Numerical Investigation of Supersonic MPD Viscous Flows with Ionization*

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An implicit time-marching method based on the LU-SGS scheme developed for self-filed magneto-plasma dynamic (MPD) fully-ionized viscous flows is extended to the method for self-field MPD viscous flows considering a finite rate of ionization. The axisymmetric compressible Navier-Stokes equations with Lorentz force and Joule heating, the equation of magnetic induction induced from Maxwell's equations with Ohm's law, and the continuity equation of electron are simultaneously solved using the present time-marching method. Partially ionized flows in an experimental MPD thruster are simulated and compared with the experiments. Also the effect of flow conditions such as the inlet temperature, the total current, and the rate of ionization to the flow field is numerically investigated.

Key Words: MPD Thruster, Numerical Method, Viscous Flows, Ionization

1. Introduction

The self-field magneto-plasma dynamic (MPD) thruster is expected as one of electric propulsion systems employed for future space missions requiring heavy-lift transfer. Figure 1 shows the schematic of the MPD thruster in which the Lorentz force is generated by the interaction between the electric current and the azimuthal magnetic field. Since it is, however, difficult to observe flows directly in experiments because of high temperature plasma and axisymmetric geometries, many numerical investigations have been developed. For examples, fully ionized 1-D, quasi-1-D, 2-D, and axisymmetric flows4, 5, 6 have been calculated. Also, partially ionized flows7, 8 have been studied. Since the characteristic time of fluid fields is 10^3 times as large as that of magnetic fields, the equation of magnetic induction derived from Maxwell's equations and Ohm's law has been solved as an elliptic Poisson equation. The magnetic field must be separately solved from the flow field. On the other hand, our group has proposed a numerical method in which both fields are solved simultaneously by using an implicit time-marching method based on LU-SGS scheme and the diagonal point-implicit scheme.9 This method has been applied to the self-field MPD viscous flows considering a finite-rate ionization10 and the calculated results have been well compared with the experimental results11.

In this paper, the previous study is further extended to

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Nomenclature

- $B$: Magnetic induction
- $E$: Vector of electric field
- $E$: Total internal energy per unit volume
- $I$: Total interelectrode current
- $j$: Vector of current density
- $m_s$: Mass of species $s$
- $n_s$: Number density of species $s$
- $p$: Static pressure
- $Re$: Reynolds number
- $R_m$: Magnetic Reynolds number ($= \frac{\sigma m \mu_0 V m_i r_e}{\nu m \nu_m}$)
- $R_p$: Magnetic pressure number ($= B^2 \mu_0 \mu_m V^2 m$)
- $r, z$: Radial and axial coordinates
- $T$: Static temperature
- $t$: Time
- $u$: Velocity
- $e_i$: Ionization energy
- $\kappa_s$: Thermal conductivity of species $s$
- $\mu_0$: Permeability of free space
- $\mu$: Viscosity
- $\nu_m$: Collision frequency of species $s$ with species $r$
- $\rho$: Density
- $\sigma$: Electrical conductivity
- $\bar{\sigma}$: Non-dimensional electrical conductivity
- $\omega_e$: Source term of electron
- $\psi$: Stream function of magnetic field ($= rB$)

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Subscripts

$h$ : Heavy particles  
$e$ : Electron

Fig.1 Schematic of a self-field MPD thruster

2. Fundamental Equations

The fundamental equations are introduced with following assumptions in this study: the propellant gas is argon. The plasma is electrically neutral. The flow is a two-dimensional axisymmetric flow. The electron temperature is in equilibrium with that of heavy particles.

Then, the fundamental equations contain six equations, which are the axisymmetric compressible Navier-Stokes equations with Lorentz force and Joule heating, the continuity equation of the electron, and the equation of magnetic induction derived from Maxwell equations with Ohm’s law.

$$
\frac{\partial Q}{\partial t} + \frac{\partial F_j}{\partial x_j} + H = \frac{\partial F_{vj}}{\partial x_j} + H + R + D
$$

where $Q$, $F_j$ and $F_{vj}$ are the vector of unknown variables, the flux vectors and the vector of the viscosities, respectively. $H$ and $H_e$ include the axisymmetric terms. $R$ and $D$ include the source term and the diffusive term of the equations of magnetic induction.

