Laminar Free Convection in Vertical Concentric Annular Ducts*

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A theoretical investigation is presented for free convection in vertical concentric annular ducts with uniform temperatures specified on inner and outer walls, taking account of the temperature dependence of the thermophysical properties of fluid. The perimeter average Nusselt numbers and dimensionless average axial velocities in the ducts of various cross sections are well correlated by using $\xi$ as a characteristic length; and the Nusselt numbers of the annular ducts show very good accordance with those of parallel plates rather than those of a circular duct. A general representation of the average Nusselt numbers is presented for wide ranges of the radius ratio and Prandtl number. When the fluid physical properties are evaluated at the average temperature of the fully developed flow, the variable property solutions correlate with the constant property solutions.

**Key Words**: Convective Heat Transfer, Free Convection, Laminar Flow, Annular Duct, Uniform Wall Temperature, Variable Thermophysical Properties

1. Introduction

Free convective heat transfer has recently gained much attention due to interest in the compact cooling of electronic equipment$^{(1),(2)}$, and in the emergency cooling of nuclear reactors$^{(3)}$. Free convection in vertical ducts is one of the fundamental problems for heat transfer equipment, and broad reviews on the field have been made by Aihara$^{(4),(5)}$ and Fuji$^{(6)}$. The authors have carried out numerical analyses on the free convection in a vertical circular tube and parallel plates, taking into account the temperature dependence of all physical properties of fluid$^{(7),(8)}$. A concentric annular duct is an important geometrical configuration for many fluid flow and heat transfer devices; and many studies have been carried out on forced convection in the annular ducts$^{(9)}$. For free convection in vertical annular ducts with open ends, however, there appear to be very few studies such as those on annular ducts of uniform heat flux by Barrow et al.$^{(10)}$

This paper describes a theoretical investigation of free convection in vertical concentric annular ducts with uniform temperatures specified on inner and outer walls, taking account of the temperature dependence of physical properties of fluid. A general representation of the average Nusselt numbers is presented under various temperature conditions and with radius ratios, which is derived from the asymptotic solutions of the duct flow.

2. Nomenclature

$c_p$ : specific heat at constant pressure  
$d_a$ : hydraulic diameter  
$G$ : dimensionless mass flow rate, Eq. (7)  
$Gr$ : Grashof number, $g\beta(T_w - T_o)r^3/\nu^2$  
$h$ : height of duct  
$h$ : perimeter average heat-transfer coefficient,  
$(r^*h_1 + h_2)/(1 + r^*)$  
$h_1, h_2$ : average heat-transfer coefficients on the inner and outer walls, respectively  
$\bar{N}u$ : perimeter average Nusselt number, Eq.(12)  
$\bar{N}u_1, \bar{N}u_2$ : average Nusselt numbers on the inner and outer walls, respectively, Eq.(11)
\[ \frac{\partial (\rho^* U)}{\partial X} + \frac{1}{R} \frac{\partial}{\partial R} (R \rho^* V) = 0 \]
\[ U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial R} = \frac{1}{\rho^* R} \frac{\partial}{\partial R} (R \rho^* \frac{\partial U}{\partial R}) \]
\[ -\frac{1}{\rho^*} \frac{dP}{dx} + \beta^* \theta, \]
\[ U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial R} = \frac{1}{\rho \rho^* c_p^* R} \frac{\partial}{\partial R} (R \rho^* \frac{\partial \theta}{\partial R}). \]

where \( T_w \) is the higher value of temperatures on the inner and outer walls. Dimensionless wall temperatures \( \theta_1 \) and \( \theta_2 \) and an inner radius ratio \( r^* \) are defined as
\[ \theta_1 = (T_{w1} - T_0)/(T_w - T_0), \]
\[ \theta_2 = (T_{w2} - T_0)/(T_w - T_0) \]
\[ r^* = r_1/r_2 \]

