Large Eddy Simulation of Homogeneous Isotropic Turbulent Flow Using the Finite Element Method\textsuperscript{*}

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This paper presents results of validation study of large eddy simulation (LES) that is applied to homogeneous isotropic turbulence in order to assess its spectral accuracy. The LES is performed by using a streamline-upwind finite element method with second order accuracy both in time and space and the results are compared with those from direct numerical simulation (DNS) based on the spectral method. The validation tests are done by using Standard Smagorinsky Model (SSM) and Dynamic Smagorinsky Model (DSM), and include following two cases: a low Reynolds number case and a higher Reynolds number case. In the former case, the Reynolds number is low enough that the computational grid is capable of resolving all the turbulence scales. In this case our interest is in whether any effects of the subgrid scale (SGS) model should appropriately be dampened out. In the latter case, a relatively large Reynolds number is selected where effects of turbulent eddies that are not resolved by the computational grid should be properly taken into account by the SGS model. It is confirmed that DSM performs better than SSM for both cases and it gives good agreement with DNS results in terms of both spatial spectra and decay of the turbulence statistics. Visualization of the computed flow fields by the DSM also reveals existence of distinct, coherent and tube-like vortical structures similar to those found in instantaneous flow fields computed by the DNS.

\textbf{Key Words:} Large Eddy Simulation, Validation, Homogeneous Turbulence, Finite Element Method, Smagorinsky Model, Dynamic Smagorinsky Model

\section{1. Introduction}

During the past decades, Large Eddy Simulation (LES) has been demonstrated to be a useful research tool for understanding the physics of turbulence as well as an accurate and sophisticated predictive method for flows of engineering interest. LES approach is conceptually intermediate between DNS (Direct Numerical Simulation) and RANS (Reynolds-Averaged Navier-Stokes) techniques. Although DNS is the exact approach to turbulence simulation, yet it is too expensive and is possible only for simple and low Reynolds number flows. Recent development of supercomputers has made it possible to carry out DNS of Navier-Stokes equations and explain the statistical properties and organized structures of turbulence\textsuperscript{(1)-(6)}, but the grid dependence is very high (proportional to $Re^{9/4}$) and calculation is fairly time consuming. Therefore, the DNS is not appropriate to the practical use at least for the near future. On the other hand, LES is less expensive and can simulate very complex flow fields in turbulence at a reasonable computational cost. Unlike the full-scale turbulence modeling of RANS technique, in LES method, large-scale motion is exactly calculated and only the effects of subgrid-scale (SGS) motions are modeled. The effects of SGS motions on the evolution of large scales are expected to be universal. Therefore, for modeling complex flow configurations often encountered in engineering applications, the use of LES is becoming increasingly common day by day as a more reliable prediction tool than RANS.
Recent important issues for LES are numerical method and SGS modeling. A literature review suggests that the numerical methods widely used for LES are either spectral method or the conventional finite difference method with structured grids\(^6\),\(^7\). However, the structured grids need to carry the fine resolution throughout the whole domain to obtain a reasonable accuracy because they do not use deformed grids. For the case of complicated flow and practical problems, use of structured grids method is often unsuitable. On the contrary, since Finite Element Method (FEM) is based on unstructured grids, this method seems very useful for engineering applications of LES to complicated flow fields.

The overall aim of our present research is to develop a general-purpose software “Front Flow” for fluid flow analysis based on LES. Front Flow numerical code is designed to perform a coupled simulation involving structural analysis and acoustical computation. This would consequently allow engineers to predict complex phenomena, which are not only of fluid flows but also of vibration, sound propagation, heat transfer, fatigue, and material production. It is also important to note that higher order numerical methods are not feasible for the engineering LES due to their complicated treatment at boundaries. The spectral accuracy is another important issue among many for sound prediction. For engineering applications of LES, the FEM with second order accuracy both in time and space seems most appropriate.

However, before its application to practical problems, it is necessary to examine the effectiveness and performance of the LES using FEM through benchmark problems. In this study, the validation tests of LES in homogeneous isotropic turbulence are done for two cases: a low Reynolds number case and a higher Reynolds number case. In the first case, the Reynolds number is low enough so that the computational grid is capable of resolving all the turbulence scales. The SGS models used for LES are the Standard Smagorinsky Model (SSM) and Dynamic Smagorinsky Model (DSM). The performance of our code is also tested using no SGS model (NMU). The LES results in all three cases are compared with DNS data, which are calculated by using the spectral method.

In the second case, a relatively large Reynolds number is selected where effects of turbulenteddies that are not resolved by the computational grid should be properly taken into account by the SGS model. In this case, only Dynamic Smagorinsky Model is used. In both cases, we also discuss instantaneous vortical structures in the computed flow fields with LES by comparing them with those in the DNS data.

