A Zonal Approach for Solving the Compressible Navier-Stokes Equations Using a TVD Finite Volume Method*

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A new zonal approach for compressible viscous flow computation using a TVD finite volume method has been developed. The two-dimensional, Reynolds-averaged Navier-Stokes equations are discretized spatially by a cell-centered finite volume formulation. The inviscid fluxes at cell interfaces are evaluated through the MUSCL-type approach of an upwind scheme. The viscous fluxes are determined in a central differencing manner. In the present approach, the computational domain is divided into nonoverlapping zones. The zonal boundaries are constructed from the cell interfaces, since the finite volume formulation with the cell-centered control volumes is used. Consequently, communication from one zone to another is accomplished by numerical fluxes through the zonal boundaries. The use of the finite volume formulation can ensure the uniqueness of the zonal boundaries and the complete conservation of the numerical fluxes across the zonal boundaries, which results in a highly accurate zonal approach. In order to demonstrate the versatility of the present zonal approach, numerical results are presented for viscous flows through a transonic turbine cascade.

Key Words: Computational Fluid Dynamics, Zonal Approach, Navier-Stokes Equations, TVD Scheme

1. Introduction

In recent years, a number of highly accurate total variation diminishing (TVD) schemes^{(1)~(3)} have been developed for solving the Euler equations using upwind-difference methods. As a result of the property that the total variation of the discrete solution does not increase with time, the TVD schemes have nonoscillatory but sharp shock capturing properties without any tuning of the numerical dissipation terms. Because of the robust nature of these TVD schemes, they are more reliable than conventional central-difference schemes with numerical dissipation terms. In addition, the TVD schemes can also be extended for the evaluation of the inviscid terms of the Navier-Stokes equations⁽⁴⁾. According to the studies on the application of the TVD schemes to the Navier-Stokes

equations by van Leer et al.⁽⁵⁾ it was found that built-in numerical dissipation terms introduced by the highly accurate TVD schemes based on Riemann solvers automatically become small in boundary layers so as not to dominate the natural diffusion.

In the application of numerical schemes to practical engineering problems, the use of a single grid system causes difficulty in grid generation because of complex geometries. To avoid this difficulty, zonal approaches with the overlaid⁽⁶⁾⁽⁷⁾ or composite^{(8)~(11)} grid system have been developed by several researchers. In the zonal approaches, as shown in Fig. 1, a computational domain is divided into zones, and grids are generated independently in each zone. The division of the computational domain introduces interfaces between the zones which are referred to as the zonal boundaries. The most important point in the zonal approach relates to the treatment of the zonal boundaries. The following treatments have been proposed: the Chimera scheme for interpolations on overlaid grids⁽⁶⁾, the Rai's scheme for a TVD finite difference formulation⁽⁸⁾, the scheme based on characteristic boundary conditions⁽¹⁰⁾ and the FDM-FEM hybrid scheme⁽¹¹⁾. Only in the Rai's scheme is the

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conservation of numerical fluxes across the zonal boundaries taken into account. However, in the Rai's scheme, the fluxes are evaluated by the finite difference method, and so the flux conservation across the zonal boundaries is not entirely satisfied.

In this paper, we propose a new zonal approach for the computation of compressible viscous flows, in which a highly accurate TVD, finite volume formulation with the composite zonal grid system is implemented in order to produce a Navier-Stokes solver with robustness and high accuracy. The emphasis in this paper is on the treatment of the zonal boundary. In the present zonal approach, the zonal boundary is constructed of interfaces of computational cells, and the communication from one zone to the other is accomplished by numerical fluxes across the zonal boundary. The complete conservation of the fluxes at the zonal boundary can be satisfied by directly evaluating the fluxes using the finite volume formulation and by ensuring the uniqueness of the zonal boundary between the two zones. The present zonal approach is applied to the computation of two-dimensional flows through a transonic turbine cascade. It is found that the approach is a highly accurate scheme for complex flow fields and that distortions of the solution near the zonal boundaries are inhibited.

2. TVD Finite Volume Formulation

2.1 Finite volume formulation

The two-dimensional, Reynolds-averaged Navier-Stokes equations are integrated over a computational cell, as shown in Fig. 2. The integrated equations can be written in a nondimensional form as

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} Q d\Omega + \oint_{\partial\Omega} H dS \\ = \frac{1}{Re} \oint_{\partial\Omega} R dS \end{aligned} \quad (1)$$

where the cell region is denoted by Ω , the cell boundary by $\partial\Omega$, and the distance along the boundary by S .