The viscous stress tensor $\tau_{ij}$ and $\beta_j$ are defined by:

$$\tau_{ij} = \mu \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right]$$

$$\beta_j = \tau_{j1} u_1 + (\kappa_h + \kappa_e) \frac{\partial T}{\partial x_j}$$

where the subscripts 1 and 2 of $x, u, r, \beta$ correspond to $z$ and $r$ directions, respectively. The viscosity $\mu$ is only that of heavy particles. The current density $j$ and electric field vector $E$ are defined by:

$$j = \begin{bmatrix} j_z \\ j_r \end{bmatrix} = \frac{1}{Re} \begin{bmatrix} \frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} \\ \frac{\partial B_\theta}{\partial z} \end{bmatrix}$$

$$E = \begin{bmatrix} E_z \\ E_r \end{bmatrix} = \begin{bmatrix} j_z - u_r B_\theta \\ j_r + u_t B_\theta \end{bmatrix}$$

The total energy $E$ is determined by:

$$E = \frac{1}{\gamma - 1} + \frac{1}{2} \rho u^2 + n_e e_i$$

where $\gamma$, $\rho$ and $e_i$ are ratio of specific heats, static pressure, and ionization energy. The number density of electron $n_e$ is given by the continuity equation of electron.

The source term of electron $\omega_e$ is defined by:

$$\omega_e = m_e \alpha(T) \left[ S_n n^{(k)} - n_e n^{(k+1)} \right]$$

Here $\alpha(T)$ and $S_n$ are given by the Hinnoy – Hirschberg recombination coefficient and Saha equation, respectively. The electric conductivity of plasma $\sigma$ is determined by:

$$\sigma = \frac{e^2 \nu}{m_e \sum \nu \nu}$$
3. Numerical Method

The implicit time-marching method based on the LU-SGS scheme and the diagonal point-implicit scheme has been proposed by our group for solving the equations of fluid and magnetic induction simultaneously. This method has been applied to the equations of fluid, magnetic induction, and the equation of electron density. Then, the convection terms in Eq.(1) are calculated by the modified flux-vector splitting scheme coupled with the higher-order MUSCL TVD extrapolation. The viscous terms are calculated by the second-order central difference. The same method is also applied to the present calculations in this study.

4. Results

The experimental MPD thruster developed by Mitsuo et al. is employed in this study. The computational grid generating in the flow field of the MPD thruster has 131 x 61 grid points as shown in Fig.2. The flow conditions are specified as follows. The mass flow rate is \( \dot{m} = 1.37 \text{g/sec} \). The inlet Mach number is \( M_{in} = 1.0 \), and the supersonic outlet flow. In magnetic conditions, when total interelectrode current \( I \) is discharged, the stream function \( \psi \) is set to \( \psi = -\mu_0 I / 2\pi \) at the inlet and insulators, is set to \( \psi = 0 \) at the axis of symmetry.

Since the electron temperature may affect the electrical conductivity and the thermal conductivity, the inlet temperature of electron should be carefully specified. However, this temperature has not been measured in the experiment. In this paper, the electron temperature is assumed to be in equilibrium with that of heavy particles. The adequate inlet temperature is numerically determined in this study as follows. The flows specifying the inlet temperature at \( T_{in} = 5000, 7500, \) and \( 10000 \text{K} \) and the corresponding electrical conductivities at \( \sigma = 900, 1620, \) and \( 2300 \text{S} \), are calculated assuming the inlet ionization rate at 0.01. Also these calculated results are compared with the flows specifying the inlet temperature at \( T_{in} = 10000 \text{K} \) with fully ionized assumption. Figure 3 shows the calculated ratios of the current into the cathode to the total discharge current along the cathode surface from the root to the tip compared with the experiment. The experimental ratio is about 60% at \( x/L = 0.5 \). The calculated results at \( T_{in} = 10000 \text{K} \) indicates the most close value to the experiment. Therefore, the inlet temperature \( T_{in} = 10000 \text{K} \) is determined to be the adequate value for the following calculations.