### 2. Nomenclature

- \(A_{ij}\) : velocity gradient tensor
- \(A_{i}(k)\) : amplitude of the vector potential
- \(C_s\) : Smagorinsky coefficient
- \(E(k)\) : turbulent energy spectrum
- \(E^\delta\) : turbulent total energy
- \(F_{ui}\) : flatness of velocity components
- \(k\) : wave number
- \(N\) : number of grids
- \(n\) : time step
- \(\bar{p}\) : grid-scale static pressure
- \(Q\) : second invariant of the velocity gradient tensor
- \(Re\) : Reynolds number
- \(Re_1\) : Taylor microscale Reynolds number
- \(S_{ui}\) : skewness of velocity components
- \(t\) : non-dimensional time
- \(\bar{u}_i\) : grid-scale velocity components
- \(u_{\text{rms}}\) : root mean square (rms) of velocity fluctuations
- \(\Delta t\) : time increment
- \(W_{ij}\) : asymmetric part of the velocity gradient tensor
- \(\Delta\) : size of the grid filter
- \(\Lambda\) : size of the test filtered width
- \(\phi(k)\) : vector potential
- \(\nabla\) : gradient operator
- \(\Theta\) : enstrophy
- \(\delta_{ij}\) : Kronecker’s delta
- \(\epsilon\) : rate of turbulent dissipation
- \(\phi\) : random function
- \(\lambda\) : Taylor microscale
- \(\nu\) : kinematic viscosity
- \(\nu_{SGS}\) : subgrid-scale eddy viscosity
- \(\nu_m\) : molecular viscosity
- \(\pi\) : periodic length
- \(\rho\) : density
- \(\tau_{ij}\) : Reynolds stress tensor
- \(\ast\) : grid filtering operation
- \(^\wedge\) : test filtering operation
- \(\langle\rangle\) : averaging operation
- \(^\ast\) : represents real part of the Fourier coefficient
- \(^\ast\) : represents imaginary part of the Fourier coefficient

### 3. Numerical Method

#### 3.1 Governing equations and SGS model

By applying the grid filter to the dimensionless continuity and Navier-Stokes equations, the governing equations of LES for incompressible flow in the Cartesian coordinates are obtained as follows:

\[
\frac{\partial \overline{u}_i}{\partial x_i} = 0
\]  
\[
\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \overline{u}_i \overline{u}_j + \tau_{ij} \right) = \frac{1}{\rho} \frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j} \left\{ \nu \left( \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) \right\}
\]  

where, \(\overline{u}_i\) represents the grid scale velocity component, \(\overline{p}\)
is the grid-scale static pressure, $\rho$ is the density and $\nu$ is the kinematic viscosity of the flow.

In Eq. (2), subgrid-scale Reynolds stress tensor $\tau_{ij}$ is defined by

$$\tau_{ij} = \tilde{u}_i \tilde{u}_j - \tilde{u}_i \tilde{u}_j$$  \hspace{1cm} (3)

$\tau_{ij}$ is unresolved term and must be modeled. In order to take the effects of eddies that are not resolved by grid (subgrid-scale eddies) into account, Standard Smagorinsky Model (SSM)\(^{(9)}\) incorporating with the wall-damping function and Dynamic Smagorinsky Model (DSM)\(^{(9)}\) with modification by Lilly\(^{(10)}\) are used in this study. The performance of our numerical code is also tested using no SGS function and Dynamic Smagorinsky Model (DSM)\(^{(9)}\) incorporating with the wall-damping (subgrid-scale eddies) into account, Standard Smagorinsky closure applied to the SGS stress defined in Eq. (3), is given by

$$\nu_{SGS}=\frac{\nu S_{ij}}{(C_{s} \Delta)^{2}(2\bar{S}_{ij})^{0.5}}$$  \hspace{1cm} (4)

and $\delta_{ij}$ is $1$ if $i=j$ and zero otherwise. The quantity $C_{s}$ is the Smagorinsky coefficient and $\Delta$ is the size of the grid filter. Since we are dealing with homogeneous isotropic turbulence, the size of the grid filter $\Delta$ is identical in each direction and it is selected as the grid interval.