Then the boundary conditions for the present problem are as follows:
\[ \text{For } L \geq X \geq 0, R = r^* : U = 0, V = 0, \theta = \theta_1 \]
\[ \text{For } L \geq X \geq 0, R = 1 : U = 0, V = 0, \theta = \theta_2 \]
\[ \text{For } X = 0, 1 > R > r^* : U = U_0, V = 0, \theta = 0, \]
\[ P = -U \delta/2 \]
\[ \text{For } X = L : P = 0 \]

Subscripts
\[ 0 : \text{inlet condition or on fluid properties evaluated at } T_0 \]
\[ 1,2 : \text{at inner or outer wall, respectively} \]
\[ ( )_{ref} : \text{on fluid properties evaluated at reference temperature } T_{ref} \]
\[ \xi : \text{based on the characteristic length } \xi \]

3. Basic Equations and Numerical Analysis

The flow geometry to be considered is shown in Fig. 1. For simplification, some assumptions are introduced as follows:

1. the flow is steady, axisymmetric, and laminar;
2. the physical properties of fluid are a function of the temperature alone.

Applying the boundary layer approximation and introducing the dimensionless quantities based on the fluid properties at \( T_0 \) and the radius of the outer wall \( r_2 \):

\[ X = x/(r_2 \delta_0), R = r/r_2, U = u/(\nu \delta_0), \]
\[ V = u_0/(r_2 \delta_0), L = l/(r_2 \delta_0), \theta = (T - T_0)/(T_w - T_0), \]
\[ P = (p - p_0)/(\rho_0 \nu^2), c_p = C_p/(C_p \delta) \beta^* = \beta/\beta_0, \]
\[ \rho^* = \rho/\rho_0, \mu^* = \mu/\mu_0, \]

yields the following governing equations of continuity, momentum, and energy:

\[ \text{Fig. 1 Physical model and coordinate system} \]
\( \text{Nu}_{ai} = \frac{h_{i} r_{i}}{\lambda_{i}} \rho \alpha d \theta / \partial R, \) \quad (9)

where \( i = 1 \) or 2, representing inner or outer wall value, respectively; and \( \alpha \) takes value of \(-1 \) for \( i = 1 \), or \(+1 \) for \( i = 2 \). The modified Rayleigh number \((Ra)_{0}\) and average Nusselt numbers on the inner and outer walls, \((\bar{Nu})_{i} \) and \((\bar{Nu})_{o} \), are defined as follows:

\[ (Ra)_{0} = (PrGr) r_{o} / l, \quad (10) \]

\[ (\bar{Nu})_{i} = \bar{h}_{i} r_{i} / \lambda_{i} = (\lambda_{i} / l) \int_{r_{0}}^{r_{i}} \bar{Nu} dX. \quad (11) \]

The perimeter average Nusselt number \( \bar{Nu} \) is defined as

\[ (\bar{Nu})_{o} = \frac{1}{r_{o}} \int_{r_{0}}^{r_{o}} \bar{h}_{o} r_{o} \rho \alpha d \theta / \partial R dR \] \quad (12)

where \( \bar{h}_{o} \) is the average specific heat \( \bar{T} \) between \( T_{o} \) and \( T_{i} \). \((\bar{Nu})_{o} \) is also determined from the following expression incorporating the individual Nusselt numbers \( \bar{Nu}_{i} \) and \( \bar{Nu}_{o} \):

\[ (\bar{Nu})_{o} = \frac{(r_{o}^{*} \bar{Nu}_{i} + \bar{Nu}_{o}) / (1 + r_{o}^{*})}. \quad (13) \]

The above expression of \((\bar{Nu})_{o}\) was used for an accuracy check of the numerical analysis.

