript 1 denotes primary instability. α is a streamwise wave number and ω is a frequency. Since temporal instability is considered, α is a real quantity, while ω is complex. The real and imaginary parts of ω represent the frequency and the temporal growth rate of the disturbance, respectively.

In the primary instability of compressible shear layers, it is shown that three-dimensional disturbances have larger growth rates than two-dimensional disturbances when $Mc > 0.6$ [2,3]. Therefore it is better to consider three-dimensional disturbances for the primary instability in such a range of Mc . However difference of growth rates between two- and three-dimensional disturbances is relatively small within the range of $Mc < 0.8$ [2]. Therefore in the present study, we deal with only two-dimensional disturbances for the primary instability. And we carry out the calculation within the range of $0.005 < Mc < 0.8$.

The basic laminar flows are assumed to be parallel shear flows, as shown schematically in Fig. 1 (a). Coordinate system is also shown. All values of the laminar flows are non-dimensionalized so that the values at the faster stream are unity. Static pressure of the laminar flow is assumed to be uniform in the flow field. The velocity is expressed as below,

$$\bar{u}(y) = 0.5(1 + \tanh(y)). \quad (9)$$

The static temperature is calculated by a Crocco-Busemann relationship, by assuming that the Prandtl number is unity.

$$\bar{T}(y) = M^2 \frac{\gamma - 1}{2} [\bar{u} - \bar{u}^2] + \bar{T}_{\text{slow}}(1 - \bar{u}) + \bar{u} \quad (10)$$

where $\bar{T}_{\text{slow}}$ is the static temperature at the slower stream. In the present study, $\bar{T}_{\text{slow}} = 1$.

2.3. Secondary instability

Second we will investigate the evolution of small disturbances in the *spatially periodic* basic flows which are obtained as a solution of the primary instability. The flow quantities are assumed to be of the form,

$$\begin{aligned} \Phi(x, y, z, t) = & \bar{\Phi}(y) \\ & + A_1 \text{Re}[\hat{\Phi}_1(y) \exp\{i(\alpha_1 x - \omega_1 t)\}] \\ & + \varepsilon [\hat{\Phi}_2(x, y) \exp\{i(\alpha_2 x + \beta_2 z - \omega_2 t)\}] \end{aligned} \quad (\varepsilon \ll 1) \quad (11)$$

The first term of the right hand side, with an over bar, represents the laminar flow quantities. The quantities of the laminar flows are just the same as the primary instability problem.

The second term of the right hand side represents the primary instability quantities. Subscript 1 denotes primary instability. We adopt neutral disturbances, i.e. disturbances whose growth rate ω_{1i} is zero, as the primary instability quantities. A_1 is amplitude of the primary instabilities. Because of the linearity of a stability analysis, amplitude of the primary instability cannot be decided. In the present study, the primary instability is assumed to saturate and to achieve an equilibrium amplitude. A_1 is chosen so that amplitude of u_1 , i.e. amplitude of streamwise velocity disturbance, becomes 2.5% of the faster stream velocity of the laminar flow. Amplitudes of other quantities, such as ρ_1 , v_1 , w_1 , T_1 are decided simultaneously, because a set of amplitudes of disturbances is decided uniquely.

The third term of the right hand side represents the disturbances of the secondary instability. Subscript 2 denotes secondary instability. A quantity with a hat denotes complex amplitude of the disturbances. α and β are streamwise and spanwise wave numbers, respectively. ω is a complex frequency. Note that the complex amplitude of the disturbance is a function of not only y direction but also x direction. The complex amplitude of the secondary instability must be periodic in x with period $\lambda_1 (=2\pi/\alpha_1)$, wavelength of the primary instability. It is required by Froquet theorem. On the other hand, no constraints are placed on α_2 and β_2 [11].