In the case of SSM, we use a constant value $C_{s}=0.2$, which is commonly used in homogeneous isotropic turbulence for obtaining good results by many researchers\(^{(11)}\). For the case of DSM, the coefficient $C_{s}$ is determined locally in time and space by computing the relations given as:

$$C_{s}^{2} = \frac{1}{2\Delta^{2}} \frac{\left( M_{ij} l_{ij} \right)}{\left( M_{ij} \right)}$$  \hspace{1cm} (5)

where; $M_{ij} = \frac{1}{2} \left( \frac{\partial M_{i}}{\partial x_{j}} + \frac{\partial M_{j}}{\partial x_{i}} \right)$

$$l_{ij} = \left( \tilde{u}_{i} \tilde{u}_{j} \right) - \alpha^{2} \left( \bar{S}_{ij} \right)$$  \hspace{1cm} (6)

in which the symbol $\langle \rangle$ represents a test-filtering operation, $\langle \rangle$ represents an averaging operation and $\Delta$ is the size of the test filtered width.

The test filtering operation of an arbitrary function is given as follows:

$$\tilde{\phi} = \phi + \frac{\tilde{\alpha}^{2}}{24} \frac{\partial^{2} \phi}{\partial x_{i} \partial x_{j}} + o(\tilde{\Delta}^{4})$$  \hspace{1cm} (7)

The test filter width, $\tilde{\Delta}$ is determined by using the value of $\alpha$ that appears in Eq. (7) and by the relation given in Eq. (10) as follows:

$$\tilde{\Delta} = 2\gamma \Delta; \hspace{1cm} 1 + \gamma^{2} = \alpha^{2}$$  \hspace{1cm} (8)

Since $\tilde{\phi}$ is a Laplacian operation, the test-filtered variable can be obtained by the bi-linear operation used in the finite element method. Therefore, in the test filtering operation the following quantities; $\tilde{u}_i$, $\tilde{u}_j$, $\tilde{S}_{ij}$ are interpolated by the bi-linear function. In averaging operation that appears in Eq. (6), the same operation as for the test filtering operation is used.

### 3.2 Numerical scheme

The Front Flow numerical code used in this study is a general-purpose fluid simulation code that calculates incompressible unsteady flows in arbitrary shaped geometry that involves moving (but not deforming) boundaries. It is designed primarily for computing unsteady flows in turbomachinery and simulating sound pressure spectra that result from unsteady fluid motion. In order to obtain accurate sound spectra in general, it is of up most importance to simulate the source, i.e., the fluid fluctuations accurately in terms of their spatial and frequency spectrum.

This code is based on an explicit streamline-upwind finite element method with the second-order accuracy both in time and space. In this numerical scheme, the spatial discretization is performed by hexahedral mesh and the coordinate system is the three dimensional Cartesian. The present FEM code assumes that the solution at $t=n\Delta t$ is known, where $\Delta t$ is the time step increment and $n$ is the time step, and the solution at the next time step, $t=(n+1)\Delta t$, can be calculated using the residual terms described in the upward finite element formulation\(^{(12),(13)}\).

The pressure algorithm is based on the ABMAC (Arbitrary Boundary Marker and Cell) method proposed by Viecelli\(^{(14)}\), where both the velocity components and static pressure are simultaneously corrected until the maximum divergence of the flow field decreases to less than a prescribed critical value. The details of the numerical methods of this code have been given by Kato & Ikegawa\(^{(12)}\) and Kato et al.\(^{(13)}\).

### 3.3 Test cases

We present the following two cases of LES for homogeneous isotropic turbulence.

In the first case (referred to as “Low Reynolds number case”), the Reynolds number is low enough so that the computational grid is capable of resolving all the turbulence scales. For this case three types of SGS model including the “no model case” (denoted by “NMU”) are tested on a uniformly distributed 64\(^{3}\) numerical grids. The NMU case should be able to serve as a reference that clarifies the accuracy of the present numerical scheme itself while in the other two cases (SSM and DSM) our interest lies in whether any effects of the subgrid scale (SGS) model are appropriately damped out.

In the second case (referred to as “Higher Reynolds number case”), LES is performed for a relatively large Reynolds number on three types of numerical grids with different number of nodes of 32\(^{3}\), 64\(^{3}\) and 128\(^{3}\). In this case only DSM is tested because the first case has revealed that DSM gives the most accurate results among the three
models tested. In this case 128³ numerical grid is expected to resolve the smallest turbulence scale while in the other two grids (64² and 32² grids) the effects of unresolved turbulent eddies must be properly modeled and our interest is whether the SGS model is capable of absorbing an appropriate amount of turbulent energy at the grid scale with least possible effects on the resolved energy spectra.

For both cases the computed results are compared with the DNS data based on a spectral method(3),(4) and the typical vortical structures found in instantaneous flow fields are also discussed. The numerical parameters for these test cases are summarized in Table 1.

