Unstructured Finite Element Method for Transient Heat Conduction of Moving Heat Source∗

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The primary objective of this study is to develop a space-time finite element formulation of heat transfer involving a moving heat source so that small time steps can be used in area of large time rates of change of temperature. The weighted residual process will be used to formulate a finite element method in a space and time domain based upon the continuous Galerkin method. A mesh refinement algorithm which will be on adaptively controlling the time step is developed and implemented for one-dimensional moving heat source simulation. A moving heat source will produce steep gradients of the temperature within and near the region of moving source. The space-time domain is divided into time-slabs and the mesh generator produces a triangular mesh that has small elements close to the front of moving source and relatively large elements away from the front. A series solution to the moving heat source problem derived will be used to compare to the numeric results obtained from the adaptive refinement technique developed in this study.

Key Words: Mesh Refinement, Heat Conduction, Galerkin, Space-Time, Finite Element, Time-Slab, Gradient, Moving Source

1. Introduction

Some classical articles on the general subject matter of moving heat sources in the heat conduction problems include Grosh, Trabant and Hawkins(1), Rosenthal(2), Rosenthal and Cameron(3). Most of them have presented basic analytical solutions of problems concerning the temperature rise near moving heat sources assumed to be points, lines, or planes. However, these studies were limited to the linear, steady-state case, i.e., the one in which the temperature field appears invariant to an observer moving along with the heat source, at the same speed. Atkinson(4) dealt with the two-dimensional problem of determining the temperature field when a rectangular rod at a constant temperature moves at constant speed in a medium with a temperature dependent thermal diffusivity using a singular perturbation method. This study was also limited to the steady-state case. Recently, Kuang and Atluri(5) applied a moving-mesh finite element method for a heat conduction problem with a heat source moving over the surface. They considered a rectangular heat source that moves at a constant velocity in a two-dimensional media. The effects of the temperature dependent material properties, and of the loss of heat to the surrounding medium through convection, are studied. Moving heat source will produce steep gradients of the temperature within and near the region of moving source. Therefore, a properly graded finite element mesh is required for a satisfactory resolution of the temperature field. Most of the current work(4), (8) – (10) in the area of finite element grid optimization is on the development of self-adaptive processors. The essential idea is to move the mesh either to minimize some quantity, such as the discretization error, or to follow some local non-uniformity, such as a shock waves in compressible flows and shear layers in laminar and turbulent flows. In the present study space and time variation could be treated uniformly by formulating space-time finite element methods(6), (11) – (14). The concept is to use the method of weighted residuals to treat space and time in a uniform manner and thus integrating both the spatial and temporal variations of the unknown quantities simultaneously. The focus will be on adaptively controlling the time step in a one-dimensional moving heat source simulation. In this way, we utilize the efficiency of finite elements by choosing a finite element mesh in the space-time domain where the finite element mesh has been adjusted to steep gradients of the solution with respect to both the space and time variables. An adaptive time-step

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control scheme of the finite element mesh for this problem will be developed.

2. Space and Time Domain

We shall now consider a domain for space and time as shown in Fig. 1. The definition of the partition of the time interval, \( I = [0, T] \) having the form \( t_0 = 0 < t_1 < t_2 < \cdots < t_N = T \). Let \( I_n = [t_n, t_{n+1}] \) and \( \Delta t_n = t_n - t_{n-1} \). In Fig. 1, we retain the definition of the open temporal partition of the time interval \( I \) given by

\[
I = \{ t : 0 < t < T \},
\]

and the space-time domain of such problems is the open spatial interval \( \Gamma \) described as follows:

\[
\Gamma = \{ x : 0 < x < L \}
\]

Then the space-time domain is the product space and time \( \Gamma \times I \) which is split up in equidistant space-time slab.

\[
\Gamma \times I = \{ (x, t) \in \Omega \times \Gamma : t > t_0 \}
\]

For the \( n \)-th slab, we define space-time domains as

\[
E_n^e = \Gamma_n \times I_n, \quad e = 1, 2, 3, \ldots, n_e.
\]