When the fluid properties are evaluated at a reference temperature \( T_{r} \), and the characteristic length is chosen as \( l_{r} \), the Rayleigh number, average Nusselt number, and dimensionless average axial velocity \( \bar{U} \) are expressed as

\[ (Ra)_{r} = (Ra) r_{i} (r_{i} / l_{r}) (Pr \rho \alpha \beta / (\mu_{r} \lambda_{r})), \quad (14) \]

\[ (\bar{Nu})_{r} = (\bar{Nu}) (r_{i} / l_{r}), \quad (15) \]

\[ (\bar{U})_{r} = (Gr_{r} / (\rho \alpha \pi (1 - r_{o}^{*}))), \quad (16) \]

The differential equations (2) and (3) with the boundary conditions in Eq. (6) are solved by a numerical method similar to that in previous studies \(^{9,10} \); the details of this are described in reference \(^{11} \).

4. Asymptotic Solutions

When one considers the case of a very small Rayleigh number or a very long duct, flow through the duct becomes fully developed both thermally and hydrodynamically; and the solution can be obtained analytically. Another limit of the flow is the case of a very large Rayleigh number or a very short duct in which the solution approaches the one of a single vertical flat plate without interaction.

In this section, we discuss these asymptotic solutions of ducts with arbitrary cross-sectional shapes and annular ducts at \( Ra \to 0 \) and \( Ra \to \infty \). In order to avoid complexity of the analysis, thermophysical properties of the fluid are assumed to be constant. The relation of the constant property solutions and variable property solutions will be discussed in the next section.

4.1 Ducts of arbitrary cross-sectional shapes with uniform wall temperature

For a vertical duct with a uniform wall temperature \( T_{w} \), the velocity distribution in the duct approaches that of a fully developed flow, and the temperature of the fluid approaches \( T_{w} \). Hence, the balance of the buoyant force and friction force is expressed as

\[ g \beta (T_{w} - T_{o}) = dp / dz. \quad (17) \]

Introducing a Fanning friction factor \( f \) and hydraulic diameter \( d_{h} \) defined as

\[ dp / dx = \frac{f}{dz}, \quad (18) \]

\[ d_{h} = 4S / F, \quad (19) \]

and the following dimensionless parameters,

\[ Re = \bar{U} d_{h} / \nu, \quad U = \bar{U} r_{2} / l_{r}, \quad D^{*} = d_{h} / r_{2}, \quad (20) \]

then, Eq. (17) reduces to a simple form such as

\[ \frac{U}{D^{*}} = \frac{f}{2}, \quad (21) \]

where \( S \) and \( F \) are the cross-sectional area and perimeter of the duct, respectively, \( \bar{U} \) is average velocity in the duct, and \( \Psi \) is a Elenbaas' parameter \(^{12} \), defined as

\[ \Psi = Re \cdot f. \quad (22) \]

For a general representation of the average Nusselt numbers of ducts of different shapes, Aihara et al. \(^{11} \) have proposed a characteristic length \( \xi \) which is defined as follows:

\[ \xi = D^{*} r_{2} / 2 \Psi^{1/3}. \quad (23) \]

If the characteristic length \( \xi \) is introduced, the dimensionless average velocity \( \bar{U}_{t} \) defined by Eq. (16) approaches the following equation for the case of \( Ra_{t} \to 0 \):

\[ \bar{U}_{t} = 2 \Psi^{-1/3}. \quad (24) \]

It can also be derived by the analysis of the energy balance between the heat transfer area and energy absorbed by the fluid that the average Nusselt number approaches the following simple equation for the case of \( Ra_{t} \to 0 \):

\[ \bar{Nu}_{t} = Ra_{t} (PrGr) \xi / l. \quad (25) \]

When the Rayleigh number is very large, the average Nusselt number should approach a similarity solution of the single vertical flat plate \(^{6} \) such as:

\[ \bar{Nu}_{t} = 0.795 \left( \frac{Pr}{1 + 2 \sqrt{Pr} + 2 Pr} \right)^{1/4} Ra_{t}^{1/4}. \quad (26) \]

4.2 Annular ducts with \( \theta_{1} = \theta_{2} \)

The Elenbaas' parameter \( \Psi = Re \cdot f \) for fully developed laminar flow through the annular ducts has been determined \(^{9} \) as:

\[ \Psi = 16(1 - r_{o}^{*})^{2} / (1 + r_{o}^{*} - 2 r_{w}^{*}), \quad (27) \]

where

\[ r_{w} = (1 - r_{o}^{*}) / (2 \ln(r_{o}^{*}))^{1/2}. \quad (28) \]