### 3.4 Initial conditions

In this study, the initial flow field for LES is calculated with the same condition and procedure that are done for DNS calculation in homogeneous isotropic turbulence based on the spectral method by Tanahashi et al.(3),(4). In this procedure the initial flow condition is assumed to have the decay of energy spectrum. We assume φ(k) is a vector potential and then the initial velocity field is determined (in Fourier space) by

\[ \hat{u}_i(k) = \nabla \times \hat{\phi}_i(k); \text{ where, } \hat{\phi}_i(k) = A_i(k) \exp[i\Theta_i(k)] \]  

(11)

\[ A_i(k) \text{ is the amplitude of the vector potential and } \Theta_i(k) \text{ is its random phase. } \Theta_i(k) \text{ is determined by a uniform random number within the range } [0,2\pi]. \]

The amplitude function \( A_i(k) \) is determined so that the following relation is satisfied:

\[ E_{ini}(k) = \sum_{|k| - 1/2 < |k| + 1/2} \nabla \times \hat{\phi}_i(k) = \sum_{|k| - 1/2 < |k| + 1/2} \nabla \times A_i(k) \]  

(12)

Here we should note that the energy does not depend on \( \Theta_i(k) \) since \( \Theta_i(k) \) is a phase. The above operation (Eq. (12)) corresponds to determining the energy distribution within a spherical shell, whose radius is \(|k|\), in the wave-number space. However, in the current study we uniformly distribute the energy as \( \frac{E_{ini}(k)}{N_{w,k}} \), where \( N_{w,k} \) is the number of the wave number vectors whose length is within the range of \(|k| - 1/2, |k| + 1/2\). Here the actual velocity condition for homogeneous turbulence in Fourier space

\[ \hat{u}_i(k) = \hat{u}_i \ast (-k) \]  

(13)

should be fulfilled. Using this procedure we have calculated the initial velocity for DNS in which the decay of initial energy is fully monitored. In addition, to obtain a reliable result for DNS and fully resolved LES in homogeneous isotropic turbulence, we are concerned about two necessary conditions such as; (i) grid spacing is less than three times of Kolmogorov microscale, and (ii) the necessary conditions for the integral length scale, Taylor microscale and Kolmogorov microscale in homogeneous isotropic turbulence (see Hinze 1975, p.225(15), the maximum possible Reynolds number (Re) of the flow is determined.

### 3.5 Reference DNS

The reference DNS is performed at 64³ and 128³ resolutions by using a spectral code(3),(4). At the end of calculations, the Taylor microscale Reynolds numbers, \( Re_t \), based on \( u_{rms} \) and Taylor microscale (l) of the DNS data are 30.5 (\( t = 3.792 \)) and 59.8 (\( t = 2.55 \)), and the maximum possible Reynolds numbers (Re) of the flow are 121.1 and 304.9, respectively.

### 4. Computed Results

In this section we have compared the LES results with the DNS results in homogeneous isotropic turbulence for detailed clarification of the purposes given in section 3.3. The comparisons of the results for the low and the high Reynolds number cases are discussed.

#### 4.1 Low Reynolds number case (Case 1)

4.1.1 Turbulent energy spectra

Three-dimensional energy spectra in the DNS and LES data at the end of calculation (\( t = 3.792 \)) are presented in Fig. 1. The energy spectrum is calculated by the definition given as:

\[ E(k) = \sum_{k = -\frac{1}{2} \leq |k| \leq \frac{1}{2}} \frac{1}{2} \hat{u}_i(k) \hat{u}_i^*(k) \]  

(14)

In Fig. 1, we can observe that the DNS spectrum shows the power decay close to \( k^{-5/3} \). The energy spectrum in LES calculation with DSM model shows almost identical decay as in the DNS calculation in the whole wave number range except for the highest wave number. On the
other hand, SSM results underestimate the DNS data and DSM results in the high wave number range. The LES results for all three cases agree well with the DNS data in the low wave number range. The maximum difference between the results of DNS and DSM in the low wave number range \((k \leq N/4)\) lies between 0.5% - 6%, and it lies between 7% - 12% in the high wave number range \((k \geq N/4)\), while that between DNS and SSM results in the low wave number range lies between 3% - 30% and it is 31% - 45% in the high wave number range, which quantitatively indicates the superiority of DSM to SSM. To the contrary, the energy spectrum obtained by NMU (No Model Case) shows over-prediction in the high wave number range with the energy spectrum obtained by NMU (No Model Case) indicates the superiority of DSM to SSM. To the contrary, the energy spectrum obtained by NMU (No Model Case) shows over-prediction in the high wave number range with a difference from DNS of 30% – 60% in this wave number range. This is solely attributed to the numerical method because any effect from SGS model is killed off. This may be caused by dispersion error associated with the current numerical scheme. However, the comparison of NMM results with the DNS data suggests that the present FEM numerical code can produce fully resolved scale turbulence even if we kill the SGS Reynolds stress term from the filtered Navier-Stokes equations.