That is, a space-time slab is the first row corresponding to the interval \( 0 \leq t \leq t_1 \), the second \( t_1 \leq t \leq t_2 \), and so on. Let \( n_e \) denote the number of space-time elements and the spatial domain be subdivided into \( n_e \) elements, \( \Gamma_n^e, e = 1, 2, 3, \ldots, n_e \). Then, for the \( n \)-th space-time element the domain is

\[
E_n^e = \Gamma_n^e \times I_n, \quad e = 1, 2, 3, \ldots, n_e.
\]

Within each space-time element the trial solution and weighting functions are approximated by \( r \)-th order interpolation polynomials \( S_r \). These functions are assumed to be continuous within each space-time slab and across the interfaces of the slabs, namely at times \( t_1, t_2, \ldots, t_{N-1} \). For the first space-time slab an initial value \( u_0 \) on \( \Gamma_1 \) has to be given, but for each following space-time slab the solution at \( \Omega_3 \) of the next space-time slab \( \Gamma_1 \). For \( E_n^e \) we take the temperature field \( u(x, t) \) to be approximated by

\[
u^e(x, t) = \sum_{i=1}^{m} N_i(x, t) u_i
\]

where \( N_i(x, t) \) are the usual shape functions defined over the region \( E_n^e \), piecewise, element by element and \( u_i \) is the discrete nodal representation of the field \( u(x, t) \). \( i \) is a summation subscript and \( m \) is the number of nodes in an element. The weighting functions \( W_k(G_k^e) \) is also defined over the space \( G_k^e \).

3. Transient Heat Conduction Problem with a Heat Source

The one-dimensional heat conduction is governed by the linear partial differential equation given in Eq. (6). This equation describes the unsteady temperature in an isotropic body occupying a domain \( \Gamma \times T \).

\[
k \frac{\partial^2 u(x, t)}{\partial x^2} + q(x, t) = \rho c_p \frac{\partial u(x, t)}{\partial t} \quad \text{in} \quad \Gamma \times T
\]

with boundary and initial conditions as follows:

\[
u(0, t) = u_b(t) \quad \text{on} \quad \Omega_1
\]

or

\[
k \frac{\partial u(L, t)}{\partial x} = -q_0(t) \quad \text{on} \quad \Omega_2
\]

\[
u(x, 0) = u_0(x) \quad \text{on} \quad \Gamma(0), \quad x \in \Gamma
\]

where \( u \) is the field \( ^{\circ} \mathrm{C} \) for \( x \in \Gamma \) at time \( t \in [0, T] \) and \( \Gamma \) is a given time; \( k \) is the thermal conductivity of materials \( (\text{W/m} \cdot \text{\circ} \mathrm{C}) \); \( u_b(t) \) is the temperature field \( ^{\circ} \mathrm{C} \); \( q_0(t) \) is the prescribed heat flux \( (\text{W/m}^2) \); \( \rho \) is the density of material \( (\text{kg/m}^3) \); \( c_p \) is the specific heat of material \( (\text{kJ/kg} \cdot \text{\circ} \mathrm{C}) \); \( q \) is the heat generation rate \( (\text{W/m}^3) \); \( h_0 \) is the heat transfer coefficient \( (\text{W/m}^2 \cdot \text{\circ} \mathrm{C}) \); \( u_\infty \) is the temperature of the bulk fluid \( ^{\circ} \mathrm{C} \); and the initial condition \( u_0 \) is a given function of \( x \). \( \Omega_1 \) and \( \Omega_2 \) denote a non-overlapping subdivision of the boundary \( \Gamma \).

4. Galerkin Finite Element Formulation

On an element basis, an algebraic relation among the \( u_i \) can be obtained by the method weighted residuals. Using the weighted residual process in which the weighting function is equal to the shape function defining the approximation, the continuous Galerkin representation for Eq. (6) can now be placed in a weighted residual formulation for seeking a function \( u^e \).