Fig. 1 Zonal grids

In the above, Re denotes the Reynolds number, and Q , H , and R are the vectors of conserved variables, inviscid fluxes and viscous fluxes, respectively, which are given by

$$\begin{aligned} Q &= \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix}, \quad H = \begin{bmatrix} \rho U \\ \rho u U + p n_x \\ \rho v U + p n_y \\ (e + p) U \end{bmatrix}, \\ R &= \begin{bmatrix} 0 \\ \tau_{xx} n_x + \tau_{xy} n_y \\ \tau_{xy} n_x + \tau_{yy} n_y \\ f_x n_x + f_y n_y \end{bmatrix} \end{aligned} \quad (2)$$

with

$$\left. \begin{aligned} U &= u n_x + v n_y \\ \tau_{xx} &= \mu(4u_x - 2v_y)/3 \\ \tau_{xy} &= \mu(u_y + v_x) \\ \tau_{yy} &= \mu(4v_y - 2u_x)/3 \\ f_x &= u\tau_{xx} + v\tau_{xy} + \mu Pr^{-1}(\gamma - 1)^{-1}(a^2)_x \\ f_y &= u\tau_{xy} + v\tau_{yy} + \mu Pr^{-1}(\gamma - 1)^{-1}(a^2)_y \end{aligned} \right\} \quad (3)$$

Here, ρ is the density, u and v are the velocity components in the x and y directions respectively, e is the total energy per unit volume, p is the pressure, n_x and n_y are the x and y components of the outward unit vector normal to the cell boundary, γ is the ratio of specific heats, and a is the sound speed. Using the molecular viscosity given by Sutherland's law, μ_i , the eddy viscosity, μ_t , the laminar Prandtl number, Pr_i , and the turbulent Prandtl number Pr_t , μ and μ/Pr are expressed as

$$\mu = \mu_i + \mu_t \quad (4)$$

$$\mu/Pr = \mu_i/Pr_i + \mu_t/Pr_t \quad (5)$$

The algebraic turbulence model of Baldwin and Lomax⁽¹²⁾ is adopted to estimate the eddy viscosity, and the boundary layer transition is determined in accordance with the criterion of the transition based on the Baldwin and Lomax model. In the present scheme, Pr_i and Pr_t are assumed to be constant. For the air, they are assumed to be 0.72 and 0.9, respectively. Only the perfect gas will be considered, in which case the pressure and sound speed can be given by

$$p = (\gamma - 1)[e - \rho(u^2 + v^2)/2] \quad (6)$$

Fig. 2 Computational cells

$$a^2 = r(r-1)[e/\rho - (u^2 + v^2)/2] \quad (7)$$

Using the cell-centered finite volume formulation with a quadrilateral structured cell system shown in Fig.2 where the vector of conserved variables averaged over the cell (i, j) , $Q_{i,j}$, is defined at the cell center, Eq.(1) is spatially discretized as

$$\begin{aligned} & \frac{\partial}{\partial t}(A_{i,j}Q_{i,j}) + (SH)_{i,j-(1/2)} + (SH)_{i+(1/2),j} \\ & + (SH)_{i,j+(1/2)} + (SH)_{i-(1/2),j} \\ & = [(SR)_{i,j-(1/2)} + (SR)_{i+(1/2),j} + (SR)_{i,j+(1/2)} \\ & + (SH)_{i-(1/2),j}]/Re \end{aligned} \quad (8)$$

where $A_{i,j}$ is the cell area, and $S_{i+1/2,j}$ is the length of the cell interface.

2.2 Evaluation of inviscid fluxes

The inviscid flux vector $H_{i+1/2,j}$ at each cell interface is computed by the following highly accurate TVD scheme. The TVD scheme used is based on the MUSCL-type approach of an upwind scheme which consists of a projection stage and an evolution stage.

