Transient Analysis of Two-Dimensional Pin Fins with Non-Constant Base Temperature*

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The transient response of a 2D pin fin with a time- and space-variation base temperature, is analyzed using the Laplace transformation method and the Duhamel’s method. The temperature distributions of the pin fin are identical with published analytical solution for the case using the method of separation of variables, while the latter method can not treat 2D problem presented in this paper. The heat flux at fin base and the actual heat flux transferred from the lateral surface and the tip surface of the pin fin to the surroundings are also obtained. For all cases analyzed in this paper, the temperature distributions and the heat flux of the fin reach a steady periodic response after $t = 2$ of dimensionless time. The results show that the effects of 2D conduction are large, particular at large time. Furthermore, as $Bi_t$ value is small, the effect of the tip convection on heat transfer is quite significant for short pin fins.

Key Words: Pin Fin, Non-Constant Base Temperature, Laplace Transfer Method, Duhamel’s Method, Heat Transfer, Transient Temperature Distribution

1. Introduction

Extended surfaces (fins) are widely used to increase the rate of heat exchange between a heated source and a colder ambient fluid. They have been used on power generators, heat exchangers, semiconductors, and electronic components.

Numerous literatures studied on the performance and design of these devices. Some representative articles can be found by Suryanaryana(1), Aziz and Na(2), Chu et al.(3) etc. Recently, Aziz etc.(4) has provided a review of the evolution of the fins. Some popular methods used for solving heat transfer in fins are the method of separation of variables(5-6), the Laplace transformation(7,8), and the numerical method(9,10). For convenient analysis, the majority of the above literature assumed temperature or heat flux of the fins base is constant, the others literatures considered periodic cases. However, the actual temperature or heat flux of the fins base depends not only on periodic oscillation but on space function as well.

This paper considers a short pin fin which is usually found in compact heat exchangers(11,12) and air-cooled devices(13,14). An analytical approach is adopted to investigate the responses of a two-dimensional pin fin with tip convection subjected to a time- and space-variation in base temperature. After applying the Laplace transformation, the general transformed solution can be obtained. The transformed solution for special case can be analytically inverted to yield the time domain result by using the residues theorem of the Laplace transformation. Then the Duhamel’s method is used to deal with the time function on boundary conditions. Finally, the temperature distributions and the heat transfer ratio of the pin fin can be obtained.

Nomenclature

$Bi_a$: lateral Biot number
$Bi_t$: tip Biot number
$G$: geometry parameter for pin fin
$h$: convective heat transfer coefficient at the lateral surface
\( h_r \): convective heat transfer coefficient at the tip surface
\( H \): ratio of convective heat transfer coefficient
\( f_0 \): zero-order Bessel function
\( f_1 \): first-order Bessel function
\( k \): thermal conductivity of pin fin
\( L \): pin fin length
\( Q(t) \): dimensionless transient heat transfer at fin base
\( Q_f(t) \): dimensionless heat transfer from lateral surface of pin to surroundings
\( Q_r(t) \): dimensionless heat transfer from tip surface of pin to surroundings
\( r \): dimensionless radial coordinate
\( R \): dimensionless radius of pin fin
\( r^* \): radial coordinate
\( R^* \): radius of pin fin
\( s \): Laplace transformation parameter
\( t \): dimensionless time
\( t^* \): time
\( T \): dimensionless transient temperature
\( T^* \): transient temperature
\( T_0^* \): initial temperature at base center \((r=0 \text{ and } t=0)\)
\( T_2^* \): temperature of surroundings
\( x \): dimensionless axial coordinate
\( x^* \): axial coordinate
\( \kappa \): thermal diffusivity

2. Analysis Considerations

The geometry of a two-dimensional pin fin is shown in Fig. 1. The material properties of the pin fin, the convective heat transfer coefficients \( h \) and \( h_r \), and the surrounding fluid temperature \( T_2^* \), with a time- and space-variable base temperature \( f^*(r^*, t^*) \) over boundary \( x^*=0 \). The governing equation and its associated initial and boundary conditions in dimensionless form are

\[
\frac{\partial^2 T(x^*, r^*, t^*)}{\partial r^*^2} + \frac{1}{r^*} \frac{\partial T(x^*, r^*, t^*)}{\partial r^*} + \frac{\partial^2 T(x^*, r^*, t^*)}{\partial x^*^2} = \frac{\partial T(x^*, r^*, t^*)}{\partial t^*} \tag{1.a}
\]

\[
T(0, r^*, t^*) = f(r^*, t^*) \tag{1.b}
\]

\[
\frac{\partial T(x^*, 0, t^*)}{\partial x^*} + B_i T(1, r^*, t^*) = 0 \tag{1.c}
\]

\[
\frac{\partial T(x^*, R^*, t^*)}{\partial r^*} + GB_i T(x^*, R^*, t^*) = 0 \tag{1.e}
\]

\[
T(x^*, R^*, t^*) = 0 \tag{1.f}
\]

where \( T(x^*, r^*, t^*) \) is dimensionless transient temperature.