We use different numerical methods for DNS and LES calculation but the DSM results suggest that the decay of turbulence in LES follows the \(k^{-5/3}\) power law as in DNS and the numerical accuracy is quite good. Clearly it reveals that the DSM works better than SSM for this FEM formulation. The performance of DSM is also good because it is not affected by the dispersion error as we assume with the current numerical method. It seems that the LES results for all the cases turn up a little at the largest wave number \(((N-2)/2 \leq k \leq N/2)\) compared to the DNS data. The exact reason is unclear but again it may happen due to dispersion error associated with the present second order FEM scheme. However, at this low Reynolds number case, it seems that the spectral accuracy of LES with DSM is as good as DNS.

### 4.1.2 Turbulence statistics

In this subsection we discuss some statistics in homogeneous turbulence. The decay of the total resolved energy, \(E^k\) is presented in Fig. 2, where

\[
E^k = \frac{1}{2} \langle |u|^2 \rangle
\]

The decay of resolved enstrophy is presented in Fig. 3, where enstrophy is defined as

\[
\Omega = \frac{1}{2} \langle |\omega|^2 \rangle
\]  

(16)

The SSM result differs significantly from the DNS data throughout the time period. The SSM result is too dissipative in the initial stage and also shows higher dissipation from DNS data throughout the analysis. The DSM and NMU results are in good agreement with the DNS data. Although DSM result slightly underestimates the DNS data in the middle stage of the calculation, yet it coincides exactly with the DNS data at the final stage of the calculation. The difference among DNS and LES results for SSM and NMU cases is about 16% in the final stage of the computation. These results again suggest that all NMU, SSM, and DSM cases can produce fully resolved scale turbulence in which SSM result is the least accurate and DSM result is the most accurate for this finite element LES.

The decay of root mean square of velocity fluctuations \(U_{rms}\) is presented in Fig. 4. The decay of \(U_{rms}\) for DSM case fully collapses with the DNS data in the whole analysis. Initially the decay of SSM results is a little faster but it is in good agreement with the DNS data after time \(t = 2.5\). While the NMU result collapses with the DNS result until \(t = 2.0\), thereafter it shows a little overestimation.
from the DNS. Finally, at the end of the computation, the \( u_{\text{rms}} \) values reach to 0.515, 0.507 and 0.518 for the case of DSM, SSM and NMU, respectively, and those values are very close to 0.501 given by the DNS data.

Skewness and flatness factors of velocity are important statistical properties that represent characteristics of turbulence. The production of the rate of dissipation of turbulent kinetic energy, or equivalently, the production of enstrophy is directly related to skewness in isotropic turbulence\(^{16}\). Skewness and flatness of a velocity component, \( u_i \) defined by the following equations are presented in Figs. 5 and 6, respectively.

\[
S_{u_i} = \frac{\left< u_i^3 \right>}{\left< |u_i|^3 \right>^{\frac{1}{2}}} ; \quad F_{u_i} = \frac{\left< u_i^4 \right>}{\left< |u_i|^2 \right>^2} \quad \text{(17)}
\]

The agreement of skewness (in Fig. 5) of LES data for all three cases with the DNS data is reasonably good until \( t = 2.0 \) and it overestimates the DNS data thereafter. In the final stage of the analysis the LES results for all three cases show 30% - 40% difference with the DNS result and the difference between DSM and SSM is about 17% at the end of the computation. However, skewness for all the cases is almost zero at \( t = 0 \) and the LES results agree well with the DNS data. On the other hand, flatness (in Fig. 6) of LES results collapses well with the DNS data throughout the analysis. At time \( t = 0 \), flatness of DNS and LES for all the three cases is about 3.0 and shows nearly the same value at the end of the calculation. The behavior of flatness suggests that turbulence velocity at the end of the calculation does not include the effects of initial condition and reaches to a fully developed state. The developments of skewness and flatness do not show any superiority among NMU, SSM and DSM results, which again suggests that our finite element flow solver can produce fully resolved scale turbulence such as in the spectral DNS.

4.1.3 Instantaneous distributions of \( C_s \) and SGS viscosity In this subsection the instantaneous distributions of the Smagorinsky coefficient, \( C_s \) computed by DSM will be discussed. The effects of SGS eddy viscosity in LES are also investigated by comparing with that computed by SSM. The contours of \( C_s \) at the end of calculation (\( t = 3.792 \)) on the \( (y,z) \)-plane at \( x = \pi \) are shown in Fig. 7. It reveals that the maximum value of \( C_s \) reaches to 0.2, but the appearance of those regions with \( C_s = 0.2 \) is localized in the whole computed flow field and \( C_s \) takes a value in the range 0.05 \( \leq C_s \leq 0.15 \) in the most regions. We also notice that in some regions \( C_s \) takes zero value.