Therefore, the characteristic length \( \xi \) of the annular duct can be calculated as follows:

\[ \xi = r_{2}(1 - r_{o}^{*}) \Psi^{-1/3}. \quad (29) \]

The variations of \( \Psi \) and \( \xi \) of the annular ducts are shown in Fig. 2. It may be noted from Fig. 2 that the
presence of a small inner wall, for example $r^* = 0.1$, affects the values of $\xi/b$ and $\Psi$; these values are closer to those of parallel plates rather than a circular tube.

In general, the average Nusselt numbers on the inner and outer walls are not the same, even though the wall temperatures are equal. It is easy to expect that the free convective heat-transfer in fully developed annular ducts is similar to that of forced convection. The ratio of the average Nusselt numbers $\eta_{\text{av}} = r^* \bar{N}_u / \bar{N}_u$ for fully developed forced convection is shown in Fig. 3, and $\eta_{\text{av}}$ can be approximated by the following equation within a ±3% deviation in the region of the figure.

$$\eta_{\text{av}} = r^* \bar{N}_u / \bar{N}_u = \exp(0.467 \ln(r^*) + 0.030 \ln(r^*)^2)$$

(30)

4.3 Annular ducts with $\theta_1 = \theta_2$

When the temperatures on the inner and outer walls are not the same, the fluid temperature in the annular duct is not uniform at $Ra_r \to 0$, but has the same distribution as that of forced convection of a fully developed flow. The temperature distribution in the annular duct is a simple heat conduction problem; hence, the average temperature is obtained as follows:

$$\bar{T} = \frac{1}{1 - r^*} \int_{r^*}^{1} \frac{u}{\bar{u}} \theta R \, dR,$$

(31)

and

$$\bar{T} = \ln(R)(\theta_1 - \theta_2) / \ln(r^*) + \theta_2.$$  

(32)

The velocity profile $u/\bar{u}$ has been given by Lunderberg et al. [13] such as:

$$\frac{u}{\bar{u}} = \frac{2(1-R^2+2r_0 \ln(R))}{(1+r^2-2r_0^2)},$$

(33)

where $r_0$ is given by Eq. (28). Substituting $\theta_1 = 1$ and $\theta_2 = 0$, the dimensionless average temperature $\bar{T}$ is calculated and shown in Fig. 3. The average temperature for given $\theta_1$ and $\theta_2$ is easily obtained by the following equation.

$$\bar{\theta} = \bar{\theta}_0(\theta_1 - \theta_2) + \bar{\theta}_1$$

(34)

In the case of a very long duct or $Ra_r \to 0$, perimeter average Nusselt number $\bar{N}_u$ approaches zero, and conduction heat-transfer between the inner and outer wall becomes dominant. Then, the asymptotic Nusselt numbers can be obtained, considering the conduction between the inner and outer walls such as

$$\bar{N}_u = \frac{(\theta_1 - \theta_2) \xi / r_2}{r^* \ln(r^*)}, \quad \bar{N}_{\text{av}} = \frac{(\theta_1 - \theta_2) \xi / r_2}{r^* \ln(r^*)}$$

(35)

Introducing the average temperature $\bar{T}$, the perimeter average Nusselt number can be obtained from a heat balance analysis similar to Eq. (25),

$$\bar{N}_u = \frac{(r^* \bar{N}_u + \bar{N}_u \bar{T})}{(r^* + 1)} = \bar{T}^* R_a.$$  

(36)

The Nusselt number for $Ra_r \to \infty$ or a very short duct is obtained in a similar manner to Eq. (26),

$$\bar{N}_u = \frac{0.795 r^* \bar{T}_1 + \bar{T}_1}{r^* + 1} \left(1 + \frac{Pr}{Pr + 2Pr}ight)^{1/4} R_a^{1/4}.$$  

(37)

5. Results and Discussion

5.1 Constant property solutions

If one takes a limit of $T_w/T_o \to 1$ in the numerical analysis of section 3, then the dimensionless properties $c_r^*, \beta^*, \cdots$ become unity, and the solution coincides with the constant property solution. In the present numerical analysis, the ratio of the inner and outer wall radii $r^*$ was chosen as 0.1 and 0.5, and the working fluid was chosen as air at 300 K ($Pr = 0.7$).