2.2.1 Projection stage To raise the order of accuracy of the scheme, the cell-centered values of the conserved variables are extrapolated toward the cell interfaces according to piecewise distributions of conserved variables in each cell, as shown in Fig. 3. The projection with a differentiable limiter⁽¹⁾ which is introduced to preserve the nonoscillatory nature of the scheme is used in the present scheme. For the projection in the i direction on the cell (i,j) , the extrapolated conserved variables, $Q_{i+1/2,j}^-$ and $Q_{i-1/2,j}^+$ are given by

$$\begin{aligned} Q_{i+1/2,j}^- &= Q_{i,j} + \{0.25s[(1-\kappa s)\Delta_- \\ & + (1+\kappa s)\Delta_+]\}_{i,j} \\ Q_{i-1/2,j}^+ &= Q_{i,j} - \{0.25s[(1-\kappa s)\Delta_+ \\ & + (1+\kappa s)\Delta_-]\}_{i,j} \end{aligned} \quad (9)$$

where

$$\begin{aligned} (\Delta_+)_{i,j} &= Q_{i+1,j} - Q_{i,j} \\ (\Delta_-)_{i,j} &= Q_{i,j} - Q_{i-1,j} \end{aligned} \quad (10)$$

$$s = (2\Delta_+\Delta_- + \varepsilon)/[(\Delta_+)^2 + (\Delta_-)^2 + \varepsilon] \quad (11)$$

In the above, s is the limiter, and ε is a small number, $\varepsilon=10^{-6}$. The parameter κ determines the spatial accuracy: $\kappa=-1$ corresponds to the second-order fully-upwind scheme, and $\kappa=1/3$ to the third-order upwind-biased scheme. For the numerical results presented in this paper, the third-order upwind-biased scheme is

Fig. 3 Projection stage

used. The formulas similar to Eq.(9) hold for the projection in the j direction, and so it is found that the cell-centered, conserved variables at the cells adjacent to the cell considered are required for the projection.

2.2.2 Evolution stage The numerical inviscid flux vector $H_{i+1/2,j}$ can be evaluated by solving the Riemann problem with the left and right states, $Q_{i+1/2,j}^-$ and $Q_{i+1/2,j}^+$, at the cell interface. In the present approach, the upwind scheme⁽¹⁴⁾ based on the approximate Riemann solver of Roe⁽¹³⁾ is adopted, in which the inviscid flux is constructed as

$$H_{i+1/2,j} = f^{Roe}(Q_{i+1/2,j}^-, Q_{i+1/2,j}^+, \hat{n}_{i+1/2,j}) \quad (12)$$

2.3 Evaluation of viscous fluxes

In order to construct the numerical viscous flux vector at the cell interfaces, it is necessary to evaluate the first-order derivatives of the velocity components and the sound speed. The first-order derivatives can be evaluated in the following central differencing manner.

For the evaluation of the first-order derivatives of the quantity u at the cell interface $(i+1/2,j)$, shown as the side SN in Fig. 4, an auxiliary cell Ω' surrounding the cell interface SN , shown by the dashed lines in Fig. 4, is introduced. Its four corners, NE , NW , SW and SE , are the midpoints of the cell interfaces. Assuming the first-order derivatives to be approximated by those averaged over the auxiliary cell, and using Green's theorem, the derivatives can be evaluated as

$$\begin{aligned} \left(\frac{\partial u}{\partial x} \right)_{i+1/2,j} &= \frac{1}{A'} \iint_{\Omega'} \frac{\partial u}{\partial x} d\Omega' = \frac{1}{A'} \int_{\partial \Omega'} u dy \\ \left(\frac{\partial u}{\partial y} \right)_{i+1/2,j} &= \frac{1}{A'} \iint_{\Omega'} \frac{\partial u}{\partial y} d\Omega' = \frac{-1}{A'} \int_{\partial \Omega'} u dx \end{aligned} \quad (13)$$

where A' and $\partial \Omega'$ denote the area and boundary of the auxiliary cell, respectively. Assuming the quantity to be constant along each side of the auxiliary cell, Eq. (13) is written as