\[
\int_{E^e} \left[ \frac{\partial}{\partial x} \left( k \frac{\partial u^e}{\partial x} \right) + q - \rho c_p \frac{\partial u^e}{\partial t} \right] W^e_k(x, t) dE^e = 0
\]

where \( E^e \) is the finite element domain for element \( e \), \( W^e_k(x, t) \) are the weight functions and \( m \) is the number of nodes in an element \( E^e \). The first term on the left-hand
side of Eq. (11) contains second order derivatives of the dependent variable \( u' \). Using the integration by parts on the first term in Eq. (11) with respect to space, the equation simplifies to:

\[
\int_{E'} \left[ \rho c_p u \frac{\partial W_i^e}{\partial t} - k_s \frac{\partial u}{\partial x} \frac{\partial W_i^e}{\partial x} \right] dE^e + \int_{E'} n_s k_s \frac{\partial u}{\partial x} W_i^e dI^e + \int_{E'} q W_i^e dE^e = 0 \quad i = 1, 2, 3, \ldots, m
\]

(12)

Note that the second integral in Eq. (12) carries information about boundary conditions. Evidently, this second integral does not contribute anything to the equation at the internal grid points. When a grid point lies on a boundary and if \( \partial u/\partial n \) is specified on that boundary, then the integral can be evaluated. The solution domain is discretized in the current problems using two-dimensional triangular finite elements with linear basis functions for a general region specified by a global coordinate system \((x, t)\). The unknown temperature is interpolated by the same functions, making this an isoparametric formulation.

\[
u'(x, t) = \frac{3}{3} \sum_{j=1}^{n} N_j(x) u_j(t)
\]

(13)

If the weighting functions \( W_i^h(x, t) \) are taken as the interpolation function \( N_j \), then Eq. (12) becomes:

\[
\sum_{j=1}^{n} \rho c_p \int_{E'} N_j \mu_j N_i dE^e - k_s \sum_{j=1}^{n} N_j \mu_j N_i \frac{\partial W_i}{\partial x} dE^e + \int_{E'} q_n N_i dI^e = \sum_{j=1}^{n} \int_{E'} q_n dE^e = 0 \quad i = 1, 2, 3
\]

(14)

Eq. (14) can be rewritten as

\[
[g]u + [f] = 0
\]

(15)

where

\[
[g] = \rho c_p \int_{E'} N_j \mu_j N_i dE^e - k_s \sum_{j=1}^{n} N_j \mu_j N_i \frac{\partial W_i}{\partial x} dE^e
\]

\[
[f] = \int_{E'} N_j h_0(u - u_{\infty}) dE^e
\]

The integration will be performed numerically in the \( \xi - \eta \) space. We need to assemble these individual element equations so that the values of \( u \) can be calculated at the end of the first time step. This continues so that during the \( n \)-th time step the values of \( u \) at \( u_{in} \) are known and \( u_{in} \) must be calculated. Define a global nodal temperature vector as

\[
[u] = [u_1, u_2, u_3, \ldots, u_n]^T
\]

(16)

\[
[Q] = [q_1, q_2, q_3, \ldots, q_n]^T
\]

(17)

The assembled equations at the \( n \)-th time level will have the matrix form

\[
[M_1][u]_n + [M_2][u]_{n+1} = [Q]_n
\]

(18)

The matrices \([M_1]\) and \([M_2]\) are tridiagonal. We may rewrite Eq. (18) in the final form as

\[
[S][u] = [Q]
\]

(19)

where

\[
[S] = [M_1] + [M_2]
\]

\[
[u] = [u_n, u_{n+1}]^T
\]

\[
[Q] = [q]^T
\]

\([S]\) is the assemblage matrix related to time level \( n \) and \( n + 1 \), and \([Q]\) is assemblage vector of nodal force. The matrices in the system show temporal coupling between times \( n \) and \( n + 1 \) with initial time \( n \).