The contours of instantaneous subgrid-scale eddy viscosity, \( \nu_{SGS} \) on the same plane as given in Fig. 7 are presented in Fig. 8 both for SSM and DSM. The SGS eddy viscosity is normalized by the molecular viscosity, \( \nu_m \) of the flow. Although the maximum value of normalized \( \nu_{SGS} \) reaches to 0.25 for DSM, it reaches to 0.46 for SSM. The value of \( \nu_{SGS} \) with DSM tends to become zero in the most regions. On the other hand, the contribution of \( \nu_{SGS} \) is very strong for SSM. We can clearly observe that the effect of SGS model is appropriately dampened out in the resolved scale turbulence for DSM.
Visualization of the computed instantaneous flow fields

In this subsection, by visualization of flows, we discuss the vortical structures computed by the LES and compare them with those computed by the DNS. There are several methods for identifying of vortical structures in turbulence, which are essentially different with each other\(^{17}\) and most of them show threshold dependence. By using DNS database, we have identified the coherent fine scale eddies and their axes without using any threshold in homogeneous isotropic turbulence\(^{4},\ 5\). In this study, we use the same method known as ‘local flow pattern method’ to visualize and discuss vortical structures in the LES data. Detail of this method is given in the previous studies\(^{4},\ 5\).

Figure 9 shows the contour surfaces of the positive second invariant of the velocity gradient tensor in the DNS and LES data for NMU, SSM and DSM cases. Of course, the visualized coherent structures depend on the threshold value of \(Q\). However, if we increase or decrease the value of \(Q\), we can also show that essentially the same tube-like structures exist in the DNS and LES data with somewhat different visualized images. In our previous study by direct filtering of DNS database\(^{18}\), we have seen that the tube-like vortical structures, which are different in sizes, exist in the grid-scale and subgrid-scale velocity fields. Our present study reveals that in the LES data we can have the coherent tube-like structures and these structures are quite unique and distinct as we can see in Fig. 9. Appearance of the vortical structures using DSM is noticeably higher than that of SSM and is close to DNS data, which again suggests that the accuracy of LES calculation with DSM is higher than that with SSM. It also reveals that the visualized structures in NMU case seem to be similar to the DNS data as well as LES data for DSM case. These observations prove that the present finite element numerical code can produce resolved scale turbulence by using SSM and DSM model and even without using any SGS model (NMU case).

4.2 A higher Reynolds number case (Case 2)

In the first test case we have demonstrated that the DSM (Dynamic Smagorinsky Model) gives the most accurate results among the three models tested for the low Reynolds number case. Therefore, in this section we discuss the results for a higher Reynolds number case, computed by DSM with different grid resolutions. The emphasis is placed on whether the SGS model is capable of absorbing an appropriate amount of turbulent energy at the grid scale with least possible effects on the resolved energy spectra.

4.2.1 Turbulent energy spectra

Figure 10 shows a comparison of the three-dimensional energy spectra of velocity fluctuations in the DNS and LES data at the end of the calculation \((t = 3.792)\). The second invariant is defined as:

\[
Q = -\frac{1}{2} (S_{ij}S_{ij} - W_{ij}W_{ij})
\]

(18)

where, \(S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)\); \(W_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)\)

(19)

are the symmetric and asymmetric part of the velocity gradient tensor,

\[
A_{ij} = \frac{\partial u_i}{\partial x_j} - S_{ij} + W_{ij}
\]

(20)