\[
\phi = \frac{\sqrt{s + G^2\eta^2} \cosh(\sqrt{s + G^2\eta^2}(1-x)) + B_i \sinh(\sqrt{s + G^2\eta^2}(1-x))}{\sqrt{s + G^2\eta^2} \cosh(\sqrt{s + G^2\eta^2}(x)) + B_i \sinh(\sqrt{s + G^2\eta^2}(x))} \tag{7.c}
\]

\[
G \text{ is a geometry parameter for pin fin} \; \text{; } B_i \text{ and } B_{ai} \text{ are Biot number for the pin tip and lateral, respectively. They are defined as } \tag{2}
\]

\[
T = T^*(x^*, r^*, t^*) - T_0^*, \quad G = \frac{L}{R^*}, \quad r = \frac{r^*}{L}, \quad R = \frac{R^*}{L}, \quad x = \frac{x^*}{L}, \quad t = \frac{t^*}{L^*}
\]

In order to study the effects of the material parameters on the heat transfer of the pin fin, the tip Biot number \( B_{ti} \) can be rewritten as follows:

\[
B_{ti} = \frac{h_L}{k} - \frac{h_R}{k} = \frac{hR^*}{R^*} = HGB_{ai} \tag{3}
\]

where \( H = h_L/h_R \), represents the ratio of the convective heat transfer coefficient to lateral convective heat transfer coefficient.

The dimensionless base temperature \( f(r, t) \) can be expressed as

\[
f(r, t) = \xi(r) \Gamma(t) \tag{4}
\]

and set \( \xi(r) = 1 - 0.1 \, r^2 \).

After taking Laplace transformation with respect to time, the solution of the above equations in the transformed domain can be derived as

\[
T(x, r, s) = \sum_{n=1}^\infty \left[ \Psi_1(r, \eta_n) + \Psi_2(r, \eta_n) \right] \Gamma(s) \phi(x, \eta_n, s) \tag{5}
\]

where the superscript symbol "— " denotes the variable in the Laplace transformation domain and \( s \) is transformation parameter; \( \eta_n \) satisfies

\[
\eta_n J(\eta_n) - B_i J(\eta_n) = 0 \tag{6}
\]

and defined

\[
\Psi_1 = \frac{2B_{ai} J_1(G\eta_n)}{(B_{ai}^2 + \eta_n^2) J_0(\eta_n)} \tag{7.a}
\]

\[
\Psi_2 = \frac{J_0(G\eta_n)}{5(\eta_n^2)} \tag{7.b}
\]


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in which $\Psi_1$ and $\Psi_2$ represent the boundary effects due to 1 and $-0.1r^2$, respectively.

As it is difficult in general to find the inverse transformation of the function $T(x, r, s)$ analytically, a numerical inversion method\(^\text{[13,16]}\) may be used. For some special cases, however, the function at any given time can be evaluated using the inversion theorem for the Laplace transformations.

3. Analysis for Special Cases

Case 1

We now consider the base temperature of the pin fin is a time-independent constant and let $I(t) = 1$; then the temperature distribution in the transformation domain, Eq. (5), can be analytically transformed into time domain by the use of the inversion theorem for the Laplace transformation\(^\text{[17]}\). From the theorem we have

\[
T(x, r, t) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \left[ \Psi_1(r, \eta_n) + \Psi_2(r, \eta_n) \right] e^{-r^2/\eta_n^2} \, dz
\]  

(8)

The singularities of the integrand are $z = 0$ and $\sqrt{z + G^2 \eta_n^2} = -i\lambda_n$, which correspond to the simple poles $z = 0$ and $z = -\lambda_n^2 + G^2 \eta_n^2$, respectively, where $\lambda_n$ satisfies

\[
B_{li} = -\lambda_n \cot (\lambda_n)
\]

(9)

Note that the relationship $\coth (i \lambda_n) = -i \cot (\lambda_n)$ is used to obtain the above equation.