Figures 4(a)(b) and (c) show calculated current contour lines for \( I = 5 \text{kA} \) (fully ionized), \( I = 2 \text{kA} \) and \( I = 5 \text{kA} \) (partially ionized). The current contour lines for \( I = 5 \text{kA} \) have been experimentally shown in Ref.11 and the calculated result is in good agreement with the experiment except for the value close to the anode. Since the distributions close to the cathode is rather important for evaluating the performance of MPD thrusters, this calculated result can be acceptable. The calculated current contour lines for \( I = 2 \text{kA} \) in Fig.4(b) are similar to those for \( I = 5 \text{kA} \) in Fig.4(c), while those for \( I = 5 \text{kA} \) in Fig.4(a) assuming a fully ionized flow are quite different from calculated results assuming a partially ionized flow both in Fig.4(b) and (c). This difference indicates that a finite rate of ionization must be at least considered in the numerical study of MPD thrusters to get real flow physics.

Figures 5 and 6 show non-dimensional axial velocities \( u_z \) and Mach number distributions along an intermediate grid line in radial direction for \( I = 2, 3, 4, \) and \( 5 \text{kA} \), respectively. It is found that flows are accelerated in all cases and the flow speed at the outlet increases as increasing in the total current \( I \). A supersonic flow is produced in cases for \( I = 3, 4, \) and \( 5 \text{kA} \). Therefore, the flow is successfully accelerated by the present MPD thruster.

Figure 7 shows the calculated pressure contours for \( I = 5 \text{kA} \). A high-pressure region is found near the cathode tip. This reason can be explained by showing the velocity vectors around the cathode. The velocity vectors for \( I = 0, 2 \) and \( 5 \text{kA} \) in Figs. 8 (a)(b) and (c) can be compared with each other. The flow is separated after the top of the charged cathode for \( I = 0 \text{kA} \) in Fig.8(a). On the other hand, no separation region can be found in both cases for \( I = 2 \) and \( 5 \text{kA} \) in Figs.8(b) and (c). Since the flow is accelerated toward the radial direction by the Lorentz force, the separation disappears and the pressure near the cathode tip increases due to the incoming flow toward the cathode tip along the cathode.

Figure 9 plots the calculated pressures at the cathode tip in comparison with the experimental and theoretical results. In Ref.11, the pressure on the cathode tip is measured and compared with theoretical values. This theoretical value is based on the assumption that current enters the end surface of cylindrical cathode in a uniform normal beam. The theoretical value is decided as follows:

\[ P_{th} = \frac{-\mu_0 I^2}{4\pi^2 r_c^2} \]  

where \( r_c \) is the cathode's radius. Although the calculated pressure for \( I < 4 \text{ [kA]} \) are overestimated from the theory, the calculated results may be in good agreement with the experiments if considering a finite error in the experiments.
Fig. 2 Scheme of computational grid

Fig. 3 Ratio of current sum

Fig. 5 Velocity distributions

Fig. 6 Mach number distributions

Fig. 7 Pressure contours (I=5kA)

(a) I=5kA (Fully ionized)

(b) I=2kA (Partially ionized)

(c) I=5kA (Partially ionized)

Fig. 4 Current contour lines

(a) I=0kA
1. The inlet temperature was carefully determined by the comparison with the experimental data of the current sum into the cathode.
2. The current contour lines are quite different between those assuming fully and partially ionized flows. Therefore, a partially ionization should be at least considered in the calculation to get real flow physics.
3. The outlet flow speed increased successfully as increasing in the total current.
4. The pressures at the cathode tip were well compared with the experiments, though the present study assumes that the electron temperature is in equilibrium with that of heavy particles.

These conclusions suggest that the present method must be a useful and economical CFD tool for developing the MPD thruster.

References