Fig. 4 Evaluation of first-order derivatives

$$\left. \begin{aligned} \int_{\partial \Omega} u dy &= u_E \Delta y_E + u_N \Delta y_N \\ &+ u_W \Delta y_W + u_S \Delta y_S \\ \int_{\partial \Omega} u dx &= u_E \Delta x_E + u_N \Delta x_N \\ &+ u_W \Delta x_W + u_S \Delta x_S \end{aligned} \right\} \quad (14)$$

with

$$\left. \begin{aligned} u_E &= u_{i+1,j}, \quad u_W = u_{i,j} \\ u_N &= (u_{i+1,j+1} + u_{i,j+1} + u_{i,j} + u_{i+1,j})/4 \\ u_S &= (u_{i+1,j} + u_{i,j} + u_{i,j-1} + u_{i+1,j-1})/4 \end{aligned} \right\} \quad (15)$$

$$\left. \begin{aligned} \Delta x_E &= x_{NE} - x_{SE}, \quad \Delta y_E = y_{NE} - y_{SE} \\ \Delta x_N &= x_{NW} - x_{NE}, \quad \Delta y_N = y_{NW} - y_{NE} \\ \Delta x_W &= x_{SW} - x_{NW}, \quad \Delta y_W = y_{SW} - y_{NW} \\ \Delta x_S &= x_{SE} - x_{SW}, \quad \Delta y_S = y_{SE} - y_{SW} \end{aligned} \right\} \quad (16)$$

Using the first-order derivatives of the velocity components and the sound speed evaluated by the above formulas, the viscous fluxes are given by Eqs. (2) and (3).

2.4 Time integration

The steady flow solutions are given by integrating Eq. (8) in time. In the present paper, a three stage Runge-Kutta scheme⁽¹⁵⁾ is applied to the integration in time. For simplicity, Eq. (8) can be written in the form

$$dQ/dt = W(Q) \quad (17)$$

In this time stepping scheme, the solution is advanced from time level n to $n+1$ as

$$\left. \begin{aligned} Q^{(0)} &= Q^n \\ Q^{(1)} &= Q^{(0)} + \alpha_1 \Delta t \cdot W(Q^{(0)}) \\ Q^{(2)} &= Q^{(0)} + \alpha_2 \Delta t \cdot W(Q^{(1)}) \\ Q^{(3)} &= Q^{(0)} + \alpha_3 \Delta t \cdot W(Q^{(2)}) \\ Q^{n+1} &= Q^{(3)} \end{aligned} \right\} \quad (18)$$

with

$$\alpha_1 = 0.21, \quad \alpha_2 = 0.5, \quad \alpha_3 = 1.0 \quad (19)$$

This scheme is second-order accurate in time for linear equations. In the above, Δt is the time step which is given according to the local time stepping as

$$\Delta t_{i,j} = \lambda (\Delta t_{CFL})_{i,j} \quad (20)$$

where λ denotes the limiting CFL number, and $(\Delta t_{CFL})_{i,j}$ is determined by the local CFL condition expressed approximately for the cell shown in Fig. 5 as follows:

$$\Delta t_{CFL} = A / [\Delta h_\xi (|\vec{q} \cdot \vec{n}_\xi| + a) + \Delta h_\eta (|\vec{q} \cdot \vec{n}_\eta| + a)] \quad (21)$$

Fig. 5 Local coordinates on a cell

Here, A is the cell area, $\vec{q}$ is the velocity, a is the sound speed, $\vec{n}_\xi$ and $\vec{n}_\eta$ are unit vectors normal to the ξ and η directions shown in Fig. 5, respectively. For the numerical results presented in this paper, the CFL number is set to 1.7.

2.5 Boundary conditions

In the present approach, boundaries of a computational domain are formed by cell interfaces, since the cell-centered finite volume formulation is used. Hence, the boundary conditions must be imposed by the fluxes through the boundaries rather than by the conserved variables at the boundaries.

At the solid wall boundary, the no-slip and adiabatic conditions are imposed. For the inviscid fluxes, the momentum fluxes due to the pressure remain, but all the other inviscid fluxes vanish. The pressure at the solid wall is set equal to the cell-centered value of pressure at the interior cell adjacent to the wall. Using the fictitious cells ($i,0$) shown by dashed lines in Fig. 6, the viscous fluxes at the wall are evaluated by applying the same scheme as that described in Section 2.3. The cell-centered values at the fictitious cells are specified as follows:

$$\left. \begin{aligned} \rho_{i,0} &= \rho_{i,1} \\ u_{i,0} &= -u_{i,1} \\ v_{i,0} &= -v_{i,1} \\ e_{i,0} &= e_{i,1} \end{aligned} \right\} \quad (22)$$

In addition, the shape of the fictitious cell is assumed to be the same as that of the adjacent interior cell.