Before submitting the quantities into the governing equations, we carried out transformation of the coordinate system. The disturbances in the primary instability convect with phase velocity $c_{ph} (= \omega_{1r}/\alpha_1)$. We can choose a convective frame of reference which convects with the phase velocity c_{ph} , with no loss of generality. The transformed coordinate system is shown in Fig. 1(b). By this transformation, the streamwise coordinate is transformed to $x' = x - c_{ph}t$ and the laminar flow velocity is also transformed to $\bar{u}' = \bar{u} - c_{ph}$. Eq.(11) becomes

$$\begin{aligned} \Phi'(x', y, z, t) = & \bar{\Phi}'(y) \\ & + A_1 \operatorname{Re}[\hat{\Phi}_1(y) \exp\{i(\alpha_1 x')\}] \\ & + \varepsilon [\hat{\Phi}_2(x', y) \exp\{i(\alpha_2 x' + \beta_2 z - \omega_2 t)\}] \\ & (\varepsilon \ll 1) \end{aligned} \quad (12)$$

In the later part of the article, the prime (') will be omitted.

3. Numerical solution by spectral method

Here we will approximate the linear operator by a finite-order matrix. We will deal with this by using a two-dimensional spectral-collocation technique [11]. By submitting Eq.(8) or Eq.(12) into Eqs. (1)-(5) and linearizing with respect to ε , we obtain linear equation systems on the evolution of the disturbances. These can be represented in the schematic forms,

$$\mathcal{A}_1 \hat{\Phi}_1 = \omega_1 \hat{\Phi}_1 \quad (13), \quad \mathcal{A}_2 \hat{\Phi}_2 = \omega_2 \hat{\Phi}_2 \quad (14)$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are 5×5 matrices and ω_1 and ω_2 are eigenvalues of the linear equation systems. Quantities which have subscript 1 denote the primary instability and subscript 2 the secondary instability.

A two-dimensional spectral method is used to represent the differentiation operators D_x and D_y

($= \partial / \partial x$ and $\partial / \partial y$) in Eqs. (13) and (14). In x direction, we adopt equidistant collocation points. In y direction, it is convenient to carry out the y expansion. Solutions are mapped from an infinite domain, $y \in (-\infty, \infty)$, to a finite computational domain, $\eta \in (0, 1)$, by an expansion function defined as below,

$$y = -a_y \cot(\pi \eta) \quad (15)$$

where a_y is an expansion parameter whose optimum value for this problem is $a_y \approx 5$.

Equation (13) is solved at N_y collocation points, resulting in solution of a $5N_y \times 5N_y$ matrix eigenvalue problem. Equation (14) is solved at $N_x \times N_y$ collocation points, resulting in solution of a $5N_x N_y \times 5N_x N_y$ matrix eigenvalue problem. The numbers of collocation points are $N_x=8$, and $N_y=50$, respectively. They were confirmed to be sufficient by verifying the value.

For the primary instability, the problem becomes one-dimensional, i.e. only y direction. The y boundary conditions are that the amplitudes of the disturbances vanish at $y \rightarrow \pm\infty$. For the secondary instability, the problem becomes two-dimensional, i.e. x and y directions. The x boundary conditions are that all perturbation quantities be periodic in x with period $\lambda_1 (=2\pi/\alpha_1)$. The y boundary conditions are that the amplitudes of the disturbances vanish at $y \rightarrow \pm\infty$. When solution, we used a standard mathematical library routine to find all eigenvalues and eigenvectors.

To find eigenvalues of Eq.(13) corresponds to solve the primary instability problem. The components of a matrix $\mathcal{A}_1$, which consists of laminar basic flow quantities, are often found in literatures [5]. Eq.(13) can be solved alone. On the other hand, Eq.(14) cannot be solved alone. The components of a matrix $\mathcal{A}_2$ include quantities of the primary instability. First it is necessary to solve the primary instability problem and to obtain the eigenvalues and eigenvectors. Second we submit these quantities into a matrix $\mathcal{A}_2$, and then solve the secondary instability problem.