Using the residue theorem to evaluate Eq. (8), it yields

\[
T(x, r, t) = \sum_{n=1}^{m} \left[ \Psi_1(r, \eta_n) + \Psi_2(r, \eta_n) \right] \left( \text{Re} \{s[0]\} + \text{Re} \{s[-\lambda_n^2 + G^2 \eta_n^2]\} \right)
\]

(10)

The residue at $z = 0$ can be evaluated by letting $z \to 0$, that is

\[
\text{Re} \{s[0]\} = \lim_{z \to 0} z \frac{d}{dz} \Phi(x, \eta_n, z) = \frac{G_\eta \cosh (G_\eta (1-x)) + B_{li} \sinh (G_\eta (1-x))}{G_\eta \cosh (G_\eta) + B_{li} \sinh (G_\eta)}
\]

(11)

The residue at $z = -\lambda_n^2 + G^2 \eta_n^2$ may be expressed as $z = z_{n, a}$, it is

\[
\text{Re} \{s[z_{n, a}]\} = \lim_{z \to z_{n, a}} \frac{\sqrt{z + G^2 \eta_n^2} \cosh (\sqrt{z + G^2 \eta_n^2} (1-x)) + B_{li} \sinh (\sqrt{z + G^2 \eta_n^2} (1-x)) e^{zt}}{\sqrt{z + G^2 \eta_n^2} \cosh (\sqrt{z + G^2 \eta_n^2} (1-x)) + B_{li} \sinh (\sqrt{z + G^2 \eta_n^2} (1-x))}
\]

(12)

The numerator and the denominator in Eq. (12) are zero as $z \to z_{n, a}$, we can rewrite the equation using L’Hospital’s rule and is given by

\[
\text{Re} \{s[z_{n, a}]\} = \lim_{z \to z_{n, a}} \frac{\sqrt{z + G^2 \eta_n^2} \cosh (\sqrt{z + G^2 \eta_n^2} (1-x)) + B_{li} \sinh (\sqrt{z + G^2 \eta_n^2} (1-x)) e^{zt}}{\sqrt{z + G^2 \eta_n^2} \cosh (\sqrt{z + G^2 \eta_n^2} (1-x)) + B_{li} \sinh (\sqrt{z + G^2 \eta_n^2} (1-x))}
\]

(13)

To obtain Eq. (13), using the relationships $\sinh (i \lambda_n) = i \sin (\lambda_n)$, $\cos (i \lambda_n) = \cos (\lambda_n)$, and Eq. (6).

Finally, substituting Eqs. (11) and (13) into Eq. (10), and then the temperature distribution can be expressed as

\[
T(x, r, t) = \sum_{n=1}^{m} \left[ \Psi_1(r, \eta_n) + \Psi_2(r, \eta_n) \right] \left( \text{Re} \{s[0]\} + \text{Re} \{s[-\lambda_n^2 + G^2 \eta_n^2]\} \right)
\]

(14)

where

\[
\alpha = \frac{G_\eta \cosh (G_\eta (1-x)) + B_{li} \sinh (G_\eta (1-x))}{G_\eta \cosh (G_\eta) + B_{li} \sinh (G_\eta)}
\]

(15.a)

\[
\beta = \frac{2 \lambda_n^2 \sin (\lambda_n \eta_n)}{(\lambda_n^2 + G^2 \eta_n^2) \cos (\lambda_n \eta_n) \sin (\lambda_n \eta_n)}
\]

(15.b)

The temperature distribution of the pin fin, Eq. (14), will be utilized in case 2.

In this case, if the base temperature $\xi (r)$ in Eq. (4) is replaced $1 - 0.1r^2$ with 1, then $\Psi_2$ in Eq. (14) will be vanished and the following expression can be obtained:

\[
T(x, r, t) = \sum_{n=1}^{m} \left[ \Psi_1(r, \eta_n) \alpha (x, \eta_n) \right.
\]

(16)

\[
- \sum_{n=1}^{m} \beta (x, \eta_n, \lambda_n) e^{-\lambda_n^2 - \cot (\lambda_n \eta_n)}
\]

The above temperature distribution of the pin fin is identical to that obtained by using the method of separation of variables\(^{[2]}\).

Case 2

In this case we assume the base temperature of the pin fin is a function of $r$ and $t$, and set $\xi (r) = 1 - 0.1r^2$ and $I(t) = 0.9 + 0.1 \cos (2\pi t)$. The temperature can be obtained from Eq.(14) and by using the Duhamel’s method, it can be expressed as

\[
T(x, r, t) = I(0) \phi (x, r, t)
\]