The perimeter average Nusselt numbers $\bar{N}_u$ of the annular ducts with $\theta_1 = \theta_2$ are shown in Fig. 4. The characteristic length in the figure is $r_2$ according to the definition of Eq. (1), and the calculated result of a circular duct [17] is also plotted. The Nusselt numbers $\bar{N}_u$ of the various cross-sectional shapes approach the single plate solution, Eq. (26), but $\bar{N}_u$ at a small Rayleigh number show quite a large difference

![Fig. 2 Variations of characteristic lengths $\xi/r_2$, $\xi/b$, and Elenbaas' shape parameter $\Psi$ of annular ducts](image1)

![Fig. 3 Variations of the ratio of the Nusselt numbers on the inner and outer walls $\eta_{\text{av}} = r^* \bar{N}_u / \bar{N}_u$, and dimensionless average temperature $\bar{\theta}$ for $\theta_1 = 1$, $\theta_2 = 0$, in the case of a fully developed flow](image2)
between each other.

In Fig. 5, the present annular duct solutions and the ones of parallel plates and circular cylinder ducts(7) are plotted by using the characteristic length  $\xi$  which was introduced in the previous section. The annular duct solutions coincide very well with the parallel plates solution as opposed to the circular tube solution even though $r^*$ is small. This fact implies that the existence of a small inner wall makes the flow pattern and the temperature distribution in the duct similar to that of parallel plates. Also it may be noted that the heat-transfer characteristics of parallel plates can be approximated by the experiment of an annular duct with rather small $r^*$, because $\xi/b$ is almost constant for $0.1 < r^* < 1$.

In consideration of Elenbaas' empirical formula of the average Nusselt number for air(12), Aihara et al.(10) derived a general representation of the average Nusselt number regardless of the duct shape as follows:

$$\bar{N}_u = Ra_l [1 - \exp (-2Ra_l)]$$

(38)

In the present analysis, the following formula of the perimeter average Nusselt number of annular ducts with consideration of the asymptotic solutions Eqs. (25) and (26) is proposed:

$$\bar{N}_u = Ra_l [1 - \exp (-A Ra_l)]$$

(39)

where $A$ is expressed as

$$A = 1.4 \left( Pr + \sqrt{Pr} + 0.5 \right) / Pr^{1/3}$$

(40)

The general representation Eq.(39) can predict the perimeter average Nusselt numbers of a concentric annular duct and parallel plates within ±10% error, and the formula is valid for $0.65 \leq Pr \leq 470$, $0.1 \leq r^* \leq 1$, and $Ra_l \leq 10^7$.

Figure 6 shows variations of the dimensionless average axial velocity $\bar{U}_l$ by using the characteristic length $\xi$. The average velocity in the form of $\bar{U}_l$ approaches $2$ at a small Rayleigh number regardless of the cross-sectional shapes of the duct. The variations of $\bar{U}_l$ are similar to those of the average Nusselt numbers in Fig. 5.