4. Results and discussions

4.1. Primary instability

First we will show the results of the primary instability problem. Figure 2 shows the relationships between wave number α_1 and temporal growth rates of the primary instability ω_{1i} . Two important features are seen, (1) as convective Mach number M_c increases, the temporal growth rates decrease, and (2) as

Fig. 2 Variations of temporal growth rates of the primary instability ω_{1i} with streamwise wave number α_1 .

Fig. 3 A typical basic flow of the secondary instability problem, (a) a velocity profile without the disturbance and (b) a velocity profile on which the primary instability quantities for $M_c=0.6$ is superimposed.

Fig. 4 Variations of temporal growth rates of the secondary instability ω_{2i} with spanwise wave number β_2 .

convective Mach number M_c increases, the values of α_1 which give the maximum growth rates also decrease. These characteristics agree with previous studies which used a linear stability analysis.

4.2. Secondary instability

Second we will show the results of the secondary instability problem. In the present study, disturbances which have spanwise orientation, i.e. $\alpha_2=0$ were investigated. This class of the secondary instability is called *transrative instability* [11].

A typical basic flow of the secondary instability problem is shown in Fig. 3. In the present study, neutral disturbances are adopted as the primary instability. Because we assume temporal instability and $\bar{T}_{slow}=1$, phase speed of the primary disturbance, c_{ph} , is always 0.5. In Fig. 3 (a), a laminar velocity profile is shown, whose coordinate system is transformed to convective frame of reference. Upon this velocity profile, primary instability quantities for $M_c=0.60$ are superimposed, resulting in Fig. 3 (b). One period λ_1 ($=2\pi/\alpha_1$) of the primary disturbance is shown. All quantities of the basic flow are uniform in z -direction, because the primary instability assumed to be two-dimensional.

Figure 4(a) shows the variations of temporal growth rates of the secondary instability ω_{2i} with spanwise wave number β_2 . Two important features are seen. (1) The maximum growth rate for each M_c is always larger than those of the corresponding primary instability. (2) As M_c increases, the values of β_2 which give the maximum growth rates decrease. Figure 4(b) shows the variations of ω_{2i} with the ratio of spanwise and streamwise wave numbers β_2/α_1 . For any M_c , the growth rate curves have a peak at almost the same β_2/α_1 , which is approximately 1.2~1.3. It means that the ratio of wavelength of the primary and secondary instabilities, λ_2/λ_1 , don't change with M_c . The reciprocal value of β_2/α_1 is the wavelength ratio, which is approximately 0.8.

It is wide known that, for the primary instability, compressibility effects on flow instability can be written in terms of M_c [1-5]. Figure 5 shows normalized maximum growth rates for the primary and secondary instabilities as a function of M_c . Maximum growth rates are normalized by the value for incompressible case ($M_c=0.005$). For comparison, a compressibility effect function, $f(M_c)$, drawn through a linear stability analysis by Zhuang et al. [4], is also shown. The growth rate of the primary instability decreases with the increase in M_c . The results of the present work agree well with $f(M_c)$. The growth rate

Fig. 5 Normalized maximum growth rates for the primary and secondary instabilities as a function of M_c .

for the secondary instability decreases more slowly than that of the primary instability, with the increase in M_c . It means that the secondary instability is less affected by compressibility than the primary instability. It is considered that the large propagation angle ($\alpha_2=0$) of the secondary instability leads to this results.

5. Conclusions

Early stages of transition to turbulence of compressible shear layers are investigated in the viewpoint of a linear stability analysis. The primary and secondary instability problems are formulated as eigenvalue problems and solved numerically using spectral methods. We obtained conclusions on characteristics of the secondary instability, as follows:

- (1) Growth rates of disturbances in the secondary instability are larger than those in the primary instability.
- (2) Growth rates of disturbances in the secondary instability have a peak when the ratio of wavelength of the primary and secondary instability is approximately 0.8.
- (3) Growth rates of disturbances in the secondary instability are less affected by flow compressibility than those in the primary instability.

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