At the inflow and outflow boundaries, the inviscid fluxes are determined by the method introduced by Sawada et al.⁽¹⁶⁾ in which the values of conserved variables corresponding to the boundary conditions are allocated to the fictitious cells just outside the boundaries to evaluate the inviscid fluxes in the same way as those through the interior cell interfaces. This boundary treatment prevents nonphysical reflections at the boundaries, since the inviscid fluxes are evaluated according to the approximate Riemann solver in which the signal propagation properties of the Euler equations are simulated. The upstream conditions are specified at the fictitious cells adjacent to the inflow boundary. At the fictitious cells adjacent to the outflow boundary, the pressure is prescribed, and the

Fig. 6 Fictitious cells at boundary

density and the velocity components are set equal to the cell-centered values of those at the adjacent interior cells.

In the next section, the treatment of the zonal boundary, which is important to the zonal approach, is described in detail.

3. Zonal Boundary Treatment

3.1 Uniqueness of zonal boundary

Consider the composite zonal grid system shown in Fig. 7. If the zonal boundary is not straight, the independent generation of zonal grids causes overlap and clearance between two zones, which means that the total area of the quadrilateral cells is not equal to the area of the computational domain. In order to avoid this problem, the common zonal boundary between the two zones is defined by the boundary of the one zone, zone 1 in Fig. 7. Consequently, for the cell $(i,1)$ of zone 2 in Fig. 7, the cell area must be evaluated by the polygon $P_0^{(2)} P_1^{(2)} P_2^{(2)} P_3^{(2)} B C$ rather than by the quadrilateral $ABCD$, and the length of the cell side $(i-1/2,1)$ must be evaluated by the line segment $CP_0^{(2)}$ rather than by the line segment CD . It should be noted that this uniqueness of the zonal boundary is important to the flux conservation. In the present approach, the cell-centered finite volume formulation is adopted, and the uniqueness of the zonal boundary is ensured, hence, the zonal boundaries are uniquely formed by the cell interfaces. This fact enables one to satisfy the complete conservation of the numerical fluxes across the zonal boundary, as described later in Section 3.4.

3.2 Inviscid fluxes at zonal boundary

3.2.1 Projection at zonal boundary The cells adjacent to the zonal boundary, such as the cell (k,L) of zone 1 and the cell $(i,1)$ of zone 2 in Fig. 7, will be

Fig. 7 Zonal boundary treatment

referred to as “zonal cells” in the following. In order to apply the projection defined by Eq. (9) to the zonal cells, it is necessary to find the cell-centered values of conserved variables at the fictitious cells adjacent to the zonal cells. At the fictitious cell $(k, L+1)$ adjacent to the zonal cell (k, L) of zone 1 in Fig. 7, the cell-centered value $Q_{k,L+1}^{(1)}$ is estimated by interpolating the value at the point I shown in Fig. 8 as follows:

$$Q_{k,L+1}^{(1)} = \frac{l_N Q_{i-1,1}^{(2)} + l_M Q_{i,1}^{(2)}}{l_M + l_N} \quad (23)$$

where

$$\left. \begin{aligned} l_M &= \overline{IM} \\ l_N &= \overline{IN} \end{aligned} \right\} \quad (24)$$

As shown in Fig. 8, the point I is defined as the intersection of the line segment MN and the extension from the cell center (k, L) in the following direction:

$$\overrightarrow{KI} = |\overrightarrow{KI}|(\overrightarrow{FE} + \overrightarrow{GH})/|\overrightarrow{FE} + \overrightarrow{GH}| \quad (25)$$

In the above, superscripts (1) and (2) represent the values at zone 1 and zone 2, respectively. The cell-centered values at the fictitious cells adjacent to the zonal cells of zone 2, such as at the fictitious cell $(i, 0)$, are interpolated in the same manner.