In Fig. 9, the visualized region is the whole calculation domain and the view point is same for all cases with a level of isosurface selected to \(Q = 10\). These figures show that a number of tube-like vortical structures are randomly oriented in the DNS data as well as in the LES data for NMU, SSM and DSM cases. Of course, the visualized coherent structures depend on the threshold value of \(Q\). However, if we increase or decrease the value of \(Q\), we can also show that essentially the same tube-like structures exist in the DNS and LES data with somewhat different visualized images. In our previous study by direct filtering of DNS database\(^{18}\), we have seen that the tube-like vortical structures, which are different in sizes, exist in the grid-scale and subgrid-scale velocity fields. Our present study reveals that in the LES data we can have the coherent tube-like structures and these structures are quite unique and distinct as we can see in Fig. 9. Appearance of the vortical structures using DSM is noticeably higher than that of SSM and is close to DNS data, which again suggests that the accuracy of LES calculation with DSM is higher than that with SSM. It also reveals that the visualized structures in NMU case seem to be similar to the DNS data as well as LES data for DSM case. These observations prove that the present finite element numerical code can produce resolved scale turbulence by using SSM and DSM model and even without using any SGS model (NMU case).
the same definition as in Fig. 1. In this figure, the DNS spectrum with $128^3$ grids is also plotted, which shows the power decay close to $k^{-5/3}$. These profiles of energy spectra exhibit that the maximum difference between the results of DNS and LES with $128^3$ grids in the low wave number range ($k \leq N/4$) lies between 0.5%–10%, and it is approximately 20%–30% in the high wave number range ($k \geq N/4$). Although the rate of decay is a little faster than DNS in the largest wave numbers, the energy spectrum of LES with $128^3$ fine grids (fully resolved case), almost collapses with the DNS data in the most part of the wave number range. For the $64^3$ grids, the decay of turbulence energy of the LES data also agrees with the DNS data. However, the energy spectrum with $32^3$ coarse grids substantially differs from the DNS and LES data for the other two cases. In this case, the decay of turbulence energy over-predicts the DNS data in the most part of the wave number range. These results suggest that the SGS model is capable of absorbing an appropriate amount of turbulent energy with the $64^3$ grids case with least possible effects on the resolved energy spectra.

4.2.2 Turbulence statistics In this subsection we discuss statistics of turbulence computed by the fine
and coarse grids. Figures 11 and 12 show the decay of the total resolved energy and enstrophy, respectively, in DNS and LES with the fine and coarse grids. It reveals that for LES with 128^3 grids, the total energy coincides with the DNS data throughout the analysis (in Fig. 11) and the enstrophy is slightly underestimated compared with the DNS data (in Fig. 12). Although the total energy of LES with 64^3 grids overestimates the DNS data as well as the LES data with 128^3 grids, the enstrophy collapses with the LES data with the fine grids at the final stage of the analysis. The LES with 64^3 grids is a little dissipative in the initial stage but the agreement with the fine grid turbulence is good at the end of the computation (in Fig. 12). The enstrophy with 32^3 grids, i.e., with the coarsest grids is highly underestimated the DNS and LES results for the other two cases throughout the analysis. To the contrary, the distributions of energy with 32^3 grids computation show higher than that of DNS (in Fig. 11), which suggests that the energy cascade is not appropriately reproduced in LES with 32^3 grids computations.

The decay of root mean square of velocity fluctuations (u_{rms}) is presented in Fig. 13. In this case, the decay of u_{rms} of LES with 128^3 grids collapses with the DNS data throughout the analysis and LES results with 64^3 grids also well compare with the DNS data. However, the LES results with 32^3 grids overestimate the DNS results significantly.

Skewness and flatness of a velocity component are presented in Figs. 14 and 15, respectively. The agreement with DNS of these statistical quantities of LES data with 128^3 grids seems good, and the behaviors are as the same as in the low Reynolds number case (Case 1). The LES results with 64^3 coarse grids also provide a reasonably good agreement with the DNS data. However, the results with 32^3 coarse grids are significantly different from DNS and LES in the other two cases. It is shown that the flatness of DNS and LES with 128^3 and 64^3 grids is almost 3 at the initial stage and at the end of calculation it collapses each other and reaches about a constant value (in Fig. 15). But the flatness for LES with 32^3 grids differs from the DNS and LES for the other two cases significantly. In summary the performance of our finite element fluid solver is not good for the coarsest grid simulation (32^3 grids case), but the agreement of LES results with 128^3 and 64^3 grids with the DNS data is good.

4.2.3 Instantaneous distribution of $C_s$ and SGS viscosity The instantaneous distributions of the Smagorinsky coefficient, $C_s$ and the effects of SGS viscosity in the resolved and grid scale turbulence are presented in this subsection. The distributions of $C_s$ at the end of the calculation ($t=2.55$) on the $(y,z)$-plane at $x=\pi$ with 128^3, 64^3 and 32^3 grids are presented by contour plots in Fig. 16. It reveals that the maximum value of $C_s$ for all three cases reaches to 0.2. However, the appearance of $C_s=0.2$ is not
Contours of instantaneous subgrid-scale eddy viscosity, $\nu_{\text{SGS}}$, on the same plane as given in Fig. 16, are presented in Fig. 17. The SGS eddy viscosity is normalized by the molecular viscosity, $\nu_m$, of the flow. The maximum value of the normalized $\nu_{\text{SGS}}$ reaches to 0.36 for $128^3$ grids, 1.33 for $64^3$ grids, and 3.99 for $32^3$ grids. However, the contribution of $\nu_{\text{SGS}}$ for $128^3$ grids is very weak and tends to become zero in the whole computed flow field. On the other hand, the contribution of $\nu_{\text{SGS}}$ for $32^3$ coarse grids is very strong. Similarly to the low Reynolds number case, the effects of SGS model (DSM) seems to be dampened out with the fine grids computation ($128^3$ grids). Comparing the results of $\nu_{\text{SGS}}$ with $128^3$ and $64^3$ grids within the range, $0.0 \leq \nu_{\text{SGS}}/\nu_m \leq 1.25$, as shown in Fig. 17, it can be observed that the effects of SGS model with $64^3$ grids also tends to be dampened out.