(17)

\[
+ \int_0^t \frac{dI(t)}{dt} \phi (x, r, t) \, dt
\]

where function $\phi (x, r, t)$ represents the solution of Eq. (1) for time-independent boundary-condition function $f(r, t) = 1 - 0.1r^2$. Therefore, the function $\phi (x, r, t)$ is same as $T(x, r, t)$, has been obtained in Eq. (14). Substituting Eq. (14) into Eq. (17), $T(x, r, t)$ can be rewritten as

\[
T(x, r, t) = \sum_{n=1}^{m} \left[ \Psi_1(r, \eta_n) + \Psi_2(r, \eta_n) \right] \times \left[ \left( 0.9 + 0.1 \cos (2\pi t) \right) \alpha (x, \eta_n) \right.
\]

(18)

\[
- \sum_{n=1}^{m} \beta (x, \eta_n, \lambda_n) \beta (\eta_n, \lambda_n, t)
\]

\[
\left. \right\} \right\}
\]

\[
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\]

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where
\[ \rho = e^{-i \eta_0 + \omega t} \]

\[ -\frac{2 \pi}{4 \pi^2 + (\lambda_0^2 + G^2 \eta_0^2)} \left\{ (\lambda_0^2 + G^2 \eta_0^2) \sin(2\pi t) - 2 \pi \cos(2\pi t) + 2 \pi e^{-i \eta_0 + \omega t} \right\} \]

The nondimensional heat flow rate at the fin base is given by
\[ Q(t) = -\frac{2 \pi R \int_0^t \frac{\partial T(x_0, r, t)}{\partial x_0} \int_{x_0}^{x_0+\eta_0} dv}{2 \pi R k(T_0 - T_\infty)} \]

Substituting Eq. (18) into Eq. (20), the nondimensional base flux can be obtained as
\[ Q(t) = -\frac{C \sum_{n=1}^{\infty} \left[ \psi_1(\eta_n) + \psi_2(\eta_n) \right]}{\sqrt{2 \pi}} \]
\[ \times \left[ (0.9 + 0.1 \cos(2\pi t)) \delta_t(\eta_n) \right] \]
\[ - \sum_{n=1}^{\infty} \beta_0(\eta_n, \lambda_n, \rho(\eta_n, \lambda_n, t)) \]

(21)

where
\[ \psi_1 = \frac{2B_{ia}^2}{(B_{ia}^2 + \eta_n^2)^2 \eta_n^2} \]
\[ \psi_2 = \frac{B_{ia}}{5(B_{ia}^2 + \eta_n^2)^2 \eta_n^2} \left( \frac{4B_{ia}}{\eta_n^2} - B_{ia} - 2 \right) \]

(22.a)

(22.b)

\[ \delta_t(\eta_n) = \frac{G_{in}(G_{in} + B_{ia} \cosh(G_{in})))}{G_{in} \cosh(G_{in} + B_{ia} \sinh(G_{in}))} \]

(22.c)

\[ \beta_0 = (\lambda_0^2 + G^2 \eta_0^2)^2 \eta_n - \cos(\lambda_n) \sin(\lambda_n) \]

(22.d)

The actual nondimensional heat flux transferred from the lateral surface and the tip surface of the pin fin to the surroundings can be expressed as \( Q_L(t) \) and \( Q_T(t) \), respectively; they are given by
\[ Q_L(t) = \frac{2 \pi R \int_0^t \frac{\partial T(x_0, r, t)}{\partial x_0} \int_{x_0}^{x_0+\eta_0} dv}{2 \pi R k(T_0 - T_\infty)} \]
\[ = B_{ia} G_1 \int_0^T T(x, r, t) \int_{x_0}^{x_0+\eta_0} dv \]

(23.a)

(23.b)

Substituting Eq. (18) into Eq. (23.a) and Eq. (23.b), respectively; they are given by
\[ Q_L(t) = B_{ia} G_1 \sum_{n=1}^{\infty} \left[ \psi_1(\eta_n) + \psi_2(\eta_n) \right] \]
\[ \times \left[ (0.9 + 0.1 \cos(2\pi t)) \delta_t(\eta_n) - \sum_{n=1}^{\infty} \beta_0(\eta_n, \lambda_n, t) \right] \]

(24.a)

\[ Q_T(t) = B_{ia} G_1 \sum_{n=1}^{\infty} \left[ \psi_1(\eta_n) + \psi_2(\eta_n) \right] \]
\[ \times \left[ (0.9 + 0.1 \cos(2\pi t)) \delta_t(\eta_n) - \sum_{n=1}^{\infty} \beta_0(\eta_n, \lambda_n, t) \right] \]

(24.b)

where
\[ \psi_1^p = \frac{2B_{ia}}{B_{ia}^2 + \eta_n^2} \]
\[ \psi_2^p = \frac{1}{5(B_{ia}^2 + \eta_n^2)^2 \eta_n^2} \left( \frac{4B_{ia}}{\eta_n^2} - B_{ia} - 2 \right) \]
\[ \delta_t^p = \frac{G_{in} \sinh^2(G_{in} + B_{ia} \cosh(G_{in})))}{G_{in} \cosh(G_{in} + B_{ia} \sinh(G_{in}))} \]