3.2.2 Evolution at zonal boundary In the case of Fig. 7, the inviscid fluxes across the zonal boundary are evaluated by applying the evolution to the zonal cell interfaces of zone 1, because the zonal boundary is defined by the boundary of zone 1. For the evolution at an interface $(k, L+1/2)$ of a zonal cell (k, L) , the right state, $Q_{k,L+1/2}^{(1)}$, is not known. However, the right states at the zonal cell interfaces of zone 2 adjacent to the cell (k, L) , $Q_{i-1,1/2}^{(2)}$ and $Q_{i,1/2}^{(2)}$, are given by the projections in zone 2. Therefore, using $Q_{i-1,1/2}^{(2)}$ and $Q_{i,1/2}^{(2)}$, $Q_{k,L+1/2}^{(1)}$ can be estimated as

$$Q_{k,L+1/2}^{(1)} = \sum_{m=1}^{M_k(1)} (W_k^{(1)})_m \cdot Q_{i_0(k)+m-1,1/2}^{(2)} \quad (26)$$

where $(W_k^{(1)})_m$ denotes the interpolation coefficient defined by

$$(W_k^{(1)})_m = \overline{P_m^{(1)} P_{m-1}^{(1)}} / S_{k,L+1/2}^{(1)} \quad (27)$$

$$S_{k,L+1/2}^{(1)} = \sum_{m=1}^{M_k(1)} \overline{P_m^{(1)} P_{m-1}^{(1)}} \quad (28)$$

In the above, $i_0(k)$ denotes the index number of the

Fig. 8 Interpolation for fictitious cells at zonal boundary

zonal cell of zone 2 adjacent to the vertex $P_0^{(1)}$ of the zonal cell (k, L) , i.e. $i_0(k) = i - 1$ in Fig. 7, m is the local number of the line segment $P_m^{(1)} P_{m-1}^{(1)}$ in the cell (k, L) , and $M_k^{(1)}$ is the total number of the line segments. Consequently, the application of the evolution to the left state $Q_{k,L+1/2}^{(1)}$ given by the projection in zone 1 and the right state $Q_{k,L+1/2}^{(1)}$ estimated by Eq. (26) yields the inviscid fluxes $H_{k,L+1/2}^{(1)}$ across the zonal boundary.

3.3 Viscous fluxes at zonal boundary

The viscous fluxes across the zonal boundary are also evaluated at the zonal cell interfaces $(k, L+1/2)$ of zone 1 in Fig. 7. The cell-centered values at the adjacent fictitious cells $(k, L+1)$ are interpolated in zone 2, as mentioned in subsection 3.2.1, and so the viscous fluxes $R_{k,L+1/2}^{(1)}$ at the zonal cell interfaces $(k, L+1/2)$ can be determined by applying the method described in section 2.3 to the zonal cell interfaces. The shape of the fictitious cell is assumed to be the same as that of the adjacent interior cell.

3.4 Flux conservation across zonal boundary

At the zonal cell (k, L) of zone 1 in Fig. 7, the numerical fluxes across the zonal cell interfaces $(k, L+1/2)$, $F_{k,L+1/2}^{(1)}$, which defined by the inviscid and viscous fluxes in the following equation, are given according to the procedures described in sections 3.2 and 3.3.

$$F_{k,L+1/2}^{(1)} = H_{k,L+1/2}^{(1)} - \frac{1}{Re} R_{k,L+1/2}^{(1)} \quad (29)$$

Hence, Eq. (8) can be applied to the zonal cells (k, L) of zone 1. On the other hand, at the zonal cells $(i, 1)$ of zone 2, the fluxes across the zonal cell interfaces $(i, 1/2)$, $F_{i,1/2}^{(2)}$, must be found. It should be noted that the conservation of the fluxes has to be satisfied across the zonal boundary. In the present approach, the flux conservation across the zonal boundary is satisfied by interpolating $F_{i,1/2}^{(2)}$ in the following manner: as shown in Fig. 9, assuming that the fluxes known at the zonal cell interfaces $(k, L+1/2)$ of zone 1, $F_{k,L+1/2}^{(1)}$, are constant along each interface, the value of $F_{i,1/2}^{(2)}$ is set equal to the shaded area shown in Fig. 9, that is, $F_{i,1/2}^{(2)}$ is evaluated as

$$\begin{aligned} & \text{DO } k=1, K \\ & \text{DO } m=1, M_k^{(1)} \\ & i = \text{Table}^{(1)}(k, m) \\ & (SF)_{i,1/2}^{(2)} = (SF)_{i,1/2}^{(2)} - (W_k^{(1)})_m \cdot (SF)_{k,L+1/2}^{(1)} \\ & \text{DO END} \\ & \text{DO END} \end{aligned} \quad (30)$$

where $\text{Table}^{(1)}(k, m)$ gives the index number i of the zonal cell of zone 2 adjacent to the line segment $P_m^{(1)} P_{m-1}^{(1)}$ in the zonal cell (k, L) of zone 1.