4.2.4 Visualization of instantaneous flows

The contour surfaces of the second invariant, $Q$, of the velocity gradient tensor at the end of computation ($t = 2.55$) in the DNS and LES data with the fine and coarse grids are presented in Fig. 18. The visualized region is the whole calculation domain for all cases. The view point of this figure is same and the level of the isosurface is 30 for all cases. A number of tube like vortical structures appear in the DNS and LES data with $128^3$ and $64^3$ grids case. However, the appearance of the visualized structures with $32^3$ grids case is very sparse. Moreover, the shape of the structures is not tube-like one and differs from those computed by the DNS, which again suggests that $32^3$ coarse grids LES does not perform well. In $64^3$ grids case, the appearance of the vortical structures decreases from the DNS. However, the shape of the structures is still tube-like one and coherent. Although the visualized structures in the DNS and LES data with $128^3$ grids are not exactly similar, we can obviously observe that the performance of LES with $128^3$ grids is good. According to the classical idea of fluid dynamics, it is known that several small scale structures compose a large scale structure. That is to say that a large scale structure contains several small scale structures in which a smaller one lies entirely inside the larger one. The tube-like coherent fine scale structures have been observed mainly in the DNS(2),(3). However, our present study demonstrates that the tube-like coherent structure can also be observed in LES with fine grids. It is also clear that even with coarse grids, we can observe the tube-like coherent structure in LES, and all of the structures with fine and coarse grids are quite distinct and unique. This result is very worthy for further research on the coherent structure in turbulence.

5. Conclusions

The validation test of large eddy simulation that is applied to homogeneous isotropic turbulence has been
Fig. 17 Instantaneous distributions of subgrid-scale (SGS) eddy viscosity at $t = 2.55$ with fine and coarse grids computed by LES with DSM. The values are normalized by the molecular viscosity (Case 2).

Fig. 18 (to be continued)
done in order to assess its spectral accuracy. The LES is performed by using a streamline-upwind finite element method with second order accuracy both in time and space. The performance of LES is tested for two cases: a low Reynolds number case and a higher Reynolds number case, and the computed results for both cases are compared with those from DNS based on a spectral method.

In the first case (the low Reynolds number case), the Reynolds number is low enough that the computational grid is capable of resolving all the turbulence scales. In this case, Standard Smagorinsky Model (SSM) and Dynamic Smagorinsky Model (DSM) are tested on a uniformly distributed $64^3$ numerical grids and the results are compared. The comparisons have also been made with the results obtained by using No SGS model (NMU case) on the same grid. The LES results at this Reynolds number suggest that our present numerical scheme can produce fully resolved scale turbulence for all the three cases. In this computation, the NMU case served as the reference that revealed the accuracy of the present numerical scheme itself. The DSM results suggest that the effects of the subgrid scale (SGS) model are appropriately dampened out in the computed flow field. It is also confirmed that DSM performs better than SSM and it gives a good agreement with DNS results in terms of both spatial spectra and decay of the turbulence statistics.

In the second case (the Higher Reynolds number case), LES is performed for a relatively large Reynolds number on three numerical grids with different number of nodes of $32^3$, $64^3$ and $128^3$. In this case only DSM is tested because in the first test case, it is revealed that the DSM gives the most accurate results among the three models tested. In this computation, it is also shown that the LES result with $128^3$ grids gives a good agreement with DNS result in terms of both spatial spectra and decay of the turbulence statistics. The decay of energy spectrum as well as the decay of turbulence statistics with $64^3$ coarse grids is very close to the LES results with $128^3$ fine grids. However, for $32^3$ coarse grids computation they are different from both $128^3$ fine grids and $64^3$ coarse grids computations. These results suggest that the SGS model is capable of absorbing an appropriate amount of turbulent energy at the grid scale ($64^3$ coarse grids case) with least possible effects on the resolved energy spectra.

Visualization of the second invariant of velocity gradient tensor, $Q$ in the computed flow fields by LES reveals the existence of distinct, coherent and tube-like vortical structures similar to those found in instantaneous flow fields computed by the DNS. In the fully resolved LES cases, the vortical structures computed by DSM are finer than those by SSM and are very close to the DNS result. Moreover, the tube-like coherent structure appears even in the coarse grid LES.

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