5. Adaptive Finite Element Mesh Scheme

Meshes of triangular elements can be devised to suit a very wide range of practical problems. The first requirement of such a mesh is that it should fit the shape of the boundary of the solution domain as closely as possible. This requirement can be met provided the boundary shape can be approximated with sufficient accuracy by a series of short straight lines which form sides of elements. As time evolves, the mesh can move, change size, or change orientation. At each time-step, new meshes can be created and old ones vanish. A mesh change performed at every \( n \) time steps, depending on the solution gradients calculated. Figure 2 shows a mesh scheme for discretizing a time slab when it is anticipated that the locations of maximum gradients lie along a line \( AB \). This would be the case if an external heat source were moving along the rod with a constant velocity. Along such a principal line the mesh should be characterized by small elements in both space and time dimensions. The elements of the mesh shall be larger the further they are away from this line. It should noted that once having established this principal line, the
domain is divided into two subregions. The principal line is divided by \( n^1_s \) nodes into \( n^0_s - 1 \) sides of equal length \( l_0 \). Meshing is performed in the right portion first and then the left portion is done in the same manner. A line parallel to line \( AB \) is established at a distance \( l_0 \) to the left along the bottom of the time slab. This sliding line is divided by \( n^1_s \) nodes where
\[
n^1_s = n^0_s - N
\] (20)
with \( N \) equal to a small integer (usually in the range 2 to 4) so that the length of the sides of the elements along this line \( l_1 \) is larger than \( l_0 \) by a ratio
\[
\gamma = l_1/l_0 = n^0_s/(n^0_s - N)
\] (21)
This ratio used to expand the layers of elements that are parallel to the line \( AB \). This process continues with each layer being wider and containing fewer elements than the first layer until the sliding line intersects the left side of the time slab instead of the top. Considering the general case where there are \( i_x \) points per horizontal row and \( n^1_s \) points per sliding row, the total numbers of nodes and elements are \((n^0_s - i_x)(i_x + 1)\) and \(2\times i_x \times (n^0_s - i_x - 1)\) respectively. This leaves either a triangular or trapezoidal region that is filled with elements of approximately equal size. This general procedure is repeated in the region to the right of the line \( AB \) to complete the triangular mesh of elements for the time slab. As the parallel layers approach the ends of the time slab several special cases can be encountered. Special coding has been developed to avoid elements with unfavorable aspect ratios or angles that would cause ill-conditioning in the resulting element matrices. The aspect ratio is the ratio of the maximum dimension to the minimum dimension of an element. Elements with aspect ratios less than 3 to 1 are known to give good result. If the largest angle between sides to close to 180°, the numerical accuracy deteriorates.

To illustrate the ability of the mesh to adapt to a moving front, consider the case when the heat source moves with a velocity \( v = 0.15 \text{ m/sec} \). The space-time domain is divided into time slabs and the mesh generator produces a triangular mesh that has small elements close to the front and large elements away from the front. The space-time finite element mesh for the six large time slabs is shown in Fig. 3. This shows the mesh that was generated for time-slabs of \( \Delta t = 1.0 \text{ second} \).

6. Numerical Solutions and Discussions

The problem of temperature distribution in a solid due to a moving heat source is of relevance in several seemingly distinct areas, such as welding, surface hardening, and continuous casting. A finite length steel bar of uniform cross section is initially at a uniform temperature of 60°C. The bar is assumed to be thermally insulated around the circumference so that heat can only flow along the longitudinal direction specified by the \( x \) coordinate. The thermophysical properties used for computation are as follows: thermal conductivity \( (k = 144.5 \text{ cal/m·s·°C}) \); This continues so that during the \( n \)-th time step the values of \( u \) at \( u_{nx} \) are known and \( u_{n+1} \) must be calculated; density \( (\rho = 7.875 \text{ g/cm}^3) \); and specific heat \( (c_p = 0.103 \text{ cal/g·°C}) \).

\[
\frac{\partial u}{\partial x} = 0^\circ \text{C}, \ h_0 = 1.450 \text{ cal/m}^2\cdot\text{s}^\circ\text{C}, \ u_{nx} = 70^\circ \text{C}
\]

The bar is also assumed to have a heat source in a zone of length \( \Delta L = 0.02 \text{ m} \) of the bar which generates heat at a rate given by source \( (p_0 = 2.4 \times 10^8 \text{ cal/m}^3\cdot\text{sec}) \). This heat generated zone starts from the right end of the bar and begins to move toward the left at a constant axial velocity \( (v = 0.15 \text{ m/sec}) \) along the bar. The domain, boundary conditions and moving heat source are illustrated in Fig. 4.