Figure 7 shows variations of average Nusselt numbers on the inner and outer walls $\bar{N}_{u1}$ and $\bar{N}_{u2}$ respectively, for the case of $\theta_1 = \theta_2 = 1$. The Nusselt number on the inner wall $\bar{N}_{u1}$, takes a larger value than that on the outer wall $\bar{N}_{u2}$, even though the temperature on each wall is the same; the difference between $\bar{N}_{u1}$ and $\bar{N}_{u2}$ increases with small $r^*$.
Figures 8(a) and (b) show variations of the average Nusselt numbers, $\bar{Nu}_{ai}$ and $\bar{Nu}_{ao}$, with different wall temperatures on the inner and outer walls. As is shown in the figures, $\bar{Nu}_{ai}$ and $\bar{Nu}_{ao}$ represent completely different characteristics compared with the case of $\theta_i = \theta_e$. The heat-transfer in the case of $\theta_i = \theta_e$ is dominated by conduction between the inner and outer walls, and the values of $\bar{Nu}_{ai}$ and $\bar{Nu}_{ao}$ approach the constant values of Eq. (35). Miyatake and Fujii demonstrated that there is a point where the average Nusselt number becomes zero in the numerical analysis of free convection in parallel plates with different wall temperatures. The same trend can be seen in Fig. 8(b).

5.2 Effect of temperature dependence of thermophysical properties of fluid

The effect of variable fluid properties on free convection in annular ducts is examined for air as a working fluid. In the numerical analysis with variable thermophysical properties, inlet temperature $T_i$ and the maximum wall temperature $T_w$ are chosen as 300 K and 600 K, respectively.

The perimeter average Nusselt numbers with various wall temperatures are compared in Fig. 9, on the basis of the fluid properties at $T_s$. In the figure, the variable property solutions (VPS) become smaller than the constant property solutions (CPS), especially at a small Rayleigh number; the difference between CPS and VPS becomes small with a decrease in the ratio of the heated area.

Aihara et al. have shown that the effect of the temperature dependence is minimized if the fluid properties are evaluated at $T_w$ for the cases of a circular duct and the parallel plates. The data in Fig. 9 are rearranged evaluating the thermophysical properties at the average temperature $T_m$ of a fully developed flow, and the relation between VPS and CPS is shown in Fig. 10. The $T_m$ is defined by Eqs. (34) and (41); $T_m$ coincides with $T_w$ in the case of $\theta_i = \theta_e$,

$$T_m = \theta(T_w - T_b) + T_b. \quad (41)$$

The VPSs with various wall temperatures show very good accordance with the CPSs.

If one considers the asymptotic solutions Eqs. (36) and (37) and the approximation Eq. (39), a general representation of the perimeter average Nusselt numbers of annular ducts (0.1 ≤ r* ≤ 1) with various wall temperatures can be derived as follows:

$$\langle \bar{Nu}\rangle_{m} = \bar{\theta}(Ra/\theta)_{m} \times \left[1 - \exp \left(-\frac{r^* \theta_i + \theta_e}{(r^* + 1)\theta}(ARa/\theta)^2\right)\right], \quad (42)$$

where A has been expressed in Eq. (40). In the case of $\theta_i = \theta_e = 1$, Eq. (42) coincides with Eq. (39). The numerical solutions and the general presentation of the perimeter average Nusselt numbers Eq. (42) are compared in Fig. 10, and good accordance between the general representation and the numerical solutions can be seen in the figure.

6. Conclusions

A theoretical investigation of laminar free con-
Fig. 10 Comparison in the perimeter average Nusselt numbers between a general representation Eq. (42) and numerical solutions on the basis of the fluid properties at the average temperature $T_m$ of fully developed flow

Convection in vertical concentric annular ducts has been carried out. The results obtained are as follows.

1. The perimeter average Nusselt numbers $Nu_t$ of the annular ducts defined by characteristic length $\xi$ show a very close correlation with those of parallel plates rather than those of a circular tube; $\xi$ divided by the duct width $b$ is almost constant for $0.1 \leq \xi^* \leq 1$.

2. The asymptotic solutions of the average Nusselt numbers at very small Rayleigh numbers are derived from the discussion of a fully developed flow by using the characteristic length $\xi$ and Elenbaas' parameter $\Psi$.

3. A general representation of the perimeter average Nusselt number is presented for wide ranges of the radius ratio $r^*$ and Prandtl number $Pr$.

4. When the fluid physical properties are evaluated at the average temperature $T_m$ of a fully developed flow, the variable property solutions correlate with the constant property solutions.

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