(25.a)

(25.b)

(25.c)

(25.d)

(25.e)

In order to study the heat flux transferred from the tip surface of the pin fin, the ratio of the heat transferred from the tip surface to the actual overall heat transferred from the pin fin to the surroundings can be defined as
\[ RQT = \frac{Q_T(t)}{Q_L + Q_T} \]

(26)

Substituting Eq. (24) into Eq. (26), the \( RQT \) value can be obtained.

4. Results and Discussion

The foregoing analysis shows that the performance of the two-dimensional pin fin depends on the lateral Biot number \( B_{ia} \), the geometric parameter \( G \), and the ratio of convective heat transfer coefficient \( H \).

The analytical results for case 2 are shown in Figs. 2 - 4. Figures 2(a) and 2(b) show the temperature at the center line \( (r=0) \) and at the surface \( (r=R) \) as a function of \( x \) and \( t \) at \( B_{ia}=0.2 \) and \( B_{ia}=0.5 \), respectively, for the cases of \( G=5 \) and \( H=1 \). It can be found that the effect of 2D conduction in lateral direction is not negligible, particular at large time. As \( B_{ia} \) increases, the 2D effects become more pronounced.

Figures 3(a) and 3(b) depict the effect of various \( B_{ia} \) and \( H \) on the heat transfer rate at fin base for \( G=2 \) and \( G=5 \), respectively. The early part of the transient is hardly affected by \( B_{ia} \) and \( H \) because the thermal energy entering the fin base is stored in the fin and little is dissipated from the fin surfaces. In this period, the heat transferred at base \( Q(t) \) for \( G=2 \) is larger than that for \( G=5 \) since the smaller \( G \) value is, the larger thermal transfer has.

For the cases of the same \( G \) values as shown in Fig. 3, as time increases, the heat flux \( Q(t) \) decreases because the thermal energy in the fin gradually increases and dissipates to the surroundings; and then the effect of the convective heat transfer on the heat transfer of the fin base is significant. For a given \( G \) value, as \( B_{ia} \) increases indicating that the heat dissipated from the lateral area of the fin is large and \( Q(t) \) becomes large while the effect by the ratio of
convective heat transfer coefficient $H$ can be negligible. Finally, for all cases analyzed in the paper, the heat flux at the fin base gradually reaches steady periodic response after $t=2$. After reaching steady situation, comparison of the Figs. 3(a) and 3(b) for the cases of $B_i=0.2$ and $H=10$, it can be found that the base flux $Q(t)$ for $G=2$ is a little larger than that for $G=5$; however, for the others cases, the $Q(t)$ for $G=2$ is a little smaller than that for $G=5$.

Figures 4(a) and 4(b) show the effect of geome-
try parameter $G$ and ratio of convective heat transfer coefficient $H$ on the heat flux ratio of the heat transfer from the fin surface to actual heat transfer from the fin to the surroundings ($RQT$). When strong tip convection is present ($H=10$), the increase in $RQT$ value with decreasing $Bi_a$ is dominant for a given $G$ value. It implies that the heat can readily flow to the fin tip and transfer to the surroundings under the conditions. Comparison of Figures 4(a) and 4(b) for given $Bi_a$ and $H$ values, as geometry parameter $G$ is small, the $RQT$ value is quite high and can not be negligible.

5. Conclusion

In this study an analytical solution of a two-dimensional pin fin with a time- and space-variation base temperature has been obtained by utilizing the Laplace transformation method and the Duhamel's method. The solution can be reduced to a special case for a 2D pin fin with a step change of base temperature; and then the result is shown to be the same as published solution obtained by the method of separation of variables.

The results show that the effects of 2D conduction are large, particular at large time. As lateral Biot number $Bi_a$ increases, the 2D effects become more pronounced. The early part of the transient heat flux at fin base $Q(t)$ is hardly affected by $Bi_a$ and the ratio of convective heat transfer coefficient $H$; as time increases, the $Q(t)$ decreases significantly. For a given geometry parameter $G$, as $Bi_a$ is large, the $Q(t)$ decreases slowly. For all cases analyzed in this paper, the $Q(t)$ gradually reaches a steady periodic response after $t=2$ of dimensionless time.

For small $Bi_a$ values, the effect of tip convection on the actual heat flux transferred from the fin to the surroundings is significant, particular for short pin fins (corresponding to small $G$ values).

References