Using unit step function \( \phi(x) \), we can express the heat source as
\[
q(x,t) = q_0(\phi(x-L+vt) - \phi(x-L+vt+\Delta t))
\] (22)
where \( q_0 \) is a strength of source and
\[
\phi(x) = \begin{cases} 
1 & \text{for } x > 0 \\
0 & \text{for } x < 0 
\end{cases}
\]
This is a linear problem for which there exists an analytical solution in series form. The series solution for the temperature $u(x,t)$ has been derived as follows:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{c_n} (u_0 - u_\infty) \frac{1}{\beta_n} \sin(\beta_n L) e^{-\alpha t} + \frac{2}{\beta_n} \frac{1}{\alpha^2 + \beta_n^2 v^2} \sin(\beta_n L + \Delta L) \cos(\beta_n v t + \beta_n \sqrt{\alpha^2 + \beta_n^2 v^2} t)$$

where $c_n = \frac{L}{2} + \frac{\sin(2\beta_n L)}{4\beta_n}$, $\tan(\beta_n t) = \frac{h_0}{k\beta_n}$, $\alpha = k \frac{\beta_n^2}{\rho c_p}$.

The solution of this problem is carried out with a starter time slab and then five time slabs with $\Delta t = 1.0$ second. The time-continuous Galerkin finite element produce temperature distribution at the end of each time slab. These results are plotted in Fig. 5. Also shown in this figure are temperature distributions calculated from the series solution. The numerical results from the time-continuous Galerkin finite element method generally agree with the series solution as shown in Fig. 5. At the top of a time slab there are twenty nodes. This indicates the level of spatial discretization. In comparison, the series solution required 200 terms to produce these results. Thus the level of discretization in the finite element method is significantly less. Because of the sharp rise of temperature in the

finite element method results indicate a fairly high level of oscillation. This oscillation is also evidenced in the oscillation in the region to the left of the front and the behind the front. The space-time finite element mesh and the solutions for the refined time-slab are given in Figs. 6 (a), (b), and 7 (a), (b) for the fifth time slab and sixth time slab, respectively. The results shown in Figs. 6 (b) and 7 (b) are interpolated on the time-slab magnifications of the equidistance levels. Figure 6 (b) shows numerical results on time slab between $t = 4.0$ sec and $t = 5.0$ sec. The distributions of temperature on time slab between $t = 5.0$ sec. and $t = 6.0$ sec. are plotted in Fig. 7 (b). In general, the results look very good. This is attributed the use of a finer mesh on the front of moving source where the steep gradients of the solution has been occurred. No effort has been made to compare these values with other computed or experimental results due to the significantly large spatial and temporal effect of gradients in the neighborhood of the moving heat source. These results suggest that even finer elements are needed to smooth out the oscillation of the finite element method. The point of these numerical results is to establish that the developed adaptive mesh generation technique will efficiently treat moving front problems that occur in many engineering situations.

Fig. 4 Model for analysis of the moving source of a finite bar

Fig. 5 Comparison of temperature distribution at $t = 2.0$ sec and $t = 3.0$ sec

Fig. 6 (a) The mesh on time-slab between $t = 4.0$ sec and $t = 5.0$ sec

Fig. 6 (b) Temperature distribution on time-slab between $t = 4.0$ sec and $t = 5.0$ sec

Fig. 7 (a) The mesh on time-slab between $t = 5.0$ sec and $t = 6.0$ sec

Fig. 7 (b) Temperature distribution on time-slab between $t = 5.0$ sec and $t = 6.0$ sec
7. Conclusions

A generalized space-time finite element weighted residual process using triangular elements has been applied to one-dimensional heat conduction of moving heat source. In this approach, the solution domain was divided into finite elements having space-time co-ordinates. A complete space-time finite element discretization is generated in which the need for any additional ordinary differential equation solver to resolve the temporal behavior of the problem is eliminated. An adaptive finite element scheme of an initial mesh on each time slab has been presented. In this method, we utilize the efficiency of finite elements by choosing a finite element mesh in the space and time domain where the finite element mesh has been adjusted to steep gradients of the solution both with respect to the space and the time variables. The gridding is employed in conjunction with a triangular finite element discretization in two dimensions. The grid is adapted at every time step, depending on the gradient computed. Numerical results have shown that the space-time Galerkin formulation is effective for localizing oscillations due to sharp gradients along with the adaptively control of mesh refinement.

References