As just stated, the complete conservation of the numerical fluxes at the zonal boundary can be satisfied by directly evaluating the fluxes using the finite vol-

ume formulation and by ensuring the uniqueness of the zonal boundary.

4. Numerical Examples

Numerical results are presented for two-dimensional, transonic flows through a VKI turbine cascade^{(17)~(19)} to demonstrate the versatility of the present zonal approach. The computational domain and grid are shown in Figs. 10 and 11 respectively. A composite zonal grid system with a C-type grid is generated, in which the periodic boundaries are treated as the zonal boundaries, that is, the grid lines are discontinuous at the periodic boundaries. The total number of computational cells is 446×50 , and the numbers of cells on the pressure and suction surfaces are 70 and 120, respectively. The minimum grid spacing at the wall is 1×10^{-5} in chord length. Computational conditions are summarized in Table 1, where β_1 is the inlet flow angle, M_1 is the inlet Mach number, M_{2is} is the isentropic exit Mach number, and Re_2 is the Reynolds number based on the chord length and the exit conditions.

Density contours and blade surface pressure distributions computed by the present zonal approach

Fig. 9 Flux conservation at zonal boundary

Fig. 10 Computational domain

are shown in Figs. 12, 13, and 14 for each computational condition. In the surface pressure distributions, the computational results are denoted by

Fig. 11 Composite zonal grid

(a) Density contours

solid lines, and the experimental results^{(17)~(19)} by symbols. It is found that the fully conservative nature of the present approach permits the smooth transition of shock waves across the zonal boundaries. The appearance of shock waves at the zonal boundaries causes no numerical instability in the present calculations. The computed surface pressure distributions are in excellent agreement with the experimental results except for a separation region for case 3, caused by the

Table 1 Computational condition

	Case 1	Case 2	Case 3
β_1	30°	30°	30°
M_1	0.268	0.270	0.282
$M_{2\infty}$	0.985	1.11	1.34
Re_2	7.8×10^5	9.0×10^5	8.0×10^5

(b) Blade surface pressure distribution

Fig. 12 Computational results for case 1

(a) Density contours

(b) Blade surface pressure distribution

Fig. 13 Computational results for case 2

(a) Density contours

(b) Blade surface pressure distribution

Fig. 15 Velocity vectors in separation region for case 3

(a) With uniqueness of zonal boundary

(b) Without uniqueness of zonal boundary

Fig. 16 Velocity vectors near zonal boundary

strong interaction between the wake shock and the boundary layer on the suction surface. Hence, the present zonal approach is found to be a highly accurate formulation. The discrepancy between the computational and experimental results in the separation region for case 3 may be due to the application of the Baldwin-Lomax turbulence model to the separation region and due to the unsteadiness caused by the separation.

The separation region for case 3 is clearly shown by velocity vectors in Fig.15. The transition of the separation region across the zonal boundary is found to be smooth, which results from the satisfaction of the complete flux conservation across the zonal boundary. Figures 16(a) and 16(b) show velocity vectors near a zonal boundary. These results are computed with and without the uniqueness of the zonal boundary described in section 3.1. Distortions of the solution exist near the zonal boundary without the uniqueness, which implies that the flow solutions are dependent on the location of the zonal boundary without the uniqueness. It should be noted that the uniqueness of the zonal boundary is important to the zonal approach.

5. Conclusions

A new zonal approach for solving the compressible Navier-Stokes equations has been developed, in which a TVD finite volume formulation with a composite zonal grid system is implemented. In the present approach, the zonal boundaries are formed by cell interfaces, and the communication from one zone to the other is accomplished by numerical fluxes across the zonal boundaries. The complete conservation of the numerical fluxes at the zonal boundary can be satisfied by directly evaluating the fluxes using the finite volume formulation and by ensuring the uniqueness of the zonal boundary. This complete conservation results in the highly accurate nature of the present zonal